ON VON NEUMANN'S VARIATION OF THE WEIERSTRASS–STONE THEOREM

João B. Prolla

RELATÓRIO TÉCNICO Nº 41/91

Abstract. Let X be a compact Hausdorff space and let $D(X)$ be the set of all continuous real-valued functions f defined on X and such that $0 \leq f(x) \leq 1$, for all $x \in X$. The set $D(X)$ is equipped with the uniform topology. We characterize the uniform closure of subsets $A \subset D(X)$ containing 0 and 1 and $\varphi\psi + (1-\varphi)\eta$, whenever they contain φ, ψ and η.

Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
IMECC – UNICAMP
Caixa Postal 6065
13.081 – Campinas – SP
BRASIL

Setembro – 1991

On von Neumann's variation of the Weierstrass–Stone Theorem

João B. Prolla
Departamento de Matemática,
IMECC — UNICAMP, Caixa Postal 6065
13081 Campinas, SP, Brazil

TO PROFESSOR B. BROSOWSKI ON THE OCCASION OF HIS 60th BIRTHDAY

§1. Introduction

Throughout this paper X is a compact Hausdorff space, and $D(X)$ is the set of all continuous function from the space X into the closed unit interval $I = \{t \in \mathbb{R};\ 0 \leq t \leq 1\}$, equipped with the topology of uniform convergence on X, determined by the metric d defined by

$$d(f,g) = \sup\{|f(x) - g(x)|\ ;\ x \in X\},$$

for every pair, f and g, of elements of $D(X)$.

We shall say that a non-empty subset A of $D(X)$ has *property VN*, if A contains the function $\varphi\psi + (1-\varphi)\eta$, whenever it contains φ, ψ and η.

The reading of von Neumann's paper [4] suggests that the following result should be true.

Theorem 1. *Consider a subset A of $D(I^n)$ which has property VN, contains the n projections, the constant functions 0 and 1 and at least one constant function with value $0 < c < 1$. Then A is uniformly dense in $D(I^n)$.*

Clearly, Theorem 1 is an easy consequence of the following description of the uniform closure of a subset $A \subset D(X)$ having property VN.

Theorem 2. *Let $A \subset D(X)$ be a non-empty subset with property VN and containing 0 and 1. Let $f \in D(X)$. Then f belongs to the uniform closure of A if, and only if, the following conditions hold:*
(1) *for each pair of points, x and y, of X such that $f(x) \neq f(y)$, there is some $\varphi \in A$ such that $\varphi(x) \neq \varphi(y)$;*
(2) *for each point $x \in X$ such that $0 < f(x) < 1$, there is some $\varphi \in A$ such that $0 < \varphi(x) < 1$.*

Now if a subset $A \subset D(X)$ contains 0 and 1 and has property VN and if φ and ψ belong to A, then clearly $1-\varphi$ and $\psi\varphi$ both belong to A. Hence A has property V of Jewett [1] and so Theorem 2 above is an easy corollary of Theorem 2, Prolla [2]. However, the proof of [2, Theorem 2] rests on a very hard result due to R. I. Jewett that says that a closed subset of $D(X)$ which has property V is a lattice. (See Theorem 1, Jewett [1].) In 1984, T. J. Ransford published a remarkably simple proof of Bishop's generalization of the Weierstrass-Stone Theorem. (See Ransford [3].) It is natural then to try to use Ransford's technique to simplify the proof of [2, Theorem 2]. However the difficulty is not overcome: one now meets the problem of proving that a closed subset A of $D(X)$ which has property V then has property VN. This is the contents of Lemma 6, Prolla [2]. But we could prove it only as a corollary of [2, Theorem 2] and so it could not be used in the proof of [2, Theorem 2] itself.

Our strategy to prove Theorem 2 above will be the following. Using Ransford's technique we first prove Theorem 3 below and then we show that Theorem 3 implies Theorem 2. This shows that in the case of property VN the use Zorn's Lemma removes the need of using the hard result that the closure of A is a lattice.

In order to state our Theorem 3 we introduce some notations. First of all, the following equivalence relation is introduced: $x \equiv y$ (mod. A) if, and only if, $\varphi(x) = \varphi(y)$ for all $\varphi \in A$. Now, if $x \in X$, then $[x]$ denotes its equivalence class (mod A). For any non-empty subset $S \subset X$, and any $f \in D(X)$, we denote by f_S the restriction of the function f to the subset S, and correspondingly, $A_S = \{g_S\,; g \in A\}$. Notice that $f_S \in D(S)$ and $A_S \subset D(S)$. When $S = [x]$, we write $f_S = f[x]$ and $A_S = A[x]$.

Theorem 3. *Let A be a non-empty subset of $D(X)$ which has property VN and contains 0 and 1. Then, for each $f \in D(X)$, there is a point $x \in X$ such that*

$$\mathrm{dist}(f; A) = \mathrm{dist}(f[x]; A[x]).$$

§2. Proof of Theorem 3

First of all notice that, for any $x \in X$, $\mathrm{dist}(f[x]; A[x]) \leq \mathrm{dist}(f; A)$. So, if $\mathrm{dist}(f; A) = 0$, then $\mathrm{dist}(f; A) = \mathrm{dist}(f[x]; A[x])$ for all points $x \in X$. Assume now $d = \mathrm{dist}(f; A) > 0$. By Zorn's Lemma there exists a minimal closed non-empty subset $S \subset X$ such that

$$\mathrm{dist}(f_S; A_S) = d.$$

We claim that $S \subset [x]$, for some point $x \in X$. If this is false, there is a pair of points $y, z \in S$ such that $\psi(y) \neq \psi(z)$, for some $\psi \in A$, and we may assume that $\psi(y) < \psi(z)$.

Choose $a < b$ such that $\psi(y) < a < b < \psi(z)$. We may assume $2a < b$. Indeed, if $k \in \{1, 2, 3, \ldots\}$ is such that $(a/b)^k < 1/2$, then $\psi^k(y) < \alpha < \beta < \psi^k(z)$, where $\alpha = a^k$ and $\beta = b^k$. Notice that $2\alpha < \beta$. On the other hand, since A has property VN and contains 0 and 1, it follows that A has property V. Hence $\psi^k \in A$. Define

$$Y = S \cap \psi^{-1}([0, b]),$$
$$Z = S \cap \psi^{-1}([a, 1]).$$

Then Y and Z are proper closed non-empty subsets of S, such that $S = Y \cup Z$. By the minimality of S, there exist v and w in A such that $d(f_Y, v_Y) < d$ and $d(f_Z, w_Z) < d$. Choose $0 < \varepsilon < 1$ so that $d(f_Y, v_Y) + \varepsilon < d$ and $d(f_Z, w_Z) + \varepsilon < d$, and then choose $0 < \delta < 1/2$ so small that $\delta < \varepsilon/2$.

Since $\frac{1}{a} - \frac{1}{b} > \frac{1}{2a} > \frac{1}{b} \geq 1$, we can choose a positive integer k so that

$$\frac{1}{b} < k < \frac{1}{a}.$$

Let m be a positive integer so large that

$$(kb)^{-m} < \delta \text{ and } (ka)^m < \delta.$$

Let $n = k^m$. Now if $0 \leq t \leq a$, then $(kt)^m < \delta$ and from Bernoulli's inequality we get

$$(1 - t^m)^n \geq 1 - (kt)^m > 1 - \delta.$$

On the other hand, if $b \leq t \leq 1$, then $(kt)^{-m} < \delta$ and once again by Bernoulli's inequality we get

$$(1 - t^m)^n \leq (1 + t^m)^{-n} \leq [1 + (kt)^m]^{-1} \leq (kt)^{-m} < \delta.$$

Let p be the polynomial $p(t) = (1 - t^m)^n$, $t \in \mathbb{R}$. Define $\varphi(x) = p(\psi(x))$, $x \in X$. Then $\varphi \in A$. Let $\eta = \varphi v + (1 - \varphi)w$. By property VN, we get $\eta \in A$. We claim that $|f(x) - \eta(x)| < d$, for all $x \in S$. To prove our claim, we consider three cases:

Case I: $x \in Y \cap Z$.
Case II: $x \in Y \backslash Z$.
Case III: $x \in Z \backslash Y$.

Case I: Let us write $f = \varphi f + (1-\varphi)f$. Then

$$\begin{aligned} |f(x)-\eta(x)| &\le \varphi(x)\,|f(x)-v(x)| + (1-\varphi(x)|f(x)-w(x)| \\ &< \varphi(x)d + (1-\varphi(x))d = d\,. \end{aligned}$$

Case II: In this case, $x \notin Z$ and therefore $\psi(x) < a$, and so $\varphi(x) > 1-\delta$. Let us write $v = \varphi v + (1-\varphi)v$. Then

$$\begin{aligned} |\eta(x)-v(x)| &= (1-\varphi(x))\,|w(x)-v(x)| \\ &\le (1-\varphi(x))2 \le 2\delta < \varepsilon\,, \end{aligned}$$

and, since $x \in Y$,

$$\begin{aligned} |f(x)-\eta(x)| &\le |f(x)-v(x)| + |v(x)-\eta(x)| \\ &< d(f_Y, v_Y) + \varepsilon < d\,. \end{aligned}$$

Case III: In this case, $x \notin Y$ and therefore $\psi(x) > b$ and so $\varphi(x) < \delta$. Let us write $w = \varphi w + (1-\varphi)w$. Then

$$\begin{aligned} |\eta(x)-w(x)| &= \varphi(x)\,|v(x)-w(x)| \\ &\le \varphi(x)2 \le 2\delta < \varepsilon\,, \end{aligned}$$

and, since $x \in Z$,

$$\begin{aligned} |f(x)-\eta(x)| &\le |f(x)-w(x)| + |w(x)-\eta(x)| \\ &< d(f_Z, w_Z) + \varepsilon < d\,. \end{aligned}$$

Hence $|f(x)-\eta(x)| < d$, for all $x \in S$ and therefore $d(f_S, \eta_S) < d$. But this contradicts the fact that $\mathrm{dist}(f_S; A_S) = d$. This contradiction shows that S must be contained in some equivalence class $[x]$. But then

$$d = \mathrm{dist}(f_S; A_S) \le \mathrm{dist}(f[x]; A[x]) \le d\,.$$

This completes the proof of Theorem 3. □

§3. Proof of Theorem 2

Conditions (1) and (2) are easily seen to be necessary for f to belong to the uniform closure of A. Conversely, let us assume that $f \in D(X)$

satisfies conditions (1) and (2). To prove that f belongs to the uniform closure of A it is equivalent to show that $\operatorname{dist}(f;A)=0$, where $\operatorname{dist}(f;A)=\inf\{d(f;g);\, g\in A\}$. Let $x\in X$ be given by Theorem 3. Now condition (1) implies that the restriction of f to $[x]$ is a constant function. Let c be its value. Let $0<\varepsilon<1$ be given. If $c=0$ (resp. $c=1$), there is some $\varphi\in A$ such that $\varphi(x)<\varepsilon$ (resp. $\varphi(x)>1-\varepsilon$). It suffices to take $\varphi=0$ (resp. $\varphi=1$). Hence $|\varphi(t)-f(t)|<\varepsilon$ for all $t\in[x]$ and consequently $\operatorname{dist}(f[x];A[x])<\varepsilon$. Assume now that $0<c<1$. By condition (2), there is some $\varphi\in A$ such that $0<\varphi(x)<1$. Choose $k\in\{1,2,3,\ldots\}$ such that $\varphi^k(x)<\varepsilon$ and let $d=1-\varphi^k(x)$. For some non-negative integer $m\in\{0,1,2,3,\ldots\}$ we have $d^{m+1}<c\le d^m$. We claim that $c-\varepsilon<d^{m+1}$. Indeed,

$$d^m-d^{m+1}=d^m(1-d)=d^m\varphi^k(x)\le\varphi^k(x)<\varepsilon\,.$$

Hence

$$c\le d^m=d^{m+1}+(d^m-d^{m+1})<d^{m+1}+\varepsilon\,.$$

Therefore $0<c-d^{m+1}<\varepsilon$. Let $\psi=(1-\varphi^k)^{m+1}$. Then $\psi\in A$ and its constant value on $[x]$ is d^{m+1}. Hence $|\psi(t)-f(t)|<\varepsilon$ for all $t\in[x]$, and consequently $\operatorname{dist}(f[x];A[x])<\varepsilon$. Since $\varepsilon>0$ was arbitrary, we see that in any case $\operatorname{dist}(f;A)=\operatorname{dist}(f[x];A[x])=0$. □

§4. Some Corollaries

Corollary 1. *Let $A\subset D(X)$ be a non-empty subset with property VN and containing 0 and 1. Assume that A separates the points of X and, for each $x\in X$, there is some $\varphi\in A$ such $0<\varphi(x)<1$. Then A is uniformly dense in $D(X)$.*

Proof. Immediate from Theorem 2. □

Corollary 2. *Let A be a closed non-empty subset of $D(X)$ with property VN and containing 0 and 1. Then A is a lattice.*

Proof. Let φ and ψ belong to A. Let $f=\max(\varphi,\psi)$. Let x and y be a pair of points of X such that $f(x)\ne f(y)$. Then at least one of the equalities $\varphi(x)=\varphi(y)$ and $\psi(x)=\psi(y)$ must be false. On the other hand, let $t\in X$ be such that $0<f(t)<1$. If $\varphi(t)\ge\psi(t)$, then $f(t)=\varphi(t)$

and therefore $0 < \varphi(t) < 1$. If $\varphi(t) < \psi(t)$, then $f(t) = \psi(t)$. and so $0 < \psi(t) < 1$. Hence (1) and (2) of Theorem 2 are satisfied, and f belongs to the uniform closure of A, i.e., A itself, since A is uniformly closed. Analogously, one shows that $g = \min(\varphi, \psi)$ belongs to A. Hence A is a lattice. □

References

[1] R. I. JEWETT, A variation on the Stone-Weierstrass theorem, *Proc. Amer. Math. Soc.* **14** (1963), 690-693.

[2] J. B. PROLLA, Uniform approximation of continuous functions with values in [0,1]. In *"Multivariate Approximation and Interpolation"* (W. Haussmann, K. Jetter, editors), International Series of Numerical Mathematics, vol. **94**, Birkhäuser Verlag, Basel 1990, pp. 241-247.

[3] T. J. RANSFORD, A short elementary proof of the Bishop-Stone-Weierstrass theorem, *Math. Proc. Camb. Phil. Soc.* **96** (1984), 309-311.

[4] J. VON NEUMANN, Probabilistic logics and the synthesis of reliable organisms from unrealiable components. In *"Automata Studies"* (C. E. Shannon, J. McCarthy, editors), Annals of Math. Studies, vol. **34**, Princeton Univ. Press, Princeton, N.J. 1956, pp. 43-98.

Abstract. Let X be a compact Hausdorff space and let $D(X)$ be the set of all continuous real-valued functions f defined on X and such that $0 \leq f(x) \leq 1$, for all $x \in X$. The set $D(X)$ is equipped with the uniform topology. We characterize the uniform closure of subsets $A \subset D(X)$ containing 0 and 1 and $\varphi\psi + (1-\varphi)\eta$, whenever they contain φ, ψ and η.

RELATÓRIOS TÉCNICOS — 1991

01/91 Um Método Numérico para Resolver Equações de Silvester e de Ricatti — *Vera Lucia da Rocha Lopes* and *José Vitório Zago.*

02/91 "Regge-Like" Relations for (Non-Evaporating) Black Holes and Cosmological Models — *Vilson Tonin-Zanchin* and *Erasmo Recami.*

03/91 The Exponential of the Generators of the Lorentz Group and the Solution of the Lorentz Force Equation — *J. R. Zeni* and *Waldyr A. Rodrigues Jr.*

04/91 Tensornorm Techniques for the (DF)-Space Problem — *Andreas Defant* and *Klaus Floret.*

05/91 Nonreversibility of Subsemigroups of Semi-Simple Lie Groups — *Luiz San Martin.*

06/91 Towards a General Theory of Convolutive Sets (With Applications to Fractals) — *Jayme Vaz Jr.*

07/91 Linearization of Holomorphic Mappings of Bounded Type — *Jorge Mujica.*

08/91 Topological Equivalence of Diffeomorphisms and Curves — *M. A. Teixeira.*

09/91 Applications of Finite Automata Representing Large Vocabularies — *Cláudio L. Lucchesi* and *Tomasz Kowaltowski.*

10/91 Torsion, Superconductivity and Massive Electrodinamics Cartan's Torsion Vector and Spin–0 Fields — *L. C. Garcia de Andrade.*

11/91 On The Continuity of Fuzzy Integrals — *G. H. Greco* and *R. C. Bassanezi.*

12/91 Optimal Chemical Control of Populations Developing Drug Resistance — *M. I. S. Costa, J. L. Boldrini* and *R. C. Bassanezi.*

13/91 Strict Monotonicity of Eigenvalues and Unique Continuation — *Djairo G. de Figueiredo* and *Jean-Pierre Gossez.*

14/91 Continuity of Tensor Product Operators Between Spaces of Bochner Integrable Functions — *Andreas Defant* and *Klaus Floret.*

15/91 Some Remarks on the Join of Spheres and their Particular Triangulations — *Davide C. Demaria* and *J. Carlos S. Kiihl.*

16/91 Sobre a Equação do Telégrafo e o Método de Riemann — *L. Prado Jr.* and *E. Capelas de Oliveira.*

17/91 Positive Solutions of Semilinear Elliptic Systems — *Ph. Clément, D. G. de Figueiredo* and *E. Mitidiere.*

18/91 The Strong Coupling Constant: Its Theoretical Derivation from a Geometric Approach to Hadron Structure — *Erasmo Recami* and *Vilson Tonin-Zanchin.*

19/91 Time Analysis of Tunnelling Processes, and Possible Applications in Nuclear Physics — *Vladislavi S. Olkhovsiky* and *Erasmo Recami*.

20/91 Procedimento, Função, Objeto ou Lógica? — *M. Cecília Calani Baranauskas*.

21/91 The Relation Between 2-Spinors and Rotations — *W. A. Rodrighes Jr.* and *J. R. Zeni*.

22/91 Boundaries for Algebras of Analytic Functions on Infinite Dimensional Banach Spaces — *R. M. Aron, Y. S. Choi, M. L. Lourenço* and *O. W. Paques*.

23/91 Factorization of Uniformly Holomorphic Functions — *Luiza A. Moraes, Otilia W. Paques* and *M. Carmelina F. Zaine*.

24/91 Métrica de Prohorov e Robustez — *Mario Antonio Gneri*.

25/91 Cálculo de Funções de Green para a Equação de Schrödinger pelo Método de Expansão Tipo Sturm-Liouville — *L. Prado Jr.* and *E. Capelas de Oliveira*.

26/91 On the Weierstrass-Stone Theorem — *João B. Prolla*.

27/91 Sull'Equazione di Laplace nell'Universo di De Sitter — *E. Capelas de Oliveira* and *G. Arcidiacono*.

28/91 The Generalized Laplace Equation in Special Projective Relativity — *E. Capelas de Oliveira* and *G. Arcidiacono*.

29/91 The Projective D'Alembert Equation — *E. Capelas de Oliveira* and *G. Arcidiacono*.

30/91 The Generalized D'Alembert Equation in Special Projective Relativity — *E. Capelas de Oliveira* and *G. Arcidiacono*.

31/91 A General Algorithm for Finding the Minimal Angle between Subspaces — *Alvaro R. De Pierro* and *Alfredo N. Iusem*.

32/91 Scalar Curvature on Fibre Bundles — *Maria Alice B. Grou*.

33/91 Sur la Dimension des Algèbres Symétriques — *Rachid Chibloun, Artibano Micali et Jean Pierre Olivier*.

34/91 An Inverse Column-Updating Method for Solving Large-Scale Nonlinear Systems of Equations — *José M. Martínez* em *Mário C. Zambaldi*.

35/91 Parallel Implementations of Broyden's Method — *Francisco A. M. Gomes* and *José M. Martínez*.

36/91 Equivalência Elementar entre Feixes — *A. M. Sette* and *X. Caicedo*.

37/91 Unique Ergodicity for Degenerate Diffusions and the Accessibility Property of Control Systems — *Luiz San Martin*.

38/91 Unobservability of the Sign Change of Spinors Under a 2π Rotation in Neutron Interferometric Experiments — *J. E. Maiorino, J. R. R. Zeni* and *W. A. Rodrigues Jr*.

39/91 Disappearance of the Numerically irrelevant Solutions (NIS) in Non-Linear Elliptic Eigenvalue problems — *Pedro C. Espinoza*.

40/91 Positive Ordered Solutions of a Analogue of Non-Linear Elliptic Eigenvalue Problems — *Pedro C. Espinoza*.