ON FIXED POINTS SETS OF INVOLUTIONS

Claudina Izepe Rodrigues

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Abstract. The purpose of this paper is to study fixed–point sets of involutions.

Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
IMECC – UNICAMP
Caixa Postal 6065
13.081 – Campinas – SP
BRASIL

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1. Introduction

The purpose of this paper is to study fixed-point sets of involutions. Let M^n be a closed n-manifold with smooth involution T. The fixed-point set of T is the disjoint union of submanifolds F^m with $0 \leq m \leq n$.

In [5] Wu Zhende studies the sets $J_n^{k_1,\ldots,k_\ell}$ of all bordism classes in the n-dimensional unoriented bordism group $\mathcal{N}_n$ with the following property: The class of an element in $J_n^{k_1,\ldots,k_\ell}$ is represented by an n-manifold M^n for which there exists an involution T such that the fixed point set of T is the disjoint union of submanifolds F^{n-k_i}, with $1 \leq i \leq \ell$. He also obtains the results:

(a) $J_n^{1,2} = \mathcal{N}_n$, if $n \geq 2$;

(b) $J_n^{2,3} = \mathcal{N}_n$, if $n \geq 4$.

In this work we are going to determine the sets $J_n^{3,4,k_1,\ldots,k_\ell}$ with $4 \leq k_1 < \cdots < k_\ell \leq n$.

The main results of this paper are:

Theorem.

(a) $J_n^{3,4} = (0)$, for $n = 4,5$.

(b) $J_6^{3,4}$ is the subgroup of $\mathcal{N}_6$ generated by the classes x_2^3 and x_6, where $x_2 = [RP^2]$ and $x_6 = [RP^6]$.

(c) $J_n^{3,4} = \mathcal{N}_n$, for $n \geq 7$.

Theorem. If $4 < k_1 < \cdots < k_\ell \leq n$, with $\ell > 0$, then $J_n^{3,4,k_1,\ldots,k_\ell} = \mathcal{N}_n$.

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2. $J_n^{3,4}$

First we recall some facts about involutions. Let $\mathcal{N}_n^{Z_2}$ be the n-dimensional unoriented bordism group of involutions. There is an exact sequence

$$0 \longrightarrow \mathcal{N}_n^{Z_2} \xrightarrow{i_n} \bigoplus_{j=0}^{n} \mathcal{N}_{n-j}(BO_j) \xrightarrow{j_n} \mathcal{N}_{n-1}(BO_1) \rightarrow 0 \qquad (2.1)$$

of $\mathcal{N}_*$-modules and homomorphisms.

Let $[M^{n-j},\nu^j]$ be a j-bundle in $\bigoplus_{j=0}^{n} \mathcal{N}_{n-j}(BO_j)$. Consider the projective space bundle $p: RP(\nu^j) \rightarrow M^{n-j}$ associated to ν. If $\lambda \rightarrow RP(\nu^j)$ is the canonical line bundle, we have that $j_n([M^{n-j},\nu^j]) = [RP(\nu^j),\lambda]$ in $\mathcal{N}_{n-1}(BO_1)$.

Now, let c be the characteristic class of the canonical line bundle $\lambda \to RP(\nu^j)$. The Stiefel-Whitney class of $RP(\nu^j)$ is given by

$$w(RP(\nu^j)) = p^*(w(M^{n-j}))(\sum_{i=0}^{j}(1+c)^{j-i}p^*(v_i)), \qquad (2.2)$$

where v_i is the ith Whitney class of the bundle ν^j. Moreover, we have the relation

$$c^j = c^{j-1}v_1+c^{j-2}v_2+\cdots+v_j. \qquad (2.3)$$

Finally, recall that $\mathcal{N}_*(BO_k)$ is a free $\mathcal{N}_*$-module with basis given by the classes $y_{i_1}\cdots y_{i_k}$, where $y_i = [RP^i,\xi_i] \in \mathcal{N}_i(BO_1)$, being ξ_i the line bundle over RP^i, and $y_{i_1}\cdots y_{i_k}$ is the class of the k-dimensional bundle $p_1^*(\xi_{i_1})\oplus\cdots\oplus p_k^*(\xi_{i_k})$ over $RP^{i_1}\times\cdots\times RP^{i_k}$ in $\mathcal{N}_{i_1+\cdots+i_k}(BO_k)$, for each sequence of integers $(i_1,\cdots,i_k)$ with $i_1 \le i_2 \le\cdots\le i_k$, where p_ℓ^* is the projection, i.e., $p_\ell: RP^{i_1}\times\cdots\times RP^{i_k} \to RP^{i_\ell}$.

Next, considering the exact sequence (2.1) for $n = 4$, we have

<u>Lemma 2.4</u>. The homomorphism j_4 restricted to $\mathcal{N}_1(BO_3)\oplus\mathcal{N}_0(BO_4)$ is monic and onto.

<u>Proof</u>. Consider the bundle $[M^1,\nu^3]$ in $\mathcal{N}_1(BO_3)$. The total Stiefel-Whitney class of $RP(\nu^3)$ is given by

$$w(RP(\nu^3)) = p^*(w(M^1))\left(\sum_{i=0}^{3}(1+c)^{3-i}p^*(v_i)\right),$$

where v_i is the ith Whitney class of ν^3. Then, it is easy to see that the first Stiefel-Whitney class of $RP(\nu^3)$ is $w_1(RP(\nu^3)) = p^*(w_1(M^1))+c+p^*(v_1)$. Therefore, $w_1^2(RP(\nu^3))c = c^3 = c^2p^*(v_1)$. Then, if $\nu^3 = [RP^1,\xi]$ is the generator of $\mathcal{N}_1(BO_3)$ we have $w_1^2(RP(\nu^3))c = c^2p^*(\alpha)$, where α is the generator of $H^1(RP^1;Z_2)$. So, the characteristic number $\langle w_1^2(RP(\nu^3))c,[RP(\nu^3)]\rangle$ is 1 for the generator.

On the other hand, if $[M^0,\nu^4]$ is an element in $\mathcal{N}_0(BO_4)$, we have

$$w(RP(\nu^4)) = p^*(w(M^0))\left(\sum_{i=0}^{4}(1+c)^{4-i}p^*(v_i)\right).$$

Then, $w_1(RP(\nu^4)) = 0$. Therefore, $\langle w_1^2(RP(\nu^4))c,[RP(\nu^4)]\rangle = 0$. These facts imply that $j_4([M^1,\nu^3])$ and $j_4([M^1,\nu^4])$ are linearly independent. Also, observe that $\langle c^3,[RP(\nu^4)]\rangle = 1$ for $\nu^4 = [pt.,4R]$. This implies that $j_4([pt.,4R] = [RP(4R),\lambda]$ doesn't bound in $\mathcal{N}_3(BO_1)$. Hence, j_4 is monic and onto, since $\mathcal{N}_1(BO_3) \oplus \mathcal{N}_0(BO_4)$ and $\mathcal{N}_3(BO_1)$ are 2-dimensional.

Theorem 2.5. $J_4^{3,4} = \{0\}$.

Proof. Since by Lemma (2.4), j_4 restricted to $\mathcal{N}_1(BO_3) \oplus \mathcal{N}_0(BO_4)$ is monic, we have that if the manifold M^4 has an involution T with fixed point set having codimensions 3 and 4, then $[M^4,T]$ bounds. Consequently, $[M^4]$ bounds.

3. $J_5^{3,4}$

Lemma 3.1.

(a) j_5 restricted to $\mathcal{N}_2(BO_3)$ is monic.

(b) j_5 restricted to $\mathcal{N}_1(BO_4)$ is monic.

Proof. (a) If the involution $[M^5,T]$ has fixed point set F of constant codimension 3, then by [3, §5], $[M^5,T]$ bounds as involution, since $\dim M^5 > 2 \dim F$. Therefore, j_5 restricted to $\mathcal{N}_2(BO_3)$ is monic.

(b) Follows similarly.

Lemma 3.2. For $[M^j,\nu^3] \in \mathcal{N}_j(BO_3)$, $j = 2,3$, the characteristic numbers $\langle w_1^4(RP(\nu^3))c^{j-2},[RP(\nu^3)]\rangle$ and $\langle c^{j+2},[RP(\nu^3)]\rangle$ are equal.

Proof. The total Stiefel-Whitney class of $RP(\nu^3)$ is given by

$$w(RP(\nu^3)) = p^*(w(M^j))\left(\sum_{i=0}^{3}(1+c)^{3-i}p^*(v_i)\right),$$

where v_i is the ith Whitney class of ν^3. So,

$$w_1(RP(\nu^3)) = p^*(w_1(M^j))+c+p^*(v_1).$$

Then

$$w_1^4(RP(\nu^3))c^{j-2} = c^4c^{j-2} = c^{j+2},$$

since $\dim M^j < 4$.

Theorem 3.3. $J_5^{3,4} = (0)$.

Proof. Observe that $\dim(\mathcal{N}_2(BO_3) \oplus \mathcal{N}_1(BO_4)) = 4$ and $\dim \mathcal{N}_4(BO_1) = 4$. Now, we are going to see that j_5 restricted to $\mathcal{N}_2(BO_3) \oplus \mathcal{N}_1(BO_4)$ is onto. By Lemma (3.1), we have that j_5 restricted to $\mathcal{N}_2(BO_3)$ is monic. Therefore, it is enough to find an element ω in $\mathcal{N}_1(BO_4)$ such that $j_5(\omega)$ isn't in the image of $j_5|_{\mathcal{N}_2(BO_3)}$. Thus, consider $\omega = [RP^1, \xi \oplus 3\mathbb{R}]$. The Stiefel-Whitney class of $RP(\xi \oplus 3\mathbb{R})$ is

$$w(RP(\xi \oplus 3\mathbb{R})) = p^*(w(RP^1))\left(\sum_{i=0}^{4} (1+c)^{4-i} p^*(v_i)\right).$$

So, $w_1(RP(\xi \oplus 3\mathbb{R})) = p^*(\alpha)$, where α is the generator of $H^1(RP^1;Z_2)$. Then, the characteristic class $w_1^4(RP(\xi \oplus 3\mathbb{R}))$ is zero. Moreover, $c^4 = c^3 p^* \alpha \neq 0$. Therefore, by Lemma (3.2) we conclude that j_5 restricted to $\mathcal{N}_2(BO_3) \oplus \mathcal{N}_0(BO_4)$ is onto. Thus, by the exact sequence (2.1), if $[M^5,T]$ has fixed point set with codimensions 3 and 4, then $[M^5,T]$ bounds as involution. Therefore, M^5 bounds and we have $J_5^{3,4} = (0)$.

4. $J_6^{3,4}$

In this section we are going to prove that $J_6^{3,4}$ is the subgroup of $\mathcal{N}_6$ generated by the classes x_2^3 and x_6, where $x_2 = [RP^2]$ and $x_6 = [RP^6]$.

Lemma 4.1. j_6 restricted to $\mathcal{N}_3(BO_3)$ is monic.

Proof. If $[M^6,T]$ has fixed point set F^3 3-dimensional, then we have that $[M^6,T] = [F^3 \times F^3, \text{twist}]$ by [3; §6]. Since $\mathcal{N}_3 = (0)$, there exists a 4-dimensional manifold V^4 such that $\partial V^4 = F^3$. The tangent bundle of V^4 restricted to ∂V^4 is the Whitney sum of the tangent bundle of ∂V^4 and the trivial 1-bundle, i.e., $\tau(V^4)|_{\partial V^4} = \tau(\partial V^4) \oplus 1 = \tau(F^3) \oplus 1$. Moreover, we have the commutative diagram

$$\begin{array}{ccc} \tau(\partial V^4)\oplus 1 & \longrightarrow & \tau(V^4) \\ \downarrow & & \downarrow \\ \partial V^4 & \xrightarrow{i} & V^4 \end{array}$$

where i is the inclusion map. Therefore, we can see that $\tau(\partial V^4) \oplus 1$ bounds in $\mathcal{N}_3(BO_4)$. Since $\mathcal{N}_3(BO_3) \cong \mathcal{N}_3(BO_4)$ by the stability theorem [2, 26.3], then $\tau(\partial V^4)$ bounds in $\mathcal{N}_3(BO_3)$. This implies that the normal bundle $\nu = \tau(F^3)$ to the fixed point set of $[F^3 \times F^3, \text{twist}]$ bounds, and consequently $[M^6,T] = [F^3 \times F^3, \text{twist}]$ bounds as involution. Hence, it follows at once that j_6 restricted to $\mathcal{N}_3(BO_3)$ is monic.

Lemma 4.2. The homomorphism j_6 restricted to

$$\mathcal{N}_3(BO_3) \oplus \mathcal{N}_2(BO_4) \text{ is onto.}$$

Proof. Since $\mathcal{N}_3(BO_3)$ is 4-dimensional and $\mathcal{N}_5(BO_1)$ is 5-dimensional, then by Lemma (4.1) it is sufficient to find an element ω in $\mathcal{N}_2(BO_4)$ such that $j_6\omega$ doesn't belong to the image of j_6 restricted to $\mathcal{N}_3(BO_3)$.

Consider $\omega = [RP^2, \xi \oplus 3\mathbb{R}]$ in $\mathcal{N}_2(BO_4)$, where ξ is the line bundle over RP^2. The total Stiefel-Whitney class of $RP(\xi \oplus 3\mathbb{R})$ is

$$w(RP(\xi \oplus 3\mathbb{R})) = p^*(w(RP^2))\left(\sum_{i=0}^{4} (1+c)^{4-i} p^*(v_i)\right),$$

where v_i is the ith Whitney class of $\xi \oplus 3\mathbb{R}$. Since $v_1 = \alpha$, with α the generator of $H^1(RP^2;Z_2)$ and $v_i = 0$ for $i > 1$, we have that $w_1(RP(\xi \oplus 3\mathbb{R})) = p^*\alpha + p^*\alpha = 0$. So, the characteristic class $w_1^4(RP(\xi \oplus 3\mathbb{R}))c$ is zero.

On the other hand, $c^5 = c^4 p^* \alpha \neq 0$ by (2.3). Therefore, by Lemma (3.2) the result follows at once.

<u>Theorem 4.3</u>. $J_6^{3,4}$ is the subgroup of $\mathcal{N}_6$ generated by the classes x_2^3 and x_6, where $x_2 = [RP^2]$ and $x_6 = [RP^6]$.

<u>Proof</u>. First observe that $\mathcal{N}_3(BO_3) \oplus \mathcal{N}_2(BO_4)$ is 7-dimensional and $\mathcal{N}_5(BO_1)$ is 5-dimensional. So, by (4.2) the kernel of j_6 restricted to $\mathcal{N}_3(BO_3) \oplus \mathcal{N}_2(BO_4)$ is 2-dimensional. Therefore, $\dim J_6^{3,4} \leq 2$.

Now, take the involutions $[RP^2 \times RP^2, \text{twist}]$ and $[RP^2, T]$, where T is given by $T: [x_0, x_1, x_2] \mapsto [-x_0, x_1, x_2]$. Thus, the involution $[RP^2 \times RP^2 \times RP^2, \text{twist} \times T]$ has fixed point set with codimensions 3 and 4. Therefore, $x_2^3 \in J_6^{3,4}$.

Next, considering RP^6 with the involution

$$T: [x_0, x_1, x_2, x_3, x_4, x_5, x_6] \mapsto [x_0, x_1, x_2, x_3, -x_4, -x_5, -x_6],$$

we see that $x_6 \in J_6^{3,4}$. Consequently, $\dim J_6^{3,4} = 2$, and the set $\{x_2^3, x_6\}$ is a basis for $J_6^{3,4}$.

5. $J_n^{3,4}$, for $n \geq 7$

Lemma 5.1. For $n = 2^r$, $n > 6$, $J_n^{3,4}$ contains an indecomposable class.

Proof. Consider the total space $RP(\xi \oplus 6\mathbb{R})$ of the associated projective space bundle of the Whitney sum $\xi \oplus 6\mathbb{R} \to RP^{2^r-6}$, where ξ is the line bundle over the projective space RP^{2^r-6}.

The total Stiefel-Whitney class of $RP(\xi \oplus 6\mathbb{R})$ is

$$w(RP(\xi \oplus 6\mathbb{R})) = (1+p^*(\alpha))^{2^r-5}(1+c)^6(1+c+p^*(\alpha)),$$

where α is the generator of $H^1(RP^{2^r-6}; Z_2)$. Then,

$$s_{2^r}[RP(\xi \oplus 6\mathbb{R})] = (c+p^*(\alpha))^{2^r}[RP(\xi \oplus 6\mathbb{R})]$$

$$= \left(\sum_{i=6}^{2^r} \binom{2^r}{i} c^i (p^*(\alpha))^{2^r-i}\right)[RP(\xi \oplus 6\mathbb{R})]$$

$$= \left(\sum_{i=6}^{2^r} \binom{2^r}{i}\right) c^6 (p^*(\alpha))^{2^r-6}[RP(\xi \oplus 6\mathbb{R})],$$

since $c^7 = c^6 p^*(\alpha)$. Therefore, $s_{2^r}[RP(\xi \oplus 6\mathbb{R})] = 2^{2^r} - \sum_{i=0}^{6} \binom{2^r}{i} \equiv 1$ mod 2, since $r \geq 3$. This implies that the class $[RP(\xi \oplus 6\mathbb{R})]$ is indecomposable.

Finally, to see that $[RP(\xi\oplus 6\mathbb{R})]$ belongs to $J_6^{3,4}$, consider on $RP(\xi\oplus 6\mathbb{R})$ the involution induced by multiplication by -1 in the fibers of $\xi\oplus 3\mathbb{R}$ and by 1 in the fibers of $3\mathbb{R}$. The fixed point set of this involution is $RP(\xi\oplus 3\mathbb{R}) \cup RP(3\mathbb{R})$, where $RP(\xi\oplus 3\mathbb{R})$ is (2^r-3)-dimensional and $RP(3\mathbb{R})$ is (2^r-4)-dimensional. Hence, the lemma follows.

<u>Theorem 5.2</u>. $J_n^{3,4} = \mathscr{N}_n$, for $n \geq 7$.

<u>Proof</u>. We may choose a set of generators for $\mathscr{N}_*$ in the following way: If $n \geq 9$, n not of the form 2^r or 2^r-1, consider x_n as in Capobianco [1; 3.1]. We have that $x_n \in J_*^{3,4}$, since $J_*^4 \subset J_*^{3,4}$ by [5; 2.2] and $x_n \in J_*^4$. For $n = 2^r$, consider x_n as in Lemma (5.1), and let x_6 be the class of RP^6. So, for $n \geq 6$, n not of the form 2^r-1, the generators x_n all belong to $J_n^{3,4}$.

Moreover, take as generators $x_2 = [RP^2]$, $x_4 = [RP^4]$ and x_5 the class of the total space bundle of the Whitney sum $\xi\oplus 3\mathbb{R}$ over RP^2, with ξ the line bundle over RP^2, i.e., $x_5 = [RP(2,0,0,0)]$.

Now, considering the involution $[RP^4\times RP^4, \text{twist}]$, we conclude that $x_4^2 \in J_*^{3,4}$, since $J_*^4 \subset J_*^{3,4}$.

Next, observe that $J_*^{1,2}\cdot J_*^2 \subset J_*^{3,4}$. Moreover, we have $J_*^2 = \{[M^n]/w_1^n = 0\}$ and $J_n^{1,2} = \mathscr{N}_n$, for $n \geq 2$, by [4; 9.2] and [5; 7.1], respectively. Therefore, x_4x_5, x_2x_5, $x_2^2x_5$ and x_2^3 belong to $J_*^{3,4}$, since x_4, x_2 belong to $J_*^{1,2}$ and $x_5, x_2^2 \in J_*^2$. Hence, for $n \geq 7$ any product of generators $x_{i_1}\cdots x_{i_k}$ with

$i_1+\cdots+i_k$ belongs to $J_n^{3,4}$. This implies that $J_n^{3,4} = \mathcal{N}_n$, for $n \geq 7$.

6. $J_n^{3,4,k_1,\cdots,k_\ell}$, for $4 < k_1 < \cdots < k_\ell \leq n$, with $\ell > 0$

First we are going to summarize the results of the previous sections in the next theorem.

Theorem 6.1.

(a) $J_n^{3,4} = (0)$, if $n = 4, 5$.

(b) $J_6^{3,4}$ is the subgroup of $\mathcal{N}_6$ generated by the classes x_2^3 and x_6.

(c) $J_n^{3,4} = \mathcal{N}_n$, for $n \geq 7$.

Now we can determine the subgroups $J_n^{3,4,k_1,\cdots,k_\ell}$ where $4 < k_1 < \cdots < k_\ell \leq n$ and $\ell > 0$. To do this, we need the following two lemmas.

Lemma 6.2. The class x_5 belongs to $J_5^{3,4,5}$.

Proof. There is the exact sequence

$$0 \to \mathcal{N}_5^{Z_2}(\{\{1\},Z_2\}) \xrightarrow{i_*} \bigoplus_{k=0}^{5} \mathcal{N}_{5-k}(BO_k) \xrightarrow{j_5} \mathcal{N}_5(BO_1) \to 0.$$

Now, consider $y_1 = [RP^2, \xi_2 \oplus 2\mathbb{R}]$ and $y_2 = [pt., 5\mathbb{R}]$, where ξ_2 is the line bundle over RP^2. So, $y_1 \in \mathcal{N}_2(BO_3)$ and $y_2 \in \mathcal{N}_0(BO_5)$.

Next, we are going to see that y_1+y_2 is in the kernel of j_5. To get this, it is enough to prove that the corresponding characteristic numbers of $j_5(y_1)$ and $j_5(y_2)$ are equal. Since $\mathcal{N}_4(BO_1) \cong \sum_{i=0}^{4} H^{4-i}(BO_1;Z_2) \otimes \mathcal{N}_i$ and $\mathcal{N}_4(BO_1)$ is 4-dimensional, one can see that the following characteristic numbers c^4, ω_2c^2, ω_4 and ω_2^2 characterize a class in $\mathcal{N}_4(BO_1)$.

Next, note that $j_5(y_1) = [RP(\xi_2 \oplus 2\mathbb{R}), \lambda_1]$ and $j_5(y_2) = [RP(5\mathbb{R}), \lambda_2]$, where λ_1 and λ_2 are the line bundles over $RP(\xi_2 \oplus 2\mathbb{R})$ and $RP(5\mathbb{R})$, respectively. So, using (2.2) and (2.3), one can obtain the following table.

	c^4	ω_2c^2	ω_4	ω_2^2
$[RP(\xi_2 \oplus 2\mathbb{R}), \lambda_1]$	1	0	1	0
$[RP(5\mathbb{R}), \lambda_2]$	1	0	1	0

This implies that $j_5(y_1+y_2) = 0$.

It then follows that there is a manifold with involution $[M^5, T]$ such that $i_*[M^5, T] = y_1+y_2$. Moreover, we have that $[M^5] = [RP(\xi_2 \oplus 3\mathbb{R})]+[RP(6\mathbb{R})]$ by [2; 24.2]. Therefore, $[M^5] = [RP(\xi_2 \oplus 3\mathbb{R})]$. Now, recall that a class α in $\mathcal{N}_5$ is characterized by the Stiefel-Whitney number $\omega_2\omega_3(\alpha)$. Using (2.1), it is easy to see that $\omega_2\omega_3[RP(\xi_2 \oplus 3\mathbb{R})] = 1$. So, $[M^5] = [RP(2,0,0,0)]$. Consequently, x_5 belongs to $J_5^{3,4,5}$.

Lemma 6.3. The class x_2x_4 belongs to $J_6^{3,4,6}$.

Proof. Consider $y_1 = [RP^2][RP^1, \xi_1 \oplus 2\mathbb{R}]$,
$y_2 = [RP^1 \times RP^1 \times RP^1, \xi_1 \oplus \xi_1 \oplus \xi_1]$, $y_3 = [RP^1 \times RP^2, \xi_1 \oplus \xi_2 \oplus \mathbb{R}]$,
$y_4 = [RP^2, \xi_2 \oplus 3\mathbb{R}]$, $y_5 = [RP^2][pt., 4\mathbb{R}]$, and $y_6 = [pt., 6\mathbb{R}]$.
Observe that $y = \sum_{i=1}^{6} y_i$ belongs to $\mathcal{N}_3(BO_3) \oplus \mathcal{N}_2(BO_4) \oplus \mathcal{N}_0(BO_6)$.

Next, we are going to prove that y is in the kernel of j_6. Since $\mathcal{N}_5(BO_1) \cong \sum_{i=0}^{5} H^{5-i}(BO_1;Z_2) \otimes \mathcal{N}_i$ and $\mathcal{N}_5(BO_1)$ is 5-dimensional, then a class in $\mathcal{N}_5(BO_1)$ is characterized by the characteristic numbers c^5, ω_2c^3, ω_4c, ω_2^2c, and $\omega_2\omega_3$. So, by (2.2) and (2.3), we can get the following characteristics numbers for each one of the classes $j_6(y_i)$, $i = 1, \cdots, 6$.

	c^5	ω_2c^3	ω_4c	ω_2^2c	$\omega_2\omega_3$
$j_6(y_1)$	0	1	1	1	0
$j_6(y_2)$	1	0	0	1	0
$j_6(y_3)$	1	0	1	0	1
$j_6(y_4)$	1	1	1	1	1
$j_6(y_5)$	0	1	0	0	0
$j_6(y_6)$	1	1	1	1	0

Analyzing these numbers, one can see that $j_6(y)$ bounds in $\mathcal{N}_5(BO_1)$. Therefore, there is a manifold with involution $[M^6, T]$ such that $i_*[M^6, T] = y$ and

$$[M^6] = [RP^2][RP(\xi_1 \oplus 3\mathbb{R})]+[RP(\xi_1 \oplus \xi_1 \oplus \xi_1 \oplus \mathbb{R})]+[RP(\xi_1 \oplus \xi_2 \oplus 2\mathbb{R})]$$

$$+ [RP([\xi_2 \oplus 4\mathbb{R})]+[RP^2][RP^4]+[RP^6],$$

by [2; 24.2].

Now, since a class α in $\mathcal{N}_6$ is characterized by the Stiefel-Whitney numbers $\omega_2^3(\alpha)$, $\omega_6(\alpha)$, and $\omega_1^6(\alpha)$; then calculating the classes above, we have

	ω_2^3	ω_6	ω_1^6
$[RP^2][RP(\xi_1 \oplus 3\mathbb{R})]$	0	0	0
$[RP(\xi_1 \oplus \xi_1 \oplus \xi_1 \oplus \mathbb{R})]$	0	0	0
$[RP(\xi_1 \oplus \xi_2 \oplus 2\mathbb{R})]$	1	0	0
$[RP(\xi_2 \oplus 4\mathbb{R})]$	0	1	1
$[RP^6]$	1	1	1

These calculations imply that $[M^6] = [RP^2][RP^4]$. Hence, x_2x_4 belongs to $J_6^{3,4,6}$.

<u>Theorem 6.4</u>. If $4 < k_1 < \cdots < k_\ell \leq n$ and $\ell > 0$, then

$$J_n^{3,4,k_1,\cdots,k_\ell} = \mathcal{N}_n.$$

<u>Proof</u>. Observe that $x_2x_4 \in J_6^{3,4,5}$, since $x_2 \in J_*^{1,2}$ and $x_4 \in J_*^{2,3}$ by [5; 7.1] and [5; 8.1], respectively. Moreover, we have that $J_n^{3,4} \subset J_n^{3,4,k_1,\cdots,k_\ell}$ (see [5; 2.2]). Therefore, by (6.1), (6.2), and (6.3), the result follows at once.

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