SIMULTANEOUS APPROXIMATION AND INTERPOLATION FOR VECTOR-VALUED CONTINUOUS FUNCTIONS

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Abstract. The purpose of this paper is to determine conditions on subsets of continuous, vector-valued functions under which (a) interpolation; (b) approximation; (c) simultaneous interpolation and approximation; are all equivalent properties.

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Simultaneous Approximation and Interpolation for Vector-Valued Continuous Functions

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§1. Introduction

Throughout this paper S is a non-empty compact Hausdorff space, E is a non-trivial real or complex normed space and E^* is its topological dual. $C(S;\ E)$ is the linear space of all continuous functions from S into E, equipped with the supremum norm

$$\|f\| = \sup\{\|f(x)\|;\ x \in S\}.$$

Let $W \subset C(S;\ E)$ be a non-empty subset. A function $\varphi \in C(S;\ \mathbb{R})$ is called a multiplier of W if $0 \leq \varphi \leq 1$ and $\varphi f + (1-\varphi)g$ belongs to W, for every pair, f and g, of elements of W.

The notion of a multiplier is due to Feyel and de La Pradelle [3], in case W is a convex cone, and was extended to arbitrary subsets by Chao-Lin [1].

In our paper [4] we proved the following Weierstrass-Stone theorem for arbitrary non-empty subsets.

Theorem 1. *Let W be a non-empty subset of $C(S;\ E)$ such that the set of all multipliers of W separates the points of S. Let $f \in C(S;\ E)$ and $\varepsilon > 0$ be given. The following statements are equivalent:*

(1) there is some $g \in W$ such that $\|f - g\| < \varepsilon$;

(2) for each $x \in S$, there is some $g_x \in W$ such that $\|f(x) - g_x(x)\| < \varepsilon$.

§2. Interpolating families

We can apply our Theorem 1 to get results on simultaneous approximation and interpolation of **vector-valued** functions. A subset $A \subset C(S;\ E)$ is an **interpolating family for** $C(S;\ E)$ if, given any finite subset $F \subset S$ and any $f \in C(S;\ E)$, there exists $g \in A$ such that $f(t) = g(t)$ for all $t \in F$.

Theorem 2. *Let $A \subset C(S;\ E)$ be an interpolating family such that the set of multipliers of A separates the points of S. Then, for every $f \in C(S;\ E)$, every $\varepsilon > 0$ and every finite subset $F \subset S$, there exists $g \in A$ such that $||f - g|| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$. In particular, A is uniformly dense in $C(S;\ E)$.*

Proof. Define $W = \{g \in A;\ f(t) = g(t) \text{ for all } t \in F\}$. Since A is an interpolating family, $W \neq \emptyset$. Now it is easy to verify that each multiplier of A is also a multiplier of W. Hence, by Theorem 1, it suffices to show that, for each $x \in S$, there exists $g_x \in W$ such that $||f(x) - g_x(x)|| < \varepsilon$. Consider the finite set $F \cup \{x\}$. Since A is an interpolating family for $C(S;\ E)$, there exists $g_x \in A$ such that $f(t) = g_x(t)$ for all $t \in F \cup \{x\}$. In particular, $f(t) = g_x(t)$ for all $t \in F$. Hence $g_x \in W$. On the other hand $f(x) = g_x(x)$ implies $||f(x) - g_x(x)|| = 0 < \varepsilon$. By Theorem 1, there exists $g \in W$ such that $||f - g|| < \varepsilon$, and $g \in W$ implies $g \in A$ and $g(t) = f(t)$ for all $t \in F$. □

§3. Linear subspaces

When $E = \mathbb{K}$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) then the conclusion of Theorem 2 is true under the hypothesis that $A \subset C(S;\ \mathbb{K})$ is a dense linear subspace. (See Deutsch [2].) This poses the question of finding dense linear subspaces of $C(S;\ E)$ for which the conclusion of Theorem 2 is valid, i. e., for which simultaneous approximation and interpolation is possible.

Theorem 3. *Let W be a linear subspace of $C(S;\ E)$ such that $A \otimes E \subset W$, where $A = \{\varphi \circ g;\ \varphi \in E^*,\ g \in W\}$. The following statements are equivalent:*

(1) W is a dense linear subspace;

(2) simultaneous approximation and interpolation from W is possible.

Proof. Obviously, we have only to prove (1) $\Rightarrow$ (2). Let $f \in C(S;\ E)$, $\varepsilon > 0$ and $F = \{x_1, \ldots, x_n\} \subset S$ be given. We first show that A is dense in $C(S;\ I\!\!K)$. Let $h \in C(S;\ I\!\!K)$ and $\varepsilon > 0$ be given. Choose $v \in E$ and $\varphi \in E^*$ such that $||\varphi|| \leq 1$ and $\varphi(v) = 1$. Let $g(x) = h(x)v$, for all $x \in S$. Then, by density of W, there is some $w \in W$ such that $||w - g|| < \varepsilon$. Let $a = \varphi \circ w$. Then $a \in A$ and $|a(x) - h(x)| = |\varphi(w(x)) - h(x)\varphi(v)| = |\varphi(w(x)) - \varphi(h(x)v))| = |\varphi(w(x) - g(x))| \leq ||\varphi|| \cdot ||w(x) - g(x)|| < \varepsilon$, for all $x \in S$.

It follows that A is an interpolating family for $C(S;\ I\!\!K)$. Indeed, if we define $T : C(S;\ I\!\!K) \to I\!\!K^n$ by $Tg = (g(x_1), \ldots, g(x_n))$ for each $g \in C(S;\ I\!\!K)$, then by density of A and continuity of T, we have

$$T(C(S;\ I\!\!K)) = T(\overline{A}) \subset \overline{T(A)} = T(A),$$

where the last equality is a consequence of the fact that $T(A)$ is a linear subspace of $I\!\!K^n$. Let $a_1, \ldots a_n \in A$ be such that $a_i(x_j) = \delta_{ij} (1 \leq i,\ j \leq n)$. Choose $\delta > 0$ so that $\delta(1 + \sum_{i=1}^{n} ||a_i||) < \varepsilon$. Since W is dense, there is some $g_1 \in W$ such that $||f - g_1|| < \delta$. Let $v_i = f(x_i) - g_1(x_i)$, $1 \leq i \leq n$. Since $A \otimes E \subset W$, it follows that

$$g_2(x) = \sum_{i=1}^{n} a_i(x) v_i, \quad x \in S,$$

belongs to W. Notice that $g_2(x_j) = v_j$ for all $1 \leq j \leq n$. Hence $g(x_j) = f(x_j)$, for all $1 \leq j \leq n$, if $g \in W$ is defined to be $g_1 + g_2$. On the other hand,

$$||f - g|| < \delta + ||g_2|| < \delta + \delta \sum_{i=1}^{n} ||a_i|| < \varepsilon. \qquad \square$$

Corollary. *Let W be as in Theorem 3 and assume that the set of multipliers of W separates the points of S. The following statements are equivalent:*

(1) W is an interpolating family for $C(S;\ E)$;

(2) W is a dense linear subspace;

(3) simultaneous approximation and interpolation from W is possible.

Proof. (1) $\Rightarrow$ (2) follows from Theorem 2, even without the hypothesis $A \otimes E \subset W$. (2) $\Rightarrow$ (3) follows from Theorem 3, while (3) $\Rightarrow$ (1) is obvious. □

§4. Polynomial algebras

When $E = \mathbb{R}$, it follows from the classical Weierstrass-Stone theorem that any interpolating subalgebra $A \subset C(S;\ \mathbb{R})$ is a dense linear subspace. Hence, for subalgebras of $C(S;\ \mathbb{R})$ the following are equivalent: (1) A is dense; (2) A is interpolating; (3) simultaneous approximation and interpolation from A is possible. Our next result shows that this remains true for the so-called **polynomial algebras**. Recall that a vector subspace $W \subset C(S;\ E)$ is called a polynomial algebra if for every $n \geq 1$ and every continuous n-linear operator $T : E^n \to E$, the composition $T(f_1, \ldots, f_n) \in W$ for all $f_1, \ldots, f_n \in W$.

Theorem 4. *Let E be a real normed space and let $W \subset C(S;\ E)$ be a polynomial algebra. The following statements are equivalent:*

(1) W is an interpolating family for $C(S;\ E)$;

(2) W is a dense linear subspace;

(3) simultaneous approximation and interpolation from W is possible.

Proof. Let $A = \{\varphi \circ g;\ \varphi \in E^*,\ g \in W\}$. Since W is a polynomial algebra, A is a subalgebra of $C(S;\ \mathbb{R})$ and $A \otimes E \subset W$. Hence (2) $\Rightarrow$ (3) follows from Theorem 3. Since (3) $\Rightarrow$ (1) is obvious, it remains to prove (1) $\Rightarrow$ (2). Now, by the Hahn-Banach theorem, (1) implies that A is interpolating in $C(S;\ \mathbb{R})$. Therefore, by the classical Weierstrass-Stone theorem, A is dense in $C(S;\ \mathbb{R})$. Since $C(S;\ \mathbb{R}) \otimes E$ is dense in $C(S;\ E)$, it follows that $A \otimes E$ is dense in $C(S;\ E)$. It remains to notice that $A \otimes E \subset W$. □

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