GLOBAL SOLUTIONS TO THE EQUATIONS FOR THE MOTION OF STRATIFIED INCOMPRESSIBLE FLUIDS

José Luiz Boldrini

and

Marko Antonio Rojas-Medar

RELATÓRIO TÉCNICO Nº 45/91

Abstract. We prove a result on global existence in time for strong solutions to the three dimensional stratified Navier-Stokes equations. These equations describe the motion of nonhomogeneous incompressible fluids. For the result, as in the usual Navier-Stokes equations, it is required small initial velocities and external force fields with a weak form of decay.

Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
IMECC - UNICAMP
Caixa Postal 6065
13.081 - Campinas - SP
BRASIL

Setembro - 1991

Global Solutions to the Equations for the Motion of Stratified Incompressible Fluids

by

José Luiz Boldrini
and
Marko Antonio Rojas-Medar
UNICAMP - IMECC
C.P. 6065
13081, Campinas, SP, Brazil

Abstract. We prove a result on global existence in time for strong solutions to the three dimensional stratified Navier-Stokes equations. These equations describe the motion of nonhomogeneous incompressible fluids. For the result, as in the usual Navier-Stokes equations, it is required small initial velocities and external force fields with a weak form of decay.

1. Introduction

In this work we will be concerned with global existence in time of strong solutions to the three dimensional stratified Navier-Stokes equations, that is, the equations fot the motion of a nonhomogeneous incompressible fluid (obtained as a mixture of miscible incompressible fluids, for instance). Being $\Omega \subset I\!R^3$ a C^3-regular bounded open set, $T > 0$ these equations are:

$$
\left\{
\begin{array}{l}
\rho \dfrac{\partial u}{\partial t} + \rho u.\nabla u - \Delta u - grad\ p = \rho f, \\
div\ u = 0, \\
\dfrac{\partial \rho}{\partial t} + u.\nabla \rho = 0 \quad \text{in } \Omega; \\
u = 0 \quad \text{on} \quad \partial\Omega \times (0,T); \\
\rho|_{t=0}(x) = \rho_0(x) \ \text{ in } \ \Omega; \\
u|_{t=0}(x) = u_0(x) \ \text{ in } \ \Omega,
\end{array}
\right. \tag{1.1}
$$

where $[0,T)$ is the interval of time being considered; Ω is the container where the fluid is in; $u(x,t) \in I\!R^3$ denotes the velocity of the fluid at a point $x \in \Omega$

and at time $t \in [0,T)$; $\rho(x,t) \in I\!R$ and $p(x,t) \in I\!R$ denote, respectively, the density and the hydrostatic pressure of the fluid; $u_0(x)$ and $\rho_0(x)$ are the initial velocity and density, respectively; $f(x,t)$ is the density by unit of mass of the external force acting on the fluid; here, without loosing generality, we have normalized the viscosity to be one; the fluid adheres to the wall $\partial\Omega$ of the container which is at rest. The expressions grad, Δ and div denote the gradient, Laplacian and divergence operators, respectively (we also denote the gradient operator by ∇ and $\frac{\partial u}{\partial t}$ by u_t); the $\underline{ith}$ component of $u.\nabla u$ is given by $(u.\nabla u)_i = \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}$; $u.\nabla\rho = \sum_{j=1}^{3} u_j \frac{\partial \rho}{\partial x_j}$. The first equation in (1.1) corresponds to the balance of linear momentum; the third to the balance of mass, and the second stokes that fluid is incompassible. The unknowns in the problem are u, ρ and p.

The classical Navier-Stokes equations corresponds to the special case where $\rho(x,t) = \rho_0$ is a positive constant; in this case the third equation in (1.1) drops out. This case has been much studied (see Ladyshenskaya [7] and Temam [12] and the references there in). Equations (1.1) have been much less studied, maybe due to their mixed parabolic-hiperbolic character. Antonzev and Kazhikov [1], Kazhikov [5], Simon [11] and Kim [6] have studied local and global existence for weak solutions to (1.1). Stronger local or global solutions were obtained by Ladyszhenskaya and Solonnikov [8] by linearization and fixed point arguments, and by Okamoto [9] by using evolution operators techniques and also fixed point arguments. The more constructive spectral semi-Galerkin method was used by Salvi [10] to obtain local strong solutions and to study conditions for regularity at $t = 0$ and by Boldrini and Rojas-Medar [2] also to obtain local strong solutions and to study their regularity for $t > 0$.

In this work we present results of global existence for strong solutions (see section 2 for the exact definitions) under certain regularity assumptions on the initial data and external force field (see section 3 for the details). In particular, as in the case of the usual Navier-stokes equations, we will require smallness of the $L^2(\Omega)$ norm of the initial velocity and of the $L^2(\Omega \times (0,T))$-norm of the force field.

Our result can be compared with the ones by Ladyzhenskaya, Solonnikov and Okamoto as follows. In their results the initial velocity can be a little less regular then in ours, but, by working with fractional powers of the Stokes

operator applied to the multipliers that we used, one could weaken our regularity requiriments. On the other hand, Ladyzhenskaya and Solonnikov require exponential decay in time of the (small) $L^q(\Omega)$-norm ($q > 3$) the external force field f; Okamoto works with f identically zero, and, in order to obtain the result with nonzero force field, at least an exponential decay in time of the $L^2(\Omega)$-norm of f would be required. We require a weaker form of decay in the sense that we allow a (small) f belonging to $L^2([0,\infty); L^2(\Omega))$. Concerning the conditions on the initial velocity, Okamoto requires small initial velocities in the $H^{\frac{3}{2}+\varepsilon}(\Omega)$-norm, $\varepsilon > 0$, and initial densities ρ_0 with small enough $L^\infty(\Omega)$ norm of $\nabla\rho_0$; Ladyzhenskaya and Solonnikov demand small initial velocities in the $W^{2-\frac{2}{q},q}(\Omega)$-norm, $q > 3$. In our result it is enough to require small initial velocity in the $H^1(\Omega)$-norm.

2. Preliminaries

In what follows we will assume Ω of class C^3. We will consider the usual Sobolev spaces

$$W^{m,q}(D) = \{f \in L^q(D); ||\partial_f^\alpha||_{L^q(D)} < +\infty, (|\alpha| \leq m)\},$$

$m = 0, 1, 2, \ldots, 1 \leq q \leq \infty, D = \Omega$ or $\Omega \times (0,T), o < T \leq +\infty$, with the usual norm. When $q = 2$, we denote $H^m(D) = W^{m,2}(D)$ and $H_0^m(D) =$ closure of $C_0^\infty(\Omega)$ in $H^m(D)$. If B is a Banach-space, we denote by $L^q([0,T), B)$ the Banach space of the B-valued functions defined in the interval $[0,T)$ that are L^q-integrable in the sense of Bochner.

Let $C_{0,\sigma}^\infty(\Omega) = \{v = (v_1, v_2, v_3) \in C_0^\infty(\Omega)^3; div\ v = 0 \text{ in } \Omega\}$; $V =$ closure of $C_{0,\sigma}^\infty(\Omega)$ in $H_0^1(\Omega)^3$, and $H =$ closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)^3$

Let P be the orthogonal projection from $L^2(\Omega)^3$ onto H obtained by the usual Helmholtz decomposition. Then the operator $A : H \to H$ given by $A = -P\Delta$ with domain $D(A) = H^2(\Omega)^n \cap V$ is called the Stokes operator. It is well known that A is a positive definite self-adjoint operator and is characterized by the relation

$$(Aw, v) = (\nabla w, \nabla v) \text{ for all } w \in D(A), v \in V.$$

From now on, we denote the inner product in H (i.e., the L^2-inner product) by $(,)$. The L^p-norm will be denoted by $|| \ ||_{L^p(\Omega)}$.

The following assumptions on the initial data will hold throughout this paper.

(A.1) The initial value for the density ρ_0 belongs to $W^{1,\infty}(\Omega)$ and

$$\text{satisfies } 0 < \alpha \leq \rho_0(x) \leq \beta < +\infty \ ae \text{ in } \Omega.$$

(A.2) The initial value u_0 belongs to $V \cap H^2(\Omega)$

Now, using the properties of P, we can reformulate problem (1.1) as follows: find $\rho \in W^{1,\infty}(\Omega \times (0,T))$ and $u \in C^1([0,T), H) \cap C((0,T), D(A))$ such that

$$(2.1) \quad \begin{cases} \dfrac{\partial \rho}{\partial t} + u.\nabla \rho = 0 \ \text{ for } ae\ (x,t) \in \Omega \times (0,T) \\ (\rho u_t, v) + (\rho u.\nabla u, v) + (Au, v) = (\rho f, v), \ \ 0 < t < T, \ \ \forall v \in H \\ u(0) = u_0, \ \ \rho(0,x) = \rho_0(x) \end{cases}$$

By using spectral semi-Galerkin approximations, Boldrini, Rojas-Medar [2] proved the following local existence theorem:

Theorem 2.1. Suppose (A.1) and (A.2) are true and that $f \in L^2(0,T,H^1(\Omega))$, $f_t \in L^2(0,T,L^2(\Omega))$. Then, there is $0 < T_1 \leq T$ such that the problem (2.1) has a unique solution in the interval $[0,T_1)$.

Remark. (i) Actually the solution has a little better regularity. For instance, it is proved that $u \in L^2(0,T_1,H^3(\Omega))$.

(ii) From the proof (2), one sees that
$T_1 = T_1(||u_0||_{H^1(\Omega)}, ||\rho_0||_{W^{1,\infty}(\Omega)})$ increases as $||u_0||_{H^1(\Omega)}$ decreases.

In the next section we will prove that under more stringent conditions, the above solution is global in time.

3. Global Existence

We have the following result:

Theorem 3.1: Suppose that (A.1) and (A.2) are true and that

$$f \in L^2_{loc}([0,\infty); H^1(\Omega)) \cap L^\infty([0,\infty); L^2(\Omega)) \cap L^2([0,\infty), L^2(\Omega)),$$

$f_t \in L^2_{loc}(0,\infty), L^2(\Omega))$. Then, if $||u_0||_{H^1(\Omega)}$ and $||f||_{L^2(\Omega \times (0,\infty))}$ are small enough, the solution described in Theorem 2.1 exists globally in time, that

is, it exists for all $t \in \mathbb{R}$ and for any $0 < T < +\infty$ it is in the required spaces.

Proof We will combine arguments used by Kim [6] with a variant of arguments used by Hey wood and Rannacher [4].

We take $\overline{u}_0 \in V \cap H^2(\Omega)$, and will prove that for small enough $\lambda \in (0,1]$, the solution $(u_\lambda, \rho_\lambda)$ to (2.1), with initial data $(\lambda\overline{u_0}, \rho_0)$ and external force λf, exists globally in time. The crucial estimate will be the one for $||\nabla u_\lambda(t)||_{L^2(\Omega)}$; and to obtain it, we proceed as follows: from the proof of the local existence theorem (Theorem 2.1), for any s in the interval of existence, we have the estimate

$$\frac{\alpha}{2}||u_\lambda(s)||^2_{L^2(\Omega)} + \int_0^s ||\nabla u_\lambda(\tau)||^2_{L^2(\Omega)} d\tau \leq \lambda^2\left[\frac{\beta}{2}||\overline{u_0}||^2_{L^2(\Omega)} + \beta\int_0^s ||f(\tau)||^2_{L^2(\Omega)} d\tau\right] \tag{3.1}$$

(this was obtained by taking u_λ as a multiplier in (2.1) (ii) with λf in place off and $\lambda\overline{u}_0$ in place of u_0).

Also, working as in Kim [6], (Proposition 2.4, p. 93), for t in the interval of existence of the solution, one obtains

$$\frac{d}{dt}||\nabla u_t(t)||^2_{L^2(\Omega)} \leq C||\nabla u_t(t)||^{10}_{L^2(\Omega)} + C\lambda^2||f(t)||^2_{L^2(\Omega)}. \tag{3.2}$$

We will show by contradiction that for λ small enough, the above two inequalities imply that $||\nabla u_\lambda(t)||$ is bounded for finite times.

In fact, suppose the opposite, that is, that for any $\lambda \in (0,1]$ the function $\psi_\lambda(t) = ||\nabla u_\lambda(t)||^2_{L^2(\Omega)}$ blows-up in a finite time $t^*(\lambda)$ (which, according to the Remark (ii) above, is necessarely larger than the $T_1 > 0$ given in Theorem 2.1, corresponding to the initial data $(\overline{u_0}, \rho_0)$).

Now, we observe that $c\psi_\lambda^5 + c_1\lambda^2 \leq 2c\psi_\lambda^5$ for $\psi_\lambda \geq (c_1/c)^{1/5}\lambda^{2/5} = l(\lambda)$, where $c_1 = c \sup_{t\in[0,T_1]} ||f(t)||_{L^2(\Omega)}$. Therefore, we have $0 \leq \psi_\lambda(t) < l(\lambda)$ or $\frac{d\psi_\lambda}{dt} \leq 2c\psi_\lambda^5$.

Now we consider the equation $\frac{d}{dt}\Phi_\lambda = 2c\Phi_\lambda^5$ and its solution $\Phi_\lambda(t)$ that blows-up exactly at $t^*(\lambda)$. We will prove that the graph of $\Phi_\lambda(t)$ stays bellow the graph of $\psi_\lambda(t)$ for $t \in \Lambda_\lambda = \{t \in [0, t^*(\lambda)), \psi_\lambda(t) \geq l(\lambda)\}$. In fact, take a sequence $\overline{t}_n > t^*(\lambda), n \in \mathbb{N}$, converging to $t^*(\lambda)$, and consider the solutions

$\varphi_{\bar{t}_n}(t)$ to the problem $\varphi' = 2c\varphi^5$ that blows-up exactly at $\bar{t}_n$. Since $\psi_\lambda(t) \to +\infty$ as $t \to t^*(\lambda)^-$ and $\varphi_{\bar{t}_n}(t)$ are finite for such t, we have $\varphi_{\bar{t}_n}(t) < \psi_\lambda(t)$ for $t < t^*(\lambda)$, close enough to $t^*(\lambda)$. On the other hand, by using well known results on differential inequalities, as well as the definitions of $\psi_\lambda(t)$ and $\varphi_{\bar{t}_n}(t)$, we see that there cannot exist $\tau \in \Delta_\lambda$ such that $\psi_\lambda(\tau) = \varphi_{\bar{t}_n}(\tau)$, and, therefore, $\varphi_{\bar{t}_n}(t) < \psi_\lambda(t)$ for $t \in \Delta_\lambda$. Since, for $0 \le t < t^*(\lambda)$, we have $\varphi_{\bar{t}_n}(t) \to \Phi_\lambda(t)$ as n goes to $+\infty$, we finally conclude that $\Phi_\lambda(t) \le \psi_\lambda(t)$ for $t \in \Delta_\lambda$.

Now, $\Phi_\lambda(t) = [8c(t^*(\lambda) - t)]^{-1/4}$ and we observe that $\Phi_\lambda(t) \ge l(\lambda)$ for $t \ge t_1(\lambda) = t^*(\lambda) - (8cl^4(\lambda))^{-1}$, and for such that t we necessarely have $\psi_\lambda(t) \ge \Phi_\lambda(t)$. Moreover, since $l(\lambda) \to 0+$ as $\lambda \to 0+$, for small enough λ we have $[8cl^4(\lambda)]^{-1} > T_1$, and we also know that $t^*(\lambda)$ grows as $\lambda \to 0+$. We conclude that $\psi_\lambda(t) \ge \Phi_\lambda(t)$ for $t \ge t^*(\lambda) - T_1$; therefore,

$$\frac{4T_1^{3/4}}{3(8c)^{1/4}} = \int_{t^*(\lambda)-T_1}^{t^*(\lambda)} \Phi_\lambda(\tau)d\tau \le \int_{t^*(\lambda)-T_1}^{t^*(\lambda)} \psi_\lambda(\tau)d\tau.$$

Now, if we take λ satisfying

$$\lambda^2\left[\frac{\beta}{2}||\overline{u}_0||^2 + \beta\int_0^\infty ||f(\tau)||^2_{L^2(\Omega)}d\tau\right] < \frac{4T_1^{3/4}}{3(8c)^{1/4}},$$

estimate (3.1) implies that

$$\begin{aligned}\int_0^s \psi_\lambda(\tau)d\tau &\le \lambda^2\left[\frac{\beta}{2}||\overline{u}_0||^2_{L^2(\Omega)} + \beta\int_0^\infty ||f(\tau)||^2_{L^2(\Omega)}d\tau\right] \\ &< \frac{4T^{3/4}}{3(8c)^{1/4}} \le \int_{t^*(\lambda)-T_1}^{t^*(\lambda)} \psi_\lambda(\tau)d\tau,\end{aligned}$$

for $0 \le s < t^*(\lambda)$, which by taking $s \to t^+(\lambda)^-$ is a contradiction. Therefore, for small enough λ, $||\nabla u_\lambda(t)||_{L^2(\Omega)}$ does not blow up in finite time and $\nabla u_\lambda \in L^\infty_{loc}([0,\infty), L^2(\Omega))$. Now, with this estimate, the proof of the local existence Theorem will furnish that $\rho_\lambda \in L^\infty([0,\infty), L^\infty(\Omega))$, $u_{\lambda,t}, \Delta u_\lambda \in L^\infty_{loc}([0,\infty), L^2(\Omega))$ and $\nabla u_{\lambda,t} \in L^2_{loc}([0,\infty), L^2(\Omega))$. These results imply that $\rho_\lambda(u_{\lambda,t} + u_\lambda.\nabla u_\lambda - \lambda f) \in L^\infty_{loc}([0,\infty), L^{3+\epsilon}(\Omega))$ for some $\epsilon > 0$. Thus, the equivalent form of (2.1) (ii):

$$Au_\lambda = P(\rho_\lambda(u_{\lambda,t} + u_\lambda.\nabla u_\lambda - \lambda f))$$

and Cattabriga's estimate for the Stokes operator, [3], will give that $u_\lambda \in L^2_{loc}([0,\infty); W^{2,3+\epsilon}(\Omega))$. Therefore, by Sobolev Imbes we have $\nabla u_\lambda \in L^1_{loc}([0,\infty); L^\infty(\Omega))$ which, by Ladyzhenskaya and Solonnikov estimates [8], imply that ∇_{ρ_λ} and $\rho_{\lambda,t} \in L^\infty_{loc}([0,\infty); L^\infty(\Omega))$. Finally, from the above argument, we observe that how small λ must be, depends only on $||\nabla \overline{u}_0||_{L^2(R)}$ (for fixed ρ_0 and f), that is, if we have another initial condition $\overline{u}_1$ such that $||\nabla \overline{u}_0|| = ||\nabla \overline{u}_1||$, the above condition on λ will be the same. In other words, the result depends on the H^1-norm of the initial velocity. □

References

[1] S.N. Antonzev, A.V. Kazhikov, Mathe,atical Study of Flows of Non Homogeneous Fluids, Novossibirks, Lectures at the University, 1973 (Russian)

[2] J.L. Boldrini, M.A. Rojas-Medar, Local existence and interior regularity for the stratified Navier-Stokes equations, to appear.

[3] L. Cattabriga, Su un problema al contorno relativo al sistema di equazioni di Stokes, Rend. Sem. Mat. Univ. Padova 31 (1961), 308-340.

[4] J.G. Heywood, R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem I: regularity of solutions and second order error estimates for spatial discretization, SIAM J. Num. Anal., 19 (1982), 275-311.

[5] A.V. Kazhikov, Solvability of the initial and boundary-value problem for the equations of motion of an inhomogeneous viscous incompressible fluid, Dok. Akad. Nank.. 216 (1974), 1008-1010. English Transl., Soviet Physis. Dokl., (1974), 331-332.

[6] J.U. Kim, Weak solutions of an initial boundary value problem for an incompressible viscous fluid, SIAM J. Math. Anal. 18 (1987), 86-96.

[7] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible. Flow, Gordon and Breach, Second Revised Edition, New York, 1986.

[8] O.A. Ladyzhenskaya, V.A. Solonnikov, Unique salvability, of an initial and boundary value problem for viscous incompressible nonhomogeneous fluids, Zap. Naučk Sem., Lenningrado Otdel Math. Inst. Steklov, 52 (1975), 52-189. English Transl., J. Soviet Math., 9(1978), 697-749.

[9] H. Okamoto, On the equation of nonstationary statified fluid motion: uniqueness and existence of the solutions, J.Fac. Sci. Univ. Tokyo, Sect. IA Math., 30 (1984), 615-643.

[10] R.Salvi, the equations of viscous incompressible nonhomogeneous fluid on the existence and regularity, to appear in the J. Australian Math. Soc.

[11] J. Simon, Éconlement d'un fluide non homogène a vee une densité initial s'annulant, C.R.Acad. Sci., 278 (1978), 1009-1012.

[12] R.Temam, Navier-Stokes Equations, North-Holland, Revised Edition, 1979.

RELATÓRIOS TÉCNICOS — 1991

01/91 **Um Método Numérico para Resolver Equações de Silvester e de Ricatti** — *Vera Lucia da Rocha Lopes* and *José Vitório Zago.*

02/91 **"Regge-Like" Relations for (Non-Evaporating) Black Holes and Cosmological Models** — *Vilson Tonin-Zanchin* and *Erasmo Recami.*

03/91 **The Exponential of the Generators of the Lorentz Group and the Solution of the Lorentz Force Equation** — *J. R. Zeni* and *Waldyr A. Rodrigues Jr.*

04/91 **Tensornorm Techniques for the (DF)-Space Problem** — *Andreas Defant* and *Klaus Floret.*

05/91 **Nonreversibility of Subsemigroups of Semi-Simple Lie Groups** — *Luiz San Martin.*

06/91 **Towards a General Theory of Convolutive Sets (With Applications to Fractals)** — *Jayme Vaz Jr.*

07/91 **Linearization of Holomorphic Mappings of Bounded Type** — *Jorge Mujica.*

08/91 **Topological Equivalence of Diffeomorphisms and Curves** — *M. A. Teixeira.*

09/91 **Applications of Finite Automata Representing Large Vocabularies** — *Cláudio L. Lucchesi* and *Tomasz Kowaltowski.*

10/91 **Torsion, Superconductivity and Massive Electrodinamics**
Cartan's Torsion Vector and Spin–0 Fields — *L. C. Garcia de Andrade.*

11/91 **On The Continuity of Fuzzy Integrals** — *G. H. Greco* and *R. C. Bassanezi.*

12/91 **Optimal Chemical Control of Populations Developing Drug Resistance** — *M. I. S. Costa, J. L. Boldrini* and *R. C. Bassanezi.*

13/91 **Strict Monotonicity of Eigenvalues and Unique Continuation** — *Djairo G. de Figueiredo* and *Jean-Pierre Gossez.*

14/91 **Continuity of Tensor Product Operators Between Spaces of Bochner Integrable Functions** — *Andreas Defant* and *Klaus Floret.*

15/91 **Some Remarks on the Join of Spheres and their Particular Triangulations** — *Davide C. Demaria* and *J. Carlos S. Kiihl.*

16/91 **Sobre a Equação do Telégrafo e o Método de Riemann** — *L. Prado Jr.* and *E. Capelas de Oliveira.*

17/91 **Positive Solutions of Semilinear Elliptic Systems** — *Ph. Clément, D. G. de Figueiredo* and *E. Mitidiere.*

18/91 **The Strong Coupling Constant: Its Theoretical Derivation from a Geometric Approach to Hadron Structure** — *Erasmo Recami* and *Vilson Tonin-Zanchin.*

19/91 **Time Analysis of Tunnelling Processes, and Possible Applications in Nuclear Physics** — *Vladislavi S. Olkhovsiky* and *Erasmo Recami.*

20/91 **Procedimento, Função, Objeto ou Lógica?** — *M. Cecília Calani Baranauskas.*

21/91 **The Relation Between 2-Spinors and Rotations** — *W. A. Rodrighes Jr.* and *J. R. Zeni.*

22/91 **Boundaries for Algebras of Analytic Functions on Infinite Dimensional Banach Spaces** — *R. M. Aron, Y. S. Choi, M. L. Lourenço* and *O. W. Paques.*

23/91 **Factorization of Uniformly Holomorphic Functions** — *Luiza A. Moraes, Otilia W. Paques* and *M. Carmelina F. Zaine.*

24/91 **Métrica de Prohorov e Robustez** — *Mario Antonio Gneri.*

25/91 **Cálculo de Funções de Green para a Equação de Schrödinger pelo Método de Expansão Tipo Sturm-Liouville** — *L. Prado Jr.* and *E. Capelas de Oliveira.*

26/91 **On the Weierstrass-Stone Theorem** — *João B. Prolla.*

27/91 **Sull'Equazione di Laplace nell'Universo di De Sitter** — *E. Capelas de Oliveira* and *G. Arcidiacono.*

28/91 **The Generalized Laplace Equation in Special Projective Relativity** — *E. Capelas de Oliveira* and *G. Arcidiacono.*

29/91 **The Projective D'Alembert Equation** — *E. Capelas de Oliveira* and *G. Arcidiacono.*

30/91 **The Generalized D'Alembert Equation in Special Projective Relativity** — *E. Capelas de Oliveira* and *G. Arcidiacono.*

31/91 **A General Algorithm for Finding the Minimal Angle between Subspaces** — *Alvaro R. De Pierro* and *Alfredo N. Iusem.*

32/91 **Scalar Curvature on Fibre Bundles** — *Maria Alice B. Grou.*

33/91 **Sur la Dimension des Algèbres Symétriques** — *Rachid Chibloun, Artibano Micali* et *Jean Pierre Olivier.*

34/91 **An Inverse Column-Updating Method for Solving Large-Scale Nonlinear Systems of Equations** — *José M. Martínez* em *Mário C. Zambaldi.*

35/91 **Parallel Implementations of Broyden's Method** — *Francisco A. M. Gomes* and *José M. Martínez.*

36/91 **Equivalência Elementar entre Feixes** — *A. M. Sette* and *X. Caicedo.*

37/91 **Unique Ergodicity for Degenerate Diffusions and the Accessibility Property of Control Systems** — *Luiz San Martin.*

38/91 **Unobservability of the Sign Change of Spinors Under a 2π Rotation in Neutron Interferometric Experiments** — *J. E. Maiorino, J. R. R. Zeni* and *W. A. Rodrigues Jr.*

39/91 **Disappearance of the Numerically irrelevant Solutions (NIS) in Non-Linear Elliptic Eigenvalue problems** — *Pedro C. Espinoza.*

40/91 **Positive Ordered Solutions of a Analogue of Non-Linear Elliptic Eigenvalue Problems** — *Pedro C. Espinoza.*

41/91 **On von Neumann's Variation of the Weierstrass–Stone Theorem** — *João B. Prolla.*

42/91 **Representable Operators and the Dunford-Pettis Theorem** — *Klaus Floret.*

43/91 **Simultaneous Approximation and Interpolation for Vector-Valued Continuous Functions** — *João B. Prolla.*

44/91 **On Applied General Equilibrium Analysis** — *José A. Scaramucci.*