# THE WEIERSTRASS-STONE THEOREM IN ABSOLUTE VALUED DIVISION RINGS

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Let $S$ be a zero-dimensional compact Hausdorff space and let $E$ be a normed space over a non-Archimedean absolute valued division ring $(I\!K, |\cdot|)$. The space $C(S;E)$ of all continuous functions from $S$ into $E$ is equipped with the uniform topology given by the supremum norm. A Weierstrass-Stone Theorem for arbitrary subsets of $C(S;E)$ is established.

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# The Weierstrass-Stone Theorem in Absolute Valued Division Rings

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## Abstract

Let $S$ be a zero-dimensional compact Hausdorff space and let $E$ be a normed space over a non-Archimedean absolute valued division ring $(\mathbb{K}, |\cdot|)$. The space $C(S; E)$ of all continuous functions from $S$ into $E$ is equipped with the uniform topology given by the supremum norm. A Weierstrass-Stone Theorem for arbitrary subsets of $C(S; E)$ is established.

## 1. The main Theorem

Let $S$ be a compact Hausdorff space which is 0-dimensional (i.e., for any point $s$ belonging to an open subset $G$, there exists a closed and open set $A$ with $s \in A \subset G$). Let $(\mathbb{K}, |\cdot|)$ be a complete non-Archimedean absolute valued division ring. Let $E$ be a non-trivial normed space over $\mathbb{K}$, and let $C(S; E)$ be the linear space of all continuous functions from $S$ into $E$, equipped with the supremum norm

$$||f|| = \sup\{||f(x)||;\ x \in S\}.$$

In a forthcoming paper [7] we show how to extend some of the results of this paper to case of a topological ring $(E, \tau)$.

**Definition 1.** A non-empty subset $M \subset C(S; \mathbb{K})$ is said to have **property** $V$ if

(1) $|\varphi(s)| \leq 1$, for every $s \in S$ and $\varphi \in M$;

(2) if $\varphi \in M$, then $1 - \varphi$ belongs to $M$;

(3) if $\varphi$ and $\psi$ belong to $M$, then $\varphi\psi \in M$.

**Definition 2.** Let $W \subset C(S; E)$ be a non-empty subset. A function $\varphi \in C(S; \mathbb{K})$ is called a **multiplier** of $W$ if

(1) $|\varphi(s)| \leq 1$, for every $s \in S$;

(2) if $f$ and $g$ belong to $W$, then $\varphi f + (1 - \varphi)g$ belongs to $W$.

Clearly, if $M$ denotes the set of all multipliers of $W$, then $M$ satisfies conditions (1) and (2) of Definition 1. The identity

$$(\varphi\psi)f + (1 - \varphi\psi)g = \varphi[\psi f + (1 - \psi)g] + (1 - \varphi)g$$

shows that $M$ satisfies condition (3) as well. Hence $M$ has property $V$. Notice that the constant functions 0 and 1 belong to $M$.

**Definition 3.** A subset $A \subset C(S; \mathbb{K})$ is said to be **separating over** $S$, if given any two distinct points, $s$ and $t$, of $S$, there exists a function $\varphi \in A$ such that $\varphi(s) \neq \varphi(t)$.

**Definition 4.** A subset $M \subset C(S; \mathbb{K})$ is said to be **strongly separating over** $S$, if given any ordered pair $(s, t) \in S \times S$, with $s \neq t$, there exists a function $\varphi \in M$ such that $\varphi(s) = 1$, $\varphi(t) = 0$ and $|\varphi(x)| \leq 1$ for all $x \in S$.

**Proposition 1.** *If $A$ is a unitary subalgebra of $C(S; I\!K)$ which is separating over $S$, then $A$ is strongly separating over $S$.*

**Proof.** Let $s \neq t$ be given in $S$. Since $A$ is vector space containing the constants, there is $a \in A$ such that $a(s) = 1$ and $a(t) = 0$. Since $a$ is continuous, $a(S)$ is a compact subset of $I\!K$. By Kaplansky's Lemma (see Lemma 1.23, Prolla [6]) there is a polynomial $p : I\!K \to I\!K$ such that $p(1) = 1$, $p(0) = 0$ and $|p(y)| \leq 1$ for all $y \in a(S)$. Let $\varphi(x) = p(a(x))$, for all $x \in S$. Then $\varphi \in A$, $\varphi(s) = 1$, $\varphi(t) = 0$, and $|\varphi(x)| \leq 1$ for all $x \in S$. Hence $A$ strongly separates the points of $S$. □

**Lemma 1.** *Let $M \subset C(S; I\!K)$ be a non-empty subset with property V, and containing the constant functions 0 and 1. Assume that $M$ is strongly separating over $S$. Let $N$ be a clopen subset of $S$. For each $\delta > 0$, there is $\varphi \in M$ such that*

(1) $|1 - \varphi(t)| < \delta$, *for all* $t \in N$,

(2) $|\varphi(t)| < \delta$, *for all* $t \notin N$.

**Proof.** If $N = S$, the function $\varphi(t) = 1$ for all $t \in S$, satisfies (1) and (2).

If $N = \emptyset$, the function $\varphi(t) = 0$ for all $t \in S$ satisfies (1) and (2). Assume that $K = S \backslash N$ is non-empty. Fix $y \in S$, $y \notin N$. For each $t \in N$, there is $\varphi_t \in M$ such that $\varphi_t(t) = 0$, $\varphi_t(y) = 1$. By continuity there exists a neighborhood $V(t)$ of $t$ such that $|\varphi_t(s)| < \delta$ for all $s \in V(t)$. By compactness of $N$ there are $t_1, \ldots, t_n \in N$ such that $N \subset V(t_1) \cup \ldots \cup V(t_n)$. Consider $\varphi_y = 1 - \varphi_{t_1} \cdot \varphi_{t_2} \cdot \ldots \cdot \varphi_{t_n}$. Then $\varphi_y \in M$ and $\varphi_y(y) = 0$, while $|1 - \varphi_y(t)| < \delta$, for all $t \in N$. Indeed, if $t \in N$, then $t \in V(t_i)$ for some $i = 1, \ldots, n$. Hence

$$|1 - \varphi_y(t)| = |\varphi_{t_i}(t)| \cdot \prod_{j \neq i} |\varphi_{t_j}(t)| < \delta\,.$$

By continuity there exists a neighborhood $W(y)$ of $y$ such that $|\varphi_y(s)| < \delta$ for all $s \in W(y)$. By compactness of $K$, there are $y_1, \ldots, y_m \in K$ such that $K \subset W(y_1) \cup \ldots \cup W(y_m)$. Let $\varphi = \varphi_{y_1} \cdot \varphi_{y_2} \cdot \ldots \cdot \varphi_{y_m}$. Clearly, $\varphi \in M$.. We claim that for each $k = 1, 2, \ldots, m$, we have

(3) $|1-\varphi_{y_1}(t)\varphi_{y_2}(t)\cdot\ldots\cdot\varphi_{y_k}(t)|<\delta$, for all $t\in N$.

Clearly, (1) follows from (3) by taking $k=m$. We prove (3) by induction. For $k=1$, (3) is clear, since $|1-\varphi_y(t)|<\delta$ for all $t\in N$ and $y\in K$. Assume (3) has been proved for $k$. To simplify notation we write $\varphi_i=\varphi_{y_i}$ for all $1\le i\le m$. Then, for each $t\in N$,

$$\begin{aligned}|1-\varphi_1(t)\cdot\ldots\cdot\varphi_{k+1}(t)| &= |1-\varphi_{k+1}(t)+\varphi_{k+1}(t)-\\ &-\varphi_1(t)\cdot\ldots\cdot\varphi_k(t)\cdot\varphi_{k+1}(t)|\\ &\le \max(|1-\varphi_{k+1}(t)|,\,|\varphi_{k+1}(t)|\cdot|1-\varphi_1(t)\cdot\ldots\cdot\varphi_k(t)|)<\delta,\end{aligned}$$

because $|1-\varphi_{k+1}(t)|<\delta$, $|\varphi_{k+1}(t)|\le 1$, and by the induction hypothesis, $|1-\varphi_1(t)\cdot\ldots\cdot\varphi_k(t)|)<\delta$. Hence (3) is true for $k+1$.

It remains to prove (2), i.e. $|\varphi(t)|<\delta$ for all $t\in K$. Now, if $t\in K$, then $t\in W(y_i)$ for some $i=1,\ldots,m$. Hence $|\varphi_i(t)|<\delta$, while $|\varphi_j(t)|\le 1$ for all $j\neq i$. Therefore $|\varphi(t)|<\delta$, and (2) is proved. □

**Theorem 1.** *Let $W$ be a non-empty subset of $C(S;E)$ such that the set $M$ of all multipliers of $W$ strongly separates the points of $S$. Let $f\in C(S;E)$ and $\varepsilon>0$ be given. The following are equivalent:*

(1) *there is some $g\in W$ such that $\|f-g\|<\varepsilon$;*

(2) *for each $x\in S$, there is some $g_x\in W$ such that $\|f(x)-g_x(x)\|<\varepsilon$.*

**Proof.** Clearly (1) ⇒ (2). Conversely, assume that (2) is true. For each $x\in S$, there is some $g_x\in W$ such that $\|f(x)-g_x(x)\|<\varepsilon$. Choose a real number $\varepsilon(x)>0$ such that $\|f(x)-g_x(x)\|<\varepsilon(x)<\varepsilon$. Let $N(x)$ be a clopen neighborhood of $x$ in $S$ such that

$$N(x)\subset\{t\in S;\ \|f(t)-g_x(t)\|<\varepsilon(x)\}.$$

Select a point $x_1\in S$ arbitrarily. Let $K=S\backslash N(x_1)$. By compactness of $K$, there exists a finite set $\{x_2,\ldots,x_m\}\subset K$ such that $K\subset N(x_2)\cup\ldots\cup N(x_m)$. Let

$$\begin{aligned}N_2&=N(x_2)\backslash N(x_1),\\ N_3&=N(x_3)\backslash(N(x_1)\cup N(x_2)),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ N_m&=N(x_m)\backslash\Big(\bigcup_{j=1}^{m-1}N(x_j)\Big).\end{aligned}$$

Then $N_2, N_3, \ldots, N_m$ are clopen subsets of $S$, such that $K \subset N_2 \cup N_3 \cup \ldots \cup N_m$, and $N_i \cap N_j = \emptyset$ for all $i \neq j$ $(2 \leq i, j \leq m)$. Let us write $g_i = g_{x_i}$ for all $i = 1, 2, \ldots, m$, and let

$$k = \max\{||f - g_1|| \, , \, ||f - g_2||, \ldots, ||f - g_m||\} .$$

Choose a number $\delta > 0$ so small that $\delta k(m-1) < \varepsilon - \varepsilon'$, where $\varepsilon' = \max\{\varepsilon(x_1), \varepsilon(x_2), \ldots, \varepsilon(x_m)\}$. By Lemma 1, there are $\varphi_2, \ldots, \varphi_m \in M$ such that

(1) $|1 - \varphi_i(t)| < \delta$, for all $t \in N_i$
(2) $|\varphi_i(t)| < \delta$, for all $t \notin N_i$

for all $i = 2, \ldots, m$. Define $N_1 = N(x_1)$, and

$$\begin{aligned} \psi_2 &= \varphi_2 \\ \psi_3 &= (1 - \varphi_2)\varphi_3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \psi_m &= (1 - \varphi_2)(1 - \varphi_3) \cdot \ldots \cdot (1 - \varphi_{m-1})\varphi_m . \end{aligned}$$

Clearly, $\psi_i \in M$, for all $i = 2, 3, \ldots, m$. Now

$$\psi_2 + \ldots + \psi_j = 1 - (1 - \varphi_2)(1 - \varphi_3) \cdot \ldots \cdot (1 - \varphi_j), \quad j = 2, \ldots, m ,$$

can be easily verified by induction. Define

$$\psi_1 = (1 - \varphi_2)(1 - \varphi_3) \ldots (1 - \varphi_m) .$$

Then $\psi_1 \in M$ and $\psi_1 + \psi_2 + \ldots + \psi_m = 1$. Notice that

(3) $|\psi_i(t)| < \delta$ , for all $t \notin N_i$ , $i = 1, 2, \ldots, m$.

Indeed, if $i \geq 2$, then $|\psi_i(t)| \leq |\varphi_i(t)|$ and (3) follows from (2). If $i = 1$, and $t \notin N(x_1)$, then $t \in K$. Hence $t \in N_j$ for some $j = 2, \ldots, m$. By (1), $|1 - \varphi_j(t)| < \delta$ and so

$$|\psi_1(t)| = |1 - \varphi_j(t)| \cdot \prod_{j \neq i} |1 - \varphi_i(t)| < \delta ,$$

because $|1 - \varphi_i(t)| \leq 1$ for all $i \neq j$. Let $g = \psi_1 g_1 + \psi_2 g_2 + \ldots + \psi_m g_m$. Then

$$g = \varphi_2 g_2 + (1 - \varphi_2)[\varphi_3 g_3 + (1 - \varphi_3)[\varphi_4 g_4 + \ldots + (1 - \varphi_{m-1})[\varphi_m g_m + (1 - \varphi_m) g_1] \ldots]] .$$

Hence $g \in W$. Let $x \in S$ be given. There is exactly one integer $1 \leq i \leq m$ such that $x \in N_i$. Call it $j$. Then $|\psi_j(x)| \cdot ||f(x) - g_j(x)|| \leq |\psi_j(x)| \cdot \varepsilon(x_j) < \varepsilon'$, since $|\psi_j(x)| \leq 1$. For all $i \neq j$, we have that $x \notin N_i$. By (3), $|\psi_i(x)| < \delta$. Hence

$$\sum_{i \neq j} |\psi_i(x)| \cdot ||(x) - g_i(x)|| \leq \delta k(m-1) < \varepsilon - \varepsilon',$$

and therefore

$$\begin{aligned} ||f(x) - g(x)|| &= ||\sum_{i=1}^{m} \psi_i(x)(f(x) - g_i(x))|| \\ &\leq \varepsilon' + \sum_{i \neq j} |\psi_i(x)| \cdot ||f(x) - g_i(x)|| < \varepsilon' + \varepsilon - \varepsilon' = \varepsilon. \quad \square \end{aligned}$$

## 2. Some Consequences

Let us recall the definition of the distance of an element $f \in C(S; E)$ from $W$:

$$\mathrm{dist}(f; W) = \inf\{||f - g|| \, ; \, g \in W\}.$$

**Theorem 2.** *Let $W$ be a non-empty subset of $C(S; E)$ such that the set $M$ of all multipliers of $W$ strongly separates the points of $S$. For each $f \in C(S; E)$ there exists $x \in S$ such that*

$$\mathrm{dist}(f; W) = \mathrm{dist}(f(x); W(x)).$$

**Proof.** If $\mathrm{dist}(f; W) = 0$, then $\mathrm{dist}(f(x); W(x)) = 0$ for every $x \in S$. Suppose now that $\mathrm{dist}(f; W) = d > 0$. By contradiction, assume that $\mathrm{dist}(f(x); W(x)) < d$ for every $x \in S$. Hence, for each $x \in S$, there is some $g_x \in W$ such that $\|f(x) - g_x(x)\| < d$. Consequently, $f$ and $d > 0$ satisfy condition (2) of Theorem 1. By Theorem 1, there exists $g \in W$ such that $\|f - g\| < d$, a contradiction, since $d = \mathrm{dist}(f; W)$. $\square$

**Theorem 3.** (Kaplansky [4]) *Let $A$ be a unitary subalgebra of $C(S; \mathbb{K})$ which is separating over $S$. Then $A$ is uniformly dense in $C(S; \mathbb{K})$.*

**Proof.** Let $E = \mathbb{K}$ and $W = A$. Notice that every element $\varphi \in A$, such that $|\varphi(x)| \leq 1$ for all $x \in S$, is a multiplier of $W$. By Proposition 1, the set $M$ of all multipliers of $W$ is strongly separating over $S$. Let now $f \in C(S; I\!K)$ be given. By Theorem 2, there exists $x \in S$ such that

$$\operatorname{dist}(f; A) = \operatorname{dist}(f(x); A(x)).$$

Since $A$ contains the constants, $A(x) = I\!K$. Hence $\operatorname{dist}(f(x); A(x)) = 0$, and therefore $\operatorname{dist}(f; A) = 0$. This shows that $A$ is uniformly dense in $C(S; I\!K)$. $\square$

**Corollary 1.** (Weierstrass Theorem) *Let $S$ be a non-empty compact subset of $I\!K$. For every $f \in C(S; I\!K)$ and every $\varepsilon > 0$, there exists a polynomial $p$ with coefficients in $I\!K$ such that $|f(x) - p(x)| < \varepsilon$, for all $x \in S$.*

**Remark.** When $I\!K$ is the field of $p$-adic numbers with the $p$-adic valuation, Theorem 3 and its Corollary 1 were proved by J. Dieudonné in 1944. (See Dieudonné [2].) In 1947, I. Kaplansky showed that a Weierstrass-Stone theorem holds for functions with values in topological rings having ideal neighborhoods of 0. (See Kaplansky [3].) Now in $I\!K$, the set $\{\lambda \in I\!K; |\lambda| < 1\}$, called the valuation ideal of $I\!K$, is an ideal neighborhood of 0. In 1950, Kaplansky showed that the methods of [3] could be extended to $(I\!K, |\cdot|)$ by proving Theorem 3. In fact, he proved a more general version of Theorem 3, by considering $S$ to be a 0-dimensional locally compact Hausdorff space, and $C_0(S; I\!K)$ the space of all those $f \in C(S; I\!K)$ vanishing at infinity, and $A \subset C_0(S; I\!K)$ a subalgebra containing for any two distinct points $s, t \in S$ a function vanishing at $s$ but not at $t$. (See Kaplansky [4]). In 1958, K. Mahler gave a constructive proof of Dieudonné's Weierstrass theorem (Corollary 1 above) for the case $S$ is the ring of $p$-adic integers $\{\lambda \in Q_p\, ; \, |\lambda|_p \leq 1\}$. (See Mahler [5].) However, Mahler's proof is based on some properties of the cyclotomic extension of $Q$. In 1974, R. Bojanic presented another proof of Mahler's result, which is entirely analytic. (See Bojanic [1].)

### 3. Simultaneous approximation and interpolation

**Definition 5.** A non-empty subset $A \subset C(S; E)$ is called an **interpolating family for** $C(S; E)$ if, for every $f \in C(S; E)$ and every finite subset

$F \subset S$, there exists $g \in A$ such that $f(x) = g(x)$ for all $x \in F$.

Let us study the problem of simultaneous approximation and interpolation. We start with scalar-valued functions, i.e., subsets of $C(S; \mathbb{K})$.

**Theorem 4.** *Let $A$ be a uniformly dense linear subspace of $C(S; \mathbb{K})$. Then, for every $f \in C(S; \mathbb{K})$, every $\varepsilon > 0$ and every finite subset $F \subset S$, there exists $g \in A$ such that $\|f - g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$.*

**Proof.** Let $F = \{x_1, \ldots, x_n\}$. Define a linear mapping $T : C(S; \mathbb{K}) \to \mathbb{K}^n$ by

$$Tg = (g(x_1), \ldots, g(x_n))$$

for each $g \in C(S; \mathbb{K})$. By density of $A$ and continuity of $T$, we have

$$T(C(S; \mathbb{K})) = T(\overline{A}) \subset \overline{T(A)}.$$

Now $T(A)$ is a linear subspace of $\mathbb{K}^n$ and therefore $T(A)$ is closed. Hence

$$T(C(S; \mathbb{K})) = T(A)$$

and $A$ is an interpolating family for $C(S; \mathbb{K})$. Therefore $a_1, \ldots, a_n$ can found in $A$ such that

$$a_i(x_j) = \delta_{ij} \quad , \quad 1 \leq i, j \leq n .$$

Choose $\delta > 0$ so that $\delta < \varepsilon$ and $\delta k < \varepsilon$, where $k = \max\{\|a_i\| \; ; \; 1 \leq i \leq n\}$. By density of $A$ there is some $g_1 \in A$ such that $\|f - g_1\| < \delta$. Let

$$v_i = f(x_i) - g_1(x_i) \quad , \quad 1 \leq i \leq n .$$

Define $g_2 = \sum_{i=1}^{n} v_i a_i$. Then $g_2 \in A$ and $g_2(x_j) = v_j$ for all $1 \leq j \leq n$. Finally , let $g = g_1 + g_2$. Then $g \in A$ and $g(x_j) = f(x_j)$ , $1 \leq j \leq n$. Moreover,

$$\|f - g\| \leq \max(\|f - g_1\|, \|g_2\|) < \varepsilon,$$

since $\|f - g_1\| < \varepsilon$ and $\|g_2\| \leq \delta \max\{\|a_i\|; 1 \leq i \leq n\}$. □

**Corollary 2.** *Let $A$ be a unitary subalgebra of $C(S; \mathbb{K})$ which is separating over $S$. Then, for every $f \in C(S; \mathbb{K})$, every $\varepsilon > 0$ and every finite subset*

*$F \subset S$, there exists $g \in A$ such that $\|f-g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$.*

**Proof.** By Theorem 3, $A$ is a uniformly dense linear subspace of $C(S; \mathbb{K})$. It remains to apply Theorem 4. □

**Remark.** The proof of Theorem 4 does not extend to subsets of $C(S; E)$. In this case we rely on Theorem 1, as our next result shows.

**Theorem 5.** *Let $A \subset C(S; E)$ be an interpolating family for $C(S; E)$ such that the set of multipliers of $A$ strongly separates the points of $S$. Then, for every $f \in C(S; E)$, every $\varepsilon > 0$ and every finite subset $F \subset S$, there exists $g \in A$ such that $\|f-g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$.*

**Proof.** Let $W = \{g \in A \,;\, f(x) = g(x) \text{ for all } x \in F\}$. Since $A$ is an interpolating family , $W \neq \emptyset$. Notice that every multiplier of $A$ is also a multiplier of $W$. Let $x \in S$ be given. Consider the finite set $F \cup \{x\}$. Since $A$ is an interpolating family for $C(S; E)$, there exists $g_x \in A$ such that $f(t) = g_x(t)$ for all $t \in F \cup \{x\}$. Therefore $g_x \in W$. Notice that $\|f(x) - g_x(x)\| = 0 < \varepsilon$. By Theorem 1 there exists $g \in W$ such that $\|f - g\| < \varepsilon$. Notice that $g \in W$ implies $g \in A$ and $f(x) = g(x)$ for all $x \in F$. □

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