A CHARACTERIZATION OF THE SET OF FIXED POINTS OF SOME SMOOTHED OPERATORS

Alfredo N. Iusem

and

Alvaro R. De Pierro

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Abstract. We characterize the set F of fixed points of an operator $T(x) = SQ(x)$, where S is a positive definite, symmetric and stochastic matrix and Q is a convex combination of orthogonal projections onto closed convex sets. We show that F is the set of minimizers of a convex function: the sum of a weighted average of the squares of the distances to the convex sets and a nonnegative quadratic related to the matrix S.

Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
IMECC – UNICAMP
Caixa Postal 6065
13.081 – Campinas – SP
BRASIL

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A characterization of the set of fixed points of some smoothed operators

*Alfredo N. Iusem**
Instituto de Matemática Pura e Aplicada
Estrada Dona Castorina 110. Rio de Janeiro RJ 22460
Brazil

*Alvaro R. De Pierro***
Instituto de Matemática, Estatística e Ciência da Computação
Universidade Estadual de Campinas
CP 6065 Campinas, SP 13081, Brazil

Abstract

We characterize the set F of fixed points of an operator $T(x) = SQ(x)$, where S is a positive definite, symmetric and stochastic matrix and Q is a convex combination of orthogonal projections onto closed convex sets. We show that F is the set of minimizers of a convex function: the sum of a weighted average of the squares of the distances to the convex sets and a nonnegative quadratic related to the matrix S.

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In [1] we discussed iterative algorithms of the form:

$$x^0 \in \mathbb{R}^n, \tag{1}$$
$$x^{k+1} = T(x^k), \tag{2}$$

with $T : \mathbb{R}^n \to \mathbb{R}^n$ of the form:

$$T(x) = SQ(x) \tag{3}$$

where $S \in \mathbb{R}^{n\times n}$ is symmetric, stochastic (meaning nonnegative and with entries s_{ij} such that $\sum_{j=1}^{n} s_{ij} = 1$ for all i) and without zeroes in the diagonal, and $Q : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies:

i) $||Q(x) - Q(y)|| \le ||x - y||$ for all $x,\ y \in \mathbb{R}^n$
ii) $||Q(x) - Q(y)|| = ||x - y||$ implies:
  a) $Q(x) - Q(y) = x - y$,
  b) $\langle x - y,\ Q(y) - y\rangle = 0$.
iii) the function $\varphi(x) = ||x - SQ(x)||$ attains its minimum in $\mathbb{R}^n$.

We proved that under such condition the sequence $\{x^k\}$ generated by (1) − (2) converges for all $x^0 \in \mathbb{R}^n$. Clearly, the limit of $\{x^k\}$ is a fixed point of T, but we did not provide in [1] a reasonable characterization of such fixed points. We present here such a characterization. We start with the following elementary observation:

Proposition 1. Take $H \in \mathbb{R}^{n\times n}$ nonsingular and symmetric and $g : \mathbb{R}^n \to \mathbb{R}$ differentiable. Define $R : \mathbb{R}^n \to \mathbb{R}^n$ as $R(x) = H(x - \nabla g(x))$. Then the set of fixed points of R is the set of the zeroes of the gradient ∇h of h defined as $h(x) = g(x) + \frac{1}{2}x^T(H^{-1} - I)x$.

Proof. $x = R(x)$ iff $0 = x - Hx + H\nabla g(x)$ iff $0 = (H^{-1} - I)x + \nabla g(x) = \nabla h(x)$.

□

We will apply Proposition 1 to the following case. Let $C_1, \ldots, C_m$ be closed convex sets in $\mathbb{R}^n$ and P_i be the orthogonal projection onto C_i. Take positive scalars λ_i $(1 \le i \le m)$ such that $\sum_{i=1}^{m} \lambda_i = 1$ and $\alpha \in (0,\ 2)$. Define:

$$\overline{P}(x) = (1 - \alpha)x + \alpha \sum_{i=1}^{m} \lambda_i P_i(x) \tag{4}$$

It has been shown in [1] that $\overline{P}$, as defined by (4), satisfies conditions i) and ii) above, and that condition iii) is also satisfied under some additional hypothesis on the sets C_i (for instance when $\bigcap_{i=1}^{m} C_i$ is nonempty, or when at least one of the C_i's is bounded, or when they are polyhedra). Consider the operator $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by:

$$T(x) = S\overline{P}(x) \tag{5}$$

where $\overline{P}$ is as in (4) and $S \in \mathbb{R}^{n\times n}$ is symmetric, stochastic and positive definite.

Theorem 1 The set of fixed points of T as defined by (5) is the set of minimizers of $f : \mathbb{R}^n \to \mathbb{R}$ defined by:

$$f(x) = \sum_{i=1}^{m} \lambda_i ||x - P_i(x)||^2 + \frac{1}{\alpha} x^T(S^{-1} - I)x \tag{6}$$

Proof: Let $g_i(x)$ be defined as $g_i(x) = \frac{1}{2}\ ||x - P_i(x)||^2$. It is well known that g is differentiable and $\nabla g_i(x) = x - P_i(x)$. It follows that $\overline{P}(x) = x - \alpha(x - \sum_{i=1}^{m} \lambda_i P_i(x)) = x - \alpha \sum_{i=1}^{m} \lambda_i (x - P_i(x)) = x - \alpha \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = x - \nabla g(x)$ with $g(x) = \alpha \sum_{i=1}^{m} \lambda_i g_i(x)$. Applying Proposition 1 with $H = S$, $R = T$ we conclude that the fixed points of T are the zeroes of the gradient of h, defined as $h(x) = g(x) + \frac{1}{2} x^T(S^{-1} - I)x = \frac{\alpha}{2} f(x)$, i. e., the zeroes of $\nabla f(x)$. The eigenvalues of S are real by symmetry, positive by positive definiteness and less than or equal to one by stochasticity. So $S^{-1} - I$ is symmetric positive semidefinite and the quadratic $x^T(S^{-1} - I)x$ is convex. Since the g_i's are clearly convex, it follows that f is convex, so that the the zeroes of its gradient are its minimizers.

□

The iterative scheme (1) – (2) with T as in (5) arises in Computerized Tomography (e.g. [2]). In this case, the objective is to find a point $x \in C = \bigcap_{i=1}^{m} C_i$, but some degree of "smoothness" of x, which represents an image, is also required. The purpose of S, as explained in [1], is to attain such "smoothing". When S is irreducible in the sense of Markov chains, as is usually the case when smoothing images, then $Sx = x$ iff all components of x are equal, i. e., x is perfectly "smooth". Note that for nonempty C the first term in the right hand side of (6) is minimized by the points in C and the second one by the points x such that $Sx = x$. So, by minimizing f one achieves a trade-off between belonging to C and being smooth, a sensible goal in the applications.

We complete the analysis of the relation between algorithm (1) - (2) (with T as in (5)) and f by showing that the algorithm not only minimizes f but does so monotonically.

Proposition 2. Take T, f as in Theorem 1. Then, for all $x \in \mathbb{R}^n$:

$$f(T(x)) \leq f(x) - \left(\frac{2-\alpha}{\alpha}\right)\|T(x) - x\|^2 \tag{7}$$

Proof. By definition of P_i:

$$\|P_i(T(x)) - T(x)\|^2 \leq \|P_i(x) - T(x)\|^2 = \|P_i(x) - x\|^2 + \|T(x) - x\|^2 - 2\langle P_i(x) - x,\, T(x) - x\rangle \tag{8}$$

Multiplying (8) by λ_i and summing on i:

$$\sum_{i=1}^{m} \lambda_i \|P_i(T(x)) - T(x)\|^2 \leq \sum_{i=1}^{m} \lambda_i \|P_i(x) - x\|^2 + \|T(x) - x\|^2 - \frac{2}{\alpha}\langle \overline{P}(x) - x,\, T(x) - x\rangle, \tag{9}$$

since $\sum_{i=1}^{m} \lambda_i (P_i(x) - x) = (\sum_{i=1}^{m} \lambda_i P_i(x)) - x = \frac{1}{\alpha}(\overline{P}(x) - x)$ by (4)

From (9):

$$\begin{aligned}
f(T(x)) &\leq f(x) + \frac{1}{\alpha}[T(x)^T (S^{-1} - I)\, T(x) - x^T (S^{-1} - I) x] \\
&+ \|T(x) - x\|^2 + \frac{2}{\alpha}\langle x,\, T(x) - x\rangle - \frac{2}{\alpha}\langle \overline{P}(x),\, T(x) - x\rangle \\
&= f(x) - \frac{1}{\alpha}\,(T(x) - x)^T (S^{-1} - I)(T(x) - x) \\
&- \frac{2}{\alpha} T(x)^T (S^{-1} - I)(T(x) - x) + \|T(x) - x\|^2 \\
&+ \frac{2}{\alpha}\langle x,\, T(x) - x\rangle - \frac{2}{\alpha}\langle \overline{P}(x),\, S S^{-1}(T(x) - x)\rangle \\
&\leq f(x) + \frac{2}{\alpha} T(x)^T (S^{-1} - I)\,(T(x) - x) + \|T(x) - x\|^2 \\
&+ \frac{2}{\alpha}\langle x,\, T(x) - x\rangle - \frac{2}{\alpha}\langle S\overline{P}(x),\, S^{-1}(T(x) - x)\rangle
\end{aligned} \tag{10}$$

using positive semidefiniteness of $S^{-1} - I$ and symmetry of S in the last inequality.
From (10):

$$\begin{aligned} f(T(x)) &\leq f(x) + \frac{2}{\alpha}T(x)^T(S^{-1} - I)\,(T(x) - x) \\ &+ \|T(x) - x\|^2 + \frac{2}{\alpha}\langle x,\, T(x) - x\rangle - \frac{2}{\alpha}T(x)^T S^{-1}(T(x) - x) \\ &= f(x) - \frac{2}{\alpha}\langle T(x), T(x) - x\rangle + \|T(x) - x\|^2 \\ &+ \frac{2}{\alpha}\langle x,\, T(x) - x\rangle \\ &= f(x) - (\frac{2-\alpha}{\alpha})\|T(x) - x\|^2. \end{aligned} \tag{11}$$

□

Corollary. If $\{x^k\}$ is the sequence defined by (1) - (2) (with T as in (5)) then $f(x^{k+1}) \leq f(x^k) - (\frac{2-\alpha}{\alpha})\|x^{k+1} - x^k\|^2$.
Proof. Immediate from Proposition 2.

□

Another specific example of Q satisfying i) - iii) which was discussed in [1] is the sequential relaxed projections operator P defined as $P = P_m^\alpha \circ \ldots \circ P_2^\alpha \circ P_1^\alpha$ where $P_i^\alpha(x) = (1-\alpha)x + \alpha P_i(x)$. We remark that the operator T with P substituting for $\overline{P}$ in (5) is not amenable to this analysis. For instance, in the case of hyperplanes, i.e., when $C_i = \{x \in \mathbb{R}^n,\ \langle a^i,\ x\rangle = b_i\}$ with $a^i \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and without relaxation ($\alpha = 1$), we get $x - P(x) = A^T(EAx - b)$ where A is the $m \times n$ matrix with rows a^i and E^{-1} is the lower triangular part of AA^T. Since E is not symmetric, $x - P(x)$ cannot be the gradient of any function g.

References:

1. A. R. De Pierro and A. N. Iusem. On the asymptotic behavior of some alternate smoothing series expansion iterative methods. Linear Algebra Appl. 130: 3 - 24 (1990).

2. G. T. Herman. Image Reconstruction from Projections: The Fundamentals of Computerized Tomography. Academic Press, New York, 1980.

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