THE MOD 2 HOMOLOGY OF BSO

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RELATÓRIO TÉCNICO Nº 36/90

Abstract. This note is about the mod 2 homology of BSO.

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Outubro - 1990

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1. Introduction

It is well known that $H_*(BO;Z_2)$ is a polynomial ring, $Z_2[x_i \mid i \geq 1]$, where $x_i \in H_i(BO;Z_2)$. The generators x_i may be chosen to come from the nonzero classes in $H_i(BO_1;Z_2)$ under the stabilization map, and in particular, $x_i = f_*[RP^i]$, where $f\colon RP^i \to BO$ classifies the usual line bundle over projective space.

The corresponding dual basis of $H^*(BO;Z_2)$ is usually denoted by s_ω, where $\omega = (i_1,\ldots,i_r)$. If the splitting principle is used to write universal Stiefel-Whitney classes w_i formally as the i-th elementary symmetric function in 1-dimensional classes $t_1, t_2, \ldots$, then $s_\omega = \Sigma\, t_1^{i_1} t_2^{i_2} \cdots t_r^{i_r}$ is the smallest symmetric function containing the given monomial. In particular, if $y = \Sigma a_{j_1 \cdots j_s} x_1^{j_1} \cdots x_s^{j_s} \in H_*(BO;Z_2)$, the coefficients are $a_{j_1 \cdots j_s} = s_{(j_1,\ldots,j_s)}[y]$.

Recently, S. Papastavridis [1] has shown that $H_*(BSO;Z_2)$ is also a polynomial ring, $Z_2[y_i \mid i > 1]$, which is described as a subring of $H_*(BO;Z_2)$ by choosing classes y_i as polynomials in the x_j. (Note: It is well known that $H^*(BSO;Z_2)$ is the quotient of $H^*(BO;Z_2)$ by the ideal generated by w_1. Dually, $H_*(BSO;Z_2)$ can be identified with a subring of $H_*(BO;Z_2)$.) Papastavridis' choices of the classes y_i are clearly algebraically independent and hence give a subring of $H_*(BO;Z_2)$ which has precisely the same dimension as $H_*(BSO;Z_2)$. The hard part of his argument is to see that the classes y_i lie in $H_*(BSO;Z_2)$.

The purpose of this paper is to simplify Papastavridis' argument. For any integer $n > 1$, one chooses a pair of integers (j,k) with $j+k = n$ by

$$\begin{cases} (j,k) = (0,n) & \text{if } n = 2^r, \\ (j,k) = (2^r, 2^{r+1}s) & \text{if } n = 2^r(2s+1),\ s > 0. \end{cases}$$

Then, let $z_n \in H_n(BO;Z_2)$ be the classes $f_*[RP^j \times RP^k]$ where $f\colon RP^j \times RP^k \to BO$ classifies the bundle $\xi_1 \oplus \xi_2 \oplus (\xi_1 \otimes \xi_2)$ with ξ_i being the usual line bundle over the i-th factor. Because the given bundle is orientable, it is clear that $z_n \in$ image $(H_n(BSO;Z_2) \to H_n(BO;Z_2))$, and our main result is

<u>Theorem</u>. $H_*(BSO;Z_2) = Z_2[z_n \mid n > 1]$.

Additionally, one has:

<u>Fact</u>. The classes $z_n = f_*[RP^j \times RP^k]$ coincide with Papastavridis' classes y_n.

I wish to express my thanks to Professor R. E. Stong for suggesting this problem and to FAPESP (Fundação de Amparo à

Pesquisa do Estado de São Paulo, Brasil) for financial support during this work.

2. Proof of the Theorem

It is clear that one has a homomorphism

$$\varphi\colon Z_2[u_n \mid n > 1] \to H_*(BSO;Z_2) \subset H_*(BO;Z_2)$$

defined by $\varphi(u_n) = z_n = f_*[RP^j \times RP^k]$, and in every dimension, $H_*(BSO;Z_2)$ and the polynomial ring have the same dimension as Z_2 vector space. To prove the theorem, it suffices to see that the classes z_n are algebraically independent. This is immediate from:

Lemma.

$$z_n = \begin{cases} x_n + \text{decomposables} & \text{if } n = 2^r(2s+1), \\ x_{n/2}^2 & \text{if } n = 2^r. \end{cases}$$

Proof. Let $H^*(RP^j \times RP^k;Z_2) = Z_2[\alpha,\beta]/(\alpha^{j+1} = 0,\ \beta^{k+1} = 0)$, where $\dim \alpha = \dim \beta = 1$. The Stiefel-Whitney class of $\xi_1 \oplus \xi_2 \oplus (\xi_1 \otimes \xi_2)$ is $(1+\alpha)(1+\beta)(1+\alpha+\beta)$. Then for $n = 2^r(2s+1)$,

$$\begin{aligned} s_n &= \alpha^n+\beta^n+(\alpha+\beta)^n \\ &= \binom{2^r(2s+1)}{2^r}\alpha^{2^r}\,\beta^{2^{r+1}s}\,, \end{aligned}$$

which is nonzero. For $n = 2^r$, $\alpha = 0$, and $H^*(RP^n;Z_2) = Z_2[\beta]/(\beta^{n+1} = 0)$, with the Stiefel-Whitney class of the bundle being $(1+\beta)^2$. Then

$$s_\omega((1+\beta)^2) = \begin{cases} 0 & \text{if } \omega \neq (\omega',\omega') \\ s_{\omega'}((1+\beta))^2 & \text{if } \omega = (\omega',\omega') \end{cases}$$

giving $z_n = x_{n/2}^2$.

3. Papastavridis' Classes

To verify that $z_n = y_n$, as defined by Papastavridis, requires a lot of unpleasant calculation. Not only is one showing that y_n belongs to $H_n(BSO;Z_2)$, but one is identifying the given class. Since this is obvious for $n = 2^r$, one need only consider $n = 2^r(2s+1)$. The goal is to verify that $s_{(a,b,c)}[z_n]$, with $0 \leq a \leq b \leq c$ is given by Papastavridis' formula,

$$\begin{cases} \binom{b-1}{2^r-a-1} & \text{if } 2^r \leq b \text{ and } 0 \leq a < 2^r \\ \binom{b-1}{\tau(2^r-a)} & \text{if } 0 < b < 2^r,\ 0 \leq a < 2^r \text{ and } a+\tau(2^r-a) \leq b \\ 0 & \text{otherwise.} \end{cases}$$

It is, of course, clear that $s_\omega[z_n] = 0$ if $\omega = (i_1,\ldots,i_r)$ with $r > 3$, since the defining bundle has dimension 3.

Here we are going to verify that $s_{(a,b,c)}[z_n]$ is given by the above formula only in case $0 < a < b < 2^r$, $c < 2^{r+1}s$. In this case we have

$$s_{(a,b,c)}[z_n] = \{\alpha^a\beta^b(\alpha+\beta)^c + \alpha^a\beta^c(\alpha+\beta)^b + \alpha^b\beta^c(\alpha+\beta)^a$$

$$+ \alpha^b\beta^a(\alpha+\beta)^c\}\ [RP^{2^r} \times RP^{2^{r+1}s}], \text{ since } \alpha^c = 0,$$

$$= \left[\binom{c}{2^r-a} + \binom{b}{2^r-a} + \binom{a}{2^r-b} + \binom{c}{2^r-b}\right] \bmod 2.$$

Next, observe that $\binom{a}{2^r-b}$ is the coefficient of x^{2^r-b} in the binomial expansion of $(1+x)^a$, and $(1+x)^a = \dfrac{(1+x)^{2^{r+1}s}}{(1+x)^{b+c-2^r}}$, with $b+c-2^r-1 > 0$. Recall that $\dfrac{1}{(1+y)^{t+1}} = \sum_{j=0}^{\infty}\binom{t+j}{j}y^j$. Then

$$\binom{a}{2^r-b} \equiv \binom{b+c-2^r-1+2^r-b}{2^r-b} \equiv \binom{c-1}{2^r-b} \bmod 2.$$

So, it is clear that

$$\binom{c}{2^r-b} + \binom{a}{2^r-b} \equiv \binom{c}{2^r-b} + \binom{c-1}{2^r-b} \equiv \binom{c-1}{2^r-b-1} = \binom{2^{r+1}s+2^r-a-b-1}{2^{r+1}s-a},$$

and this is the coefficient of $x^{2^{r+1}s-a}$ in the binomial expansion of $(1+x)^{c-1}$, where

$$(1+x)^{c-1} = \frac{(1+x)^{2^{r+1}s}(1+x)^{2^r}}{(1+x)^{a+b+1}}$$

$$= \left\{\underbrace{\sum_{m=0}^{s}\binom{s}{m}x^{2^{r+1}s-2^{r+1}m}}_{(A)} + \underbrace{\sum_{m=0}^{s}\binom{s}{m}x^{2^{r+1}s-2^{r+1}m+2^r}}_{(B)}\right\}\underbrace{\left\{\sum_{\ell=0}^{\infty}\binom{a+b+\ell}{\ell}x^\ell\right\}}_{(C)}.$$

Now, if we take the product of any term in (A) by the complementary term in (C), the coefficient is

$$\binom{s}{m}\binom{2^{r+1}m+b}{2^{r+1}m-a} \equiv 0 \bmod 2,$$

since $b < 2^r$. (Obs: if $m = 0$ in (A), $2^{r+1}s > 2^{r+1}-a$). Next, observe that all the powers in the expansion (B) for $m > 0$ are between $2^{r+1}(s-1)+2^r$ and $2^{r+1}s$, moreover, we have $2^{r+1}(s-1)+2^r < 2^{r+1}s-a < 2^{r+1}s$, since $2^r > a > 0$. For $m = 0$, the power is $2^{r+1}s+2^r$, so it is bigger than $2^{r+1}s$. Therefore,

$$\binom{c}{2^r-b} + \binom{a}{2^r-b} = \left\{\sum_{m=1}^{s}\binom{s}{m}\binom{2^{r+1}m-2^r+b}{2^{r+1}m-2^r-a}\right\}$$

$$= \left\{\sum_{m=1}^{s}\binom{s}{m}\right\}\binom{b}{2^r-a} \equiv \binom{b}{2^r-a} \bmod 2.$$

Thus, we can see immediately that

$$s_{(a,b,c)}[z_n] = \binom{c}{2^r-a} \bmod 2.$$

Now, writing $2^r-a = 2^j+t$, $0 \le t < 2^j$ with $\tau(2^r-a) = t$ as in Papastavridis [1], we have $c = 2^{r+1}s+2^j+t-b$. So, we can look at $\binom{c}{2^r-a}$ as the coefficient of x^{2^r-a} in the binomial expansion of

$$(1+x)^c = \frac{(1+x)^{2^{r+1}s}(1+x)^{2^j}}{(1+x)^{b-t}}.$$

Hence, it follows that

$$s_{(a,b,c)}[z_n] = \left\{\binom{2^j+b-1}{2^j+t} + \binom{b-1}{t}\right\} \bmod 2.$$

Next, we can write $b = a+b'$ with $0 < b' < 2^r - a = 2^j + t$, since $a < b < 2^r$. Thus, we get

$$\binom{2^j+b-1}{2^j+t} = \binom{2^j+2^j(2^{r-j}-1)-t+b'-1}{2^j+t} \equiv \begin{cases} \binom{b-1}{t} \bmod 2 & \text{if } 0 < b' \leq t \\ 0 \bmod 2 & \text{if } t < b' < 2^j+t \end{cases}$$

and

$$\binom{b-1}{t} = \binom{2^j(2^{r-j}-1)-t+b'-1}{t} \equiv 0 \bmod 2 \text{ if } t < b' < 2t.$$

Finally, since

$$0 < b' < 2t \iff a+2\tau(2^r-a) > b$$

and

$$2t \leq b' < 2^j+t \iff a+2\tau(2^r-a) \leq b,$$

we conclude that

$$s_{(a,b,c)}[z_n] = \begin{cases} 0 \bmod 2 & \text{if } a+2\tau(2^r-a) > b \\ \binom{b-1}{\tau(2^r-a)} \bmod 2 & \text{if } a+2\tau(2^r-a) \leq b \end{cases}$$

as in [1].

Since we have checked one of the more difficult cases, one now can write down the other possibilities for (a,b,c) using similar calculations.

Reference

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