INVOLUTIONS AND STATIONARY POINT FREE Z_4–ACTIONS.

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Abstract. This paper studies fixed point sets of involutions and Z_2–fixed point sets of stationary point free Z_4–actions.

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1. Introduction

This paper studies fixed point sets of involutions and $\mathbb{Z}_2$–fixed point sets of stationary point free $\mathbb{Z}_4$–actions.

In section 2, the interest is to determine which bordism classes in the unoriented bordism ring $\mathcal{N}_*$ can be realized as the fixed point set of an involution on a n–dimensional manifold. Denoting by I_n the subgroup of these classes in $\mathcal{N}_*$, we are going to prove that $I_n = \underset{j\leq n}{\oplus} \mathcal{N}_j$ if n is even; and for n odd I_n is the set of classes in $\underset{j\leq n}{\oplus} \mathcal{N}_j$ with zero Euler characteristic mod 2.

In section 3, the $\mathbb{Z}_2$–fixed sets of stationary point free $\mathbb{Z}_4$–actions will be studied. Let $\mathcal{N}_m^{\mathbb{Z}_4}$ (st. pt. free) be the m–dimensional bordism group of manifolds with stationary point free $\mathbb{Z}_4$–action. Considering a $\mathbb{Z}_4$–action restricted to $\mathbb{Z}_2$ we get an involution, and the fixed set of this involution with the action induced by the $\mathbb{Z}_4$–action is an element in the bordism group of free involutions.

We are going to study the following question: Which classes in the bordism group of free involutions $\mathcal{N}_*^{\mathbb{Z}_2}$ (free) can be realized as the $\mathbb{Z}_2$–fixed point set of an $\mathbb{Z}_4$–action in $\mathcal{N}_m^{\mathbb{Z}_4}$ (st. pt. free)?

Denoting by $I_m^{\mathbb{Z}_2}$ the set of these classes and considering $A_m = (\underset{j\leq m}{\oplus} \mathcal{N}_j) \cap \mathcal{X}_*$, where $\mathcal{X}_*$ is the set of classes in $\mathcal{N}_*$ with zero Euler characteristic mod 2, the main result of this section is the following theorem.

Theorem.

(a) For m odd,

$$I_m^{\mathbb{Z}_2} = \underset{\substack{j=1 \\ j\,odd}}{\overset{m}{\oplus}} \mathcal{N}_j^{\mathbb{Z}_2}(free) + A_m[S^0,-1] + (\overset{m-1}{\underset{j=0}{\oplus}} \mathcal{N}_j)[S^1,-1]$$

(b) For m even

$$I_m^{\mathbb{Z}_2} = \bigoplus_{\substack{j=0 \\ j\,even}}^{m} \mathcal{N}_j^{\mathbb{Z}_2}(free) + A_m[S^0, -1] + A_{m-1}[S^1, -1]$$

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2. Involutions

Let $\mathcal{N}_*$ be the unoriented bordism ring of smooth manifolds and $\mathcal{N}_*^{\mathbb{Z}_2}$ the unrestricted bordism group of smooth manifolds with involution.

Being given a closed manifold M^n with an involution T, the fixed point set of $[M^n, T]$ is a disjoint union of closed manifolds $F^j, 0 \leq j \leq n$.

Consider the homomorphism

$$F_n : \mathcal{N}_n^{\mathbb{Z}_2} \longrightarrow \bigoplus_{j \leq n} \mathcal{N}^j$$

which assigns to $[M^n, T]$ the class $\bigoplus_{j \leq n} [F^j]$, where the disjoint union $\bigcup_{j \leq n} F^j$ is the fixed point set of T. Denote by I_n the image of F_n. In what follows, we are going to determine the image of the homomorphism F_n. To do this, we need the following lemmas.

Lemma 2.1 Let $[M^n]$ be in $\mathcal{N}_n$. If $\mathcal{X}[M^n] \equiv 0$ mod 2, then for every integer $k \geq 0$, there exists a $(n+k)$–manifold with involution $[W^{n+k}, T]$ such that the fixed point set is bordant to M^n.

proof. First, the lemma holds for $k = 0$ since the involution $[M^n, id]$ fixes M^n.

Now, suppose that $k \geq 1$. By [3, 4.5], we have that the bordism class of M^n admits a representative fibred over the circle since $\mathcal{X}[M^n] \equiv 0$ mod 2, i.e., there exists a closed manifold F^{n-1} with involution t such that $[M^n] = [(F^{n-1} \times S^1)/(t \times -1)]$. Then, considering the manifold with involution

$$[W^{n+k}, T] = [(F^{n-1} \times S^{k+1})/(t \times -1), 1 \times T'] + [F^{n-1} \times RP^{k+1}, t \times T'']$$

where $T' : (x_0, x_1, x_2, \ldots, x_{k+1}) \longmapsto (x_0, x_1, -x_2, \ldots, -x_{k+1})$
and $T'' : [x_0, x_1, x_2, \ldots, x_{k+1}] \longmapsto [x_0, x_1, -x_2, \ldots, -x_{k+1}]$,
it is easy to see that the fixed point set of the involution T is bordant to $(F^{n-1} \times S^1)/(t \times -1)$. Hence, the class $[M^n]$ is represented by a manifold which is the fixed point set of $[W^{n+k}, T]$. Therefore, the lemma holds at all.

Lemma 2.2. Fix an interger $k \geq 0$. Let $[M^m]$ and $[N^n]$ be in $\mathcal{N}_m$ and $\mathcal{N}_n$ respectively, for $m, n \leq 2k+1$. If $\mathcal{X}[M^m] + \mathcal{X}[N^n] \equiv 0 \bmod 2$, then there exists a $(2k+1)$ -manifold with involution $[W^{2k+1}, T]$ such that the class of the fixed point set of T is bordant to $[M^m] + [N^n]$.

proof. If $\mathcal{X}[M^m] \equiv \mathcal{X}[N^n] \equiv 0 \bmod 2$, it is clear that there exists $[W^{2k+1}, T]$ with fixed point set bordant to $[M^m] + [N^n]$, by (2.1).

Thus, we only need to consider the case $\mathcal{X}[M^m] \equiv \mathcal{X}[N^n] \equiv 1 \bmod 2$. In this case, we have $m = 2j$ and $n = 2l$, since $\mathcal{X}_m = \mathcal{N}_m$ if m is odd.

We may suppose $j \leq l$. Consider the involution

$$[W_1^{2k+1}, T_1] = [M^m \times RP^{2l-2j+1} \times RP^{2k-2l}, 1 \times t_1 \times t_2],$$

where $t_1 : [x_0, \ldots, x_{2l-2j+1}] \longmapsto [-x_0, x_1, \ldots, x_{2l-2j+1}]$
and $t_2 : [x_0, \ldots, x_{2k-2l}] \longmapsto [-x_0, x_1, \ldots, x_{2k-2l}]$.
The fixed point set F of $1 \times t_1 \times t_2$ is

$$\begin{aligned} F &= M^m \times (RP^0 \cup RP^{2l-2j}) \times (RP^0 \cup RP^{2k-2l-1}) \\ &= M^m \cup (M^m \times RP^{2k-2l-1}) \cup (M^m \times RP^{2l-2j}) \cup \\ &\qquad (M^m \times RP^{2l-2j} \times RP^{2k-2l-1}). \end{aligned}$$

Therefore, $[F] = [M^m] + [M^m \times RP^{2l-2j}]$ since $2k - 2l - 1$ is odd.

Now, note that $\mathcal{X}[M^m \times RP^{2l-2j} \cup N^n] \equiv 0 \bmod 2$, since $\mathcal{X}[M^m] \equiv \mathcal{X}[RP^{2l-2j}] \equiv \mathcal{X}[N^n] \equiv 1 \bmod 2$. Then, there exists an involution $[W_2^{2k+1}, T_2]$ with fixed point set in the class $[M^m \times RP^{2l-2j}] + [N^n]$, by (2.1).

Finally, the class of the fixed point set of

$$[W, T] = [W_1^{2k+1}, T_1] + [W_2^{2k+1}, T_2] \text{ is } [M^m] + [N^n].$$

Theorem 2.3

(a) The homomorphism F_n is onto for $n = 2k$ even,

$$i.e.,\ I_n = \bigoplus_{j \leq n} \mathcal{N}_j$$

(b) The image of F_n is the subgroup of classes in $\bigoplus_{j \leq n} \mathcal{N}_j$ with zero Euler characteristic, if $n = 2k+1$ odd.

proof. (a) First considering the involution $[M^n, id]$ we see that the class $[M^n]$ belongs to I_n. This means that $\mathcal{N}_n \subset I_n$. Now, by Capobianco [2, pp. 339] we have that $\mathcal{N}_j \subset I_n$ for $k \leq j \leq 2k$ and $j \neq 2k-1$. For $j = 2k-1$, the lemma (2.1) implies that $\mathcal{N}_{2k-1} \subset I_n$ since $\mathcal{N}_{2k-1} = \mathcal{X}_{2k-1}$.

Finally, it remains to show that $\mathcal{N}_j \subset I_n$ for $0 \leq j \leq k$. To prove this, take $[M^j]$ in $\mathcal{N}_j$. Consider the involution $[RP^{2k-2j}, T]$ where $T : [x_0, \ldots, x_{2k-2j}] \longmapsto [-x_0, x_1, \ldots, x_{2k-2j}]$. So, the class of the fixed point set of the involution $[RP^{2k-2j} \times M^j \times M^j, T \times twist]$ is $[RP^0 \times M^j] + [RP^{2k-2j-1} \times M^j]$ which is bordant to $[M^j]$ since $2k-2j-1$ is odd. Then, $\mathcal{N}_j \subset I_n$ for $o \leq j \leq k$

(b) By [3, 27.2], the image is contained in $\mathcal{X}_*$, i.e., the subgroup with zero Euler characteristic. We use now the lemma (2.2) and (2.1) to conclude that the classes in $\bigoplus_{j \leq 2k+1} \mathcal{N}_j$ with zero Euler characteristic are in the image. Hence, the theorem follows at once.

3. Stationary point free Z_4–actions.

Let $\mathcal{N}_*^{Z_4}$ (st. pt. free) the unoriented bordism group of stationary point free Z_4–actions and $\mathcal{N}_*^{Z_2}$ (free) the unoriented bordism group of free involutions.

Consider the homomorphism

$$F_m^{Z_2} : \mathcal{N}_m^{Z_4}(st.pt.free) \longrightarrow \bigoplus_{j \leq m} \mathcal{N}_j^{Z_2}(free)$$

which assigns to $[M^m, T]$ the class of the Z_2 – fixed point set of $[M^m, T]$. Recall the restriction homomorphism

$$\rho : \mathcal{N}_m^{Z_4}(st.pt.free) \longrightarrow \mathcal{N}_m^{Z_2}$$

assigning to $[M,T]$ the involution $[M,T^2]$. The fixed point set of $[M,T^2]$ is the disjoint union of closed submanifolds $\bigcup_{j\leq m} F^j$. Then, considering $t_j \equiv T/_{F^j}, j=0,\ldots,m$, we have $F_m^{\mathbb{Z}_2}([M,T]) = \oplus_{j\leq m} [F^j,t_j]$.

In this section we are going to study the image of the homomorphism $F_m^{\mathbb{Z}_2}$.

Now, let $\mathcal{N}_*^{\mathbb{Z}_4}$ (st. pt. free, free) be the relative bordism group of stationary point free $\mathbb{Z}_4$–actions on manifolds with boundary for which the action is free on the boundary. There exist the isomorphism

$$\mathcal{N}_*^{\mathbb{Z}_4}(st.pt.free,free) \cong \overset{*}{\underset{k=0}{\oplus}} \mathcal{N}_{*-k}^{\mathbb{Z}_2}(free)(BO_k(C^\infty))$$

by [1, pp. 85], and the sequence

$$0 \rightarrow \mathcal{N}_*^{\mathbb{Z}_4}(st.pt.free) \xrightarrow{i_*} \overset{*}{\underset{k=0}{\oplus}} \mathcal{N}_{*-k}^{\mathbb{Z}_2}(free)(BO_k(C^\infty)) \xrightarrow{\partial} \mathcal{N}_*^{\mathbb{Z}_4}(free) \rightarrow 0$$

of $\mathcal{N}_*$–modules and homomorphisms is split exact, where ∂ is the boundary homomorphism.

Further, for all k odd, we have the isomorphism

$$\varphi : \mathcal{N}_*^{\mathbb{Z}_4}(free) \otimes_{\mathcal{N}_*} \mathcal{N}_*(BSO_k) \longrightarrow \mathcal{N}_*^{\mathbb{Z}_2}(free)(BO_k(C^\infty)) \qquad (3.1)$$

which assigns to $[N,t]\times[P,\xi]$ the class of

$$[(N\times D\xi)/(t^2\times -1), t\times 1]. \text{ (see[5, 4.1])}.$$

Also, we have the homomorphism

$$\overline{F}_{\mathbb{Z}_2} : \mathcal{N}_*^{\mathbb{Z}_4}(st.pt.free,free) \longrightarrow \mathcal{N}_*^{\mathbb{Z}_2}(free).$$

mapping the class $[M,T]$ into the class of Z_2–fixed point set of $[M,T]$, and the restriction homomorphism

$$\rho : \mathcal{N}_*^{\mathbb{Z}_4}(free) \longrightarrow \mathcal{N}_*^{\mathbb{Z}_2}(free)$$

mapping the class $[M,T]$ into the class $[M,T^2]$.

Next, considering the homomorphism

$$\rho\circ\partial : \overset{m}{\underset{k=0}{\oplus}} \mathcal{N}_{m-k}^{\mathbb{Z}_2}(free)(BO_k(C^\infty)) \xrightarrow{\partial} \mathcal{N}_m^{\mathbb{Z}_4}(free) \xrightarrow{\rho} \mathcal{N}_m^{\mathbb{Z}_2}(free)$$

for m even, we are going to analyze the kernel of $\rho o \partial$ restricted to the summands with k odd.

Theorem 3.2. For m even, if α is in the kernel of the homomorphism $\rho o \partial$ restricted to the summands with k odd, then the $\mathbb{Z}_2$–fixed point set of α belongs to $\mathcal{X}_*[S^0,-1]+\mathcal{X}_{*-1}[S^1,-1]$

proof. First, by [5; 5.1], $\overline{F}_{\mathbb{Z}_2}$ restricted to the summands with k odd maps into $\mathcal{N}_*[S^0,-1]+\mathcal{N}_{*-1}[S^1,-1]$. Now, we are going to prove that if an element x belongs to the kernel of $\rho o \partial$ restricted to the summands with k odd, then the $\mathbb{Z}_2$–fixed point set of x is in $\mathcal{X}_*[S^0,-1]+\mathcal{X}_{*-1}[S^1,-1]$.

For k odd, we have the isomorphism

$$\mathcal{N}_*^{\mathbb{Z}_2}(free)(BO_k(C^\infty)) \simeq \mathcal{N}_*^{\mathbb{Z}_4}(free)\otimes_{\mathcal{N}_*}\mathcal{N}_*(BSO_k)$$

(see [5; 4.1]); and recall that $\mathcal{N}_*^{\mathbb{Z}_4}(free)$ is freely generated as an $\mathcal{N}_*$ module by extensions of the antipodal action on even dimensional spheres and by restrictions of circle actions on odd dimensional spheres. Therefore, for k odd, we can take as generators of $\mathcal{N}_{m-k}^{\mathbb{Z}_2}(free)(BO_k(C^\infty))$ the classes

$$y_{(2l,J)} = ([S^{2l}\times_{\mathbb{Z}_2}\mathbb{Z}_4, 1\times i],[RP^J,\xi^J])$$

and

$$y_{(2l+1,J')} = ([S^{2l+1}, i],[RP^{J'},\xi^{J'}])$$

where $[RP^J,\xi^J]$ and $[RP^{J'},\xi^{J'}]$ are generators of $\mathcal{N}_{n-2l}(BSO_k)$ and $\mathcal{N}_{n-2l-1}(BSO_k)$, respectively. (obs. $m=n+k$).

Thus, as in [5; 6.2] we have

$$\rho\circ\partial(\alpha)=\begin{cases} 0 & \text{if } \alpha = y_{(2\ell,J)} \\ [S^{2\ell+1},-1][S(\xi^{J'}),-1] & \text{if } \alpha = y_{(2\ell+1,J')}\end{cases}$$

Moreover, the $\mathbb{Z}_2$–fixed point set of the generators are

$$\overline{F}_{\mathbb{Z}_2}(\alpha)=\begin{cases} [RP^{2l}\times RP^J][S^0,-1] & \text{if } \alpha = y_{(2l,J)} \\ [\mathbb{C}P^l\times RP^{J'}][S^1,-1] & \text{if } \alpha = y_{(2l+1,J')}\end{cases}$$

Now, taking the map $f: RP^{2l+1}\times RP^{J'} \longrightarrow RP^\infty$ that classifies the bundle $[RP^{2l+1}\times RP^{J'},\gamma_1\otimes\gamma_2]$ with γ_1 the line bundle over RP^{2l+1}

and γ_2 the line bundle over $RP^{J'}$, we have that the Whitney number $< cw_{m-2}, \sigma_{m-1} >$ of the map f, where $c = \alpha_{2l+1} \times 1$ and α_{2l+1} is the generator of $H^1(RP^{2l+1}; \mathbb{Z}_2)$, is given by

$$\begin{aligned} < cw_{m-2}, \sigma_{m-1} > &= < (\alpha_{2l+1} \times 1) w_{m-2}, \sigma_{m-1} > \\ &= < (\alpha_{2l+1} \times 1) \binom{2l+2}{2l} \alpha_{2l+1}^{2l} \times \mathcal{X}(RP(\xi^{J'}), \sigma_{m-1} > \\ &= < (\alpha_{2l+1} \times 1) \binom{2l+2}{2l} \alpha_{2l+1}^{2l} \times \mathcal{X}(RP^{J'}) \mathcal{X}(RP^{k-1}), \sigma_{m-1} > \end{aligned}$$

Further, we have $\mathcal{X}(\mathbb{C}P^l \times RP^{J'}) \equiv \binom{l+1}{l} \beta^l \times \mathcal{X}(RP^{J'})$ mod 2, where β is the generator of $H^2(\mathbb{C}P^l; \mathbb{Z}_2)$.

Next, observe that $\mathcal{X}(\mathbb{C}P^l \times RP^{J'}) \equiv < cw_{m-2}, \sigma_{m-1} >$ and $\mathcal{X}(RP^{2l} \times RP^J) \equiv 0$ mod 2, since the dimension of $RP^{2l} \times RP^J$ is $2l + (n - 2l) = n$ odd.

Finally, it is easy to see that these facts don't depend on k, since k is odd. Hence, if
$x = \Sigma(a_{l,J} y_{(2l,J)} + b_{l,J'} y_{(2l+1,J')})$, with $a_{l,J}, b_{l,J'} \in \mathbb{Z}_2$ is in the kernel of $cw_{m-2} o\rho \circ \partial$ restricted to the summands with k odd, then we can see that the $\mathbb{Z}_2$–fixed point set of x is in $\mathcal{X}_*[S^0, -1] + \mathcal{X}_{*-1}[S^1, -1]$.

Next, consider the homomorphism

$$cw_{m-2} o\rho \circ \partial : \bigoplus_{k \leq m} \mathcal{N}^{\mathbb{Z}_2}_{m-k}(free)(BO_k) \longrightarrow \mathbb{Z}_2$$

where $cw_{m-2} : \mathcal{N}^{\mathbb{Z}_2}_{m-1}(free) \longrightarrow \mathbb{Z}_2$ maps α into the Whitney number $< cw_{m-2}, [\alpha] >$.

Theorem 3.3. For m even, the homomorphism $cw_{m-2} o\rho \circ \partial$ restricted to the summands with k even is the zero homomorphism.

proof. Take $m = n + k, k = 2j$ even. Let ξ^k be a k–bundle over M^n with M^n having a $\mathbb{Z}_4$–action such that restricts to $\mathbb{Z}_2$ acts trivially. Further, this $\mathbb{Z}_4$–action is covered by a $\mathbb{Z}_4$–action on the total space of ξ^k and the induced $\mathbb{Z}_2$–action acts by multiplication by -1 in the fibers of ξ^k covering a free $\mathbb{Z}_2$–action on the base.

Observe that $\rho \circ \partial([\xi^k, M^n]) = [RP(\xi^k), \lambda]$, where $RP(\xi^k)$ is the associated $(m-1)$–dimensional projective space and λ is the canonical line bundle over $RP(\xi^k)$. Next, the total Stiefel–Whitney class of $RP(\xi^k)$ is given by

$$W(RP(\xi^k)) = W(M) + (\sum_{i=0}^{k}(1+c)^{k-i}v_i)$$

where $v = \sum_{i=0}^{k} v_i$ is the total Whitney class of ξ^k. Moreover, we have the relation $\sum_{i=0}^{k} c^{k-i} v_i = 0$.

Therefore, the Whitney number $< cw_{m-2}, [RP(\xi^k)] >$ is

$$\begin{aligned} < cw_{m-2}, [RP(\xi^k)] > \; &= \; < cw_n(M)\{\binom{k}{k-2}c^{k-2} + \binom{k-1}{k-3}c^{k-3}v_1 \\ &\qquad + \cdots + v_{k-2}\}, [RP(\xi^k)] > \\ &+ \; < cw_{n-1}(M)\{\binom{k}{k-1}c^{k-1} + \binom{k-1}{k-2}c^{k-2}v_1 \\ &\qquad + \cdots + v_{k-1}\}, [RP(\xi^k)] > \end{aligned}$$

Now, since $k = 2j$ and M is n–dimensional, we have

$$\begin{aligned} < cw_{m-2}, [RP(\xi^k)] > \; &\equiv \; < jw_n(M)c^{k-1} + v_1 w_{n-1}(M)c^{k-1}, [RP(\xi^k)] > \\ &\equiv \; j\mathcal{X}[M] + < v_1 w_{n-1}(M), [M] > \\ &\equiv \; < v_1 w_{n-1}, (M), [M] > \end{aligned}$$

since $\mathcal{X}[M] \equiv 0$ mod 2 due the fact that we have a free $\mathbb{Z}_2$–action on M.

Next, we are going to see that $< v_1 w_{n-1}(M), [M] > \equiv 0$ mod 2. First, recall that $v_1 = w_1$ (det ξ^k), where det ξ^k is the determinant bundle of ξ^k. Moreover, we have det $\xi^k = \wedge^k \xi^k$ the k–exterior power of the bundle ξ^k. So, we can see that the $\mathbb{Z}_4$–action T on ξ^k induce a $\mathbb{Z}_2$–action on det ξ^k. In fact, let $x = x_1 \wedge x_2 \wedge \cdots \wedge x_k$ be in $\wedge^k \xi^k$ with $x_i \in \xi^k$. Then $T^2(x) = (-x_1) \wedge (-x_2) \wedge \cdots \wedge (-x_k) = x$ since k is even.

Therefore, we get the commutative diagram

$$\begin{array}{ccc} \det \xi^k & \longrightarrow & (\det \xi^k)/\mathbb{Z}_2 \\ \downarrow & & \downarrow \\ M & \xrightarrow{\pi} & M/\mathbb{Z}_2 \end{array}$$

with $\det \xi^k$ having a $\mathbb{Z}_2$–action covering a free $\mathbb{Z}_2$–action on M. Thus,

$$\begin{aligned} < v_1 w_{n-1}(M), [M] > &= < w_1(\det \xi^k) w_{n-1}(M), [M] > \\ &= < \pi^*(w_1((\det \xi^k)/\mathbb{Z}_2) w_{n-1}(M/\mathbb{Z}_2)), [M] > \\ &= < w_1((\det \xi^k)/\mathbb{Z}_2) w_{n-1}(M/\mathbb{Z}_2), \pi_*[M] > \\ &\equiv 0 \bmod 2, \end{aligned}$$

since $\pi_*[M] = 2[M/\mathbb{Z}_2] \equiv 0 \bmod 2$.

Theorem 3.4. For m even, if α is in the kernel of the boundary homomorphism ∂, then the $\mathbb{Z}_2$–fixed point set of α is in $\bigoplus_{\substack{j=0 \\ j\,even}} \mathcal{N}_j^{\mathbb{Z}_2}(free) + \mathcal{X}_*[S^0, -1] + \mathcal{X}_{*-1}[S^1, -1]$

proof. We have that $cw_{m-2} o p o \partial(\alpha) = 0$, since $\partial(\alpha) = 0$

Therefore, by (3.2) and (3.3) the result follows at once.

Lemma 3.5 Let $[N^n, t]$ be in $\mathcal{N}_*^{\mathbb{Z}_2}$ (free). For $k \geq 0$, there exists a stationary point free $\mathbb{Z}_4$–action $[W^{n+2k}, T]$ such that the $\mathbb{Z}_2$–fixed point set is $[N^n, t]$.

proof. Suppose $k > 0$ and consider the $\mathbb{Z}_4$–action $[RP^{2k} \times N, T \times t]$, where $T : [x_0, x_1, \ldots, x_{2k}] \mapsto [x_0, -x_2, x_1, \ldots, -x_{2k}, x_{2k-1}]$. The $\mathbb{Z}_2$–fixed point set is the class $[N, t] + [RP^{2k-1} \times N, i \times t]$ which is equal to $[N, t]$ since the free involution $[RP^{2k-1}, i]$ bounds as involution and then $[RP^{2k-1} \times N, i \times t]$ bounds as free involution.

Finally, for $k = 0$, taking $[N, t]$ as stationary point free $\mathbb{Z}_4$–action, the $\mathbb{Z}_2$–fixed point set is $[N, t]$.

Next, denote by $I_m^{\mathbb{Z}_2}$ the image of the homomorphism $F_m^{\mathbb{Z}_2}$. Considering $A_m = (\underset{j\leq m}{\oplus} \mathcal{N}_j) \cap \mathcal{X}_*$, we have the following lemma.

Lemma 3.6 $A_m[S^0, -1] + A_{m-1}[S^1, -1] \subset I_m^{\mathbb{Z}_2}$

proof. If $[N] \in A_m$, by theorem (2.3) there exists an involution $[W_1^m, t_1]$ with the fixed point set bordant to N. Thus, the stationary point free $\mathbb{Z}_4$–action $[W_1^m \times_{\mathbb{Z}_2} \mathbb{Z}_4, t_1 \times i]$ has $\mathbb{Z}_2$–fixed point set bordant to $[N][S^0, -1]$. Therefore, $A_m[S^0, -1] \subset I_m^{\mathbb{Z}_2}$.

Now, if $[M] \in A_{m-1}$, again by (2.3) there exists an involution $[W_2^{m-1}, t_2]$ such that the fixed point set is $[M]$. Then, the $\mathbb{Z}_4$–action $[(W_2^{m-1} \times S^1)/(t_2 \times -1), 1 \times i]$ has the class $[M][S^1, -1]$ as $\mathbb{Z}_2$–fixed point set. Hence, $A_{m-1}[S^1, -1] \subset I_m^{\mathbb{Z}_2}$ and the lemma holds.

Now, we can state the main result of this section

Theorem 3.7.

(a) For m odd,

$$I_m^{\mathbb{Z}_2} = \underset{\substack{j=1\\ j\,odd}}{\overset{m}{\oplus}} \mathcal{N}_j^{\mathbb{Z}_2}(free) + A_m[S^0, -1] + (\overset{m-1}{\underset{j=0}{\oplus}} \mathcal{N}_j)[S^1, -1]$$

(b) For m even,

$$I_m^{\mathbb{Z}_2} = \underset{\substack{j=0\\ j\,even}}{\overset{m}{\oplus}} \mathcal{N}_j^{\mathbb{Z}_2}(free) + A_m[S^0, -1] + A_{m-1}[S^1, -1]$$

proof. First, since m is odd, then using [5; 5.1] and [3; 27.2], it is easy to see that

$$I_m^{\mathbb{Z}_2} \subset \underset{\substack{j=1\\ j\,odd}}{\overset{m}{\oplus}} \mathcal{N}_j^{\mathbb{Z}_2}(free) + \mathcal{X}_*[S^0, -1] + \mathcal{N}_*[S^1, -1]$$

Now, if j is odd then $m - j$ is even and the lemma (3.5) implies that $\mathcal{N}_j^{\mathbb{Z}_2}(free) \subset I_m^{\mathbb{Z}_2}$.

Further, note that if we have N^{j-1}, j *odd*, then $[N^{j-1}][S^1, -1]$ belongs to $I_m^{\mathbb{Z}_2}$ by lemma (3.5) since the codimension is even; and if j is even $[N^{j-1}][S^1, -1]$ belongs to $I_m^{\mathbb{Z}_2}$ by lemma (3.6) since $\mathcal{X}(N^{j-1}) \equiv 0 \bmod 2$.

Hence, applying again the lemma (3.6) the part (a) of the theorem follows at once.

(b) By theorem (3.4) we have

$$I_m^{\mathbb{Z}_2} \subset \bigoplus_{\substack{j=0 \\ j even}} \mathcal{N}_j^{\mathbb{Z}_2}(free) + \mathcal{X}_*[S^0, -1] + \mathcal{X}_{*-1}[S^1, -1].$$

Now, considering j even, the lemma (3.5) implies that $\mathcal{N}_j^{\mathbb{Z}_2}(free) \subset I_m^{\mathbb{Z}_2}$ since the codimension is even. Therefore, applying the lemma (3.6) we have the result.

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