

**A METHOD TO SOLVE MATRICIAL  
EQUATIONS OF THE TYPE**

$$\sum_{i=1}^p A_i X B_i = C$$

*Vera Lúcia Rocha Lopes*  
and  
*José Vitório Zago*

**RELATÓRIO TÉCNICO Nº 22/90**

**Abstract:** A functional formulation of the Conjugate Gradient method is presented to find the unique solution of matricial equations of the type  $\sum_{i=1}^p A_i X B_i = C$  where  $A_i$  and  $B_i$  are symmetric matrices of dimension  $n \times n$  and  $m \times m$  respectively and  $C$  is a  $n \times m$  matrix. We give sufficient conditions on  $A_i$  and  $B_i$  for the application of the method and present results for a discretization of the heat equation.

Instituto de Matemática, Estatística e Ciência da Computação  
Universidade Estadual de Campinas  
13.081, Campinas, S.P.  
BRASIL

O conteúdo do presente Relatório Técnico é de única responsabilidade dos autores.

**Maió - 1990**

# A Method to Solve Matricial Equations of the Type $\sum_{i=1}^p A_i X B_i = C$

*Vera Lúcia Rocha Lopes*

*José Vitório Zago*

Departamento de Matemática Aplicada

IMECC-UNICAMP

Caixa Postal 6065

13.081 - Campinas, SP., Brasil

**Abstract:** A functional formulation of the Conjugate Gradient method is presented to find the unique solution of matricial equations of the type  $\sum_{i=1}^p A_i X B_i = C$  where  $A_i$  and  $B_i$  are symmetric matrices of dimension  $n \times n$  and  $m \times m$  respectively and  $C$  is a  $n \times m$  matrix. We give sufficient conditions on  $A_i$  and  $B_i$  for the application of the method and present results for a discretization of the heat equation.

**Keywords** Matricial equations. Conjugate Gradient Method.

## 1. INTRODUCTION

The Conjugate Gradient method has been successfully employed to solve linear systems  $Ax = b$  where  $A$  is symmetric positive definite.

In this paper we present a functional formulation of the method to be applied to matricial equations of the type

$$(1) \quad \sum_{i=1}^p A_i X B_i = C$$

when  $A_i$  is a  $n \times n$ ,  $B_i$  is a  $m \times m$  matrix for every  $i$  and  $C$  is a  $n \times m$  matrix.

The method can be applied when solving equation (1) is equivalent to minimizing a quadratic functional  $V : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ .

In Section 2 we present a functional version of the Conjugate Gradient method.

In Section 3 we define the functional  $V$ , show the equivalence between solving equation (1) and minimizing  $V$  and we set sufficient conditions on  $A_i$  and  $B_i$  under which the method applies.

In Section 4 we give examples of matricial equations of type (1) which can be solved by our method. We show numerical results obtained from a particular discretization of the heat equation.

## 2. A FUNCTIONAL FORM OF THE CONJUGATE GRADIENT METHOD

Let  $Z$  be a Hilbert Space with inner product  $\langle \cdot, \cdot \rangle$  and  $V : Z \rightarrow \mathbb{R}$  a functional on it

$$V(x) = \frac{1}{2}Q(x, x) - P(x) + k$$

where  $Q : Z \times Z \rightarrow \mathbb{R}$  is a positive-definite, symmetric continuous bi-linear form and

$$P : Z \rightarrow \mathbb{R} \text{ a continuous linear form.}$$

If we know how to compute  $\text{grad } V(x)$  and  $\text{grad } P(x)$  the Conjugate Gradient algorithm to compute

Min  $V(x)$  is the following :

### Algorithm 2.1.

Let an arbitrary  $x_0 \in Z$  and a given  $\varepsilon > 0$

Let  $d_0 = -g_0 = -\text{grad } V(x_0)$ . If  $\|d_0\| \leq \varepsilon$  then  $x_0$  is an approximate solution.

Otherwise for  $k = 0, 1, \dots$

$$x_{k+1} = x_k + \alpha_k d_k$$

where

$$\alpha_k = \frac{\langle g_k, d_k \rangle}{\langle d_k, G_k \rangle} \text{ with } G_k = g_k + \text{grad } P(x_k)$$

Compute  $g_{k+1} = \text{grad } V(x_{k+1})$ . If  $\|g_{k+1}\| \leq \varepsilon$  then  $x_{k+1}$  is an approximate solution. Otherwise  $d_{k+1} = -g_{k+1} + \beta_k d_k$  where

$$\beta_k = \frac{\langle g_{k+1}, G_k \rangle}{\langle d_k, G_k \rangle}$$

### 3. THE FUNCTIONAL $V$ AND THE CONJUGATE GRADIENT METHOD

Let  $Z = \mathbb{R}^{n \times m}$  and  $A_i \in \mathbb{R}^{n \times n}$ ,  $B_i \in \mathbb{R}^{n \times m}$   $i = 1, 2, \dots, p$

$$X \text{ and } C \in Z$$

We define  $V : Z \rightarrow \mathbb{R}$  as

$$(2) \quad V(X) = 0.5 \sum_{i=1}^p \text{tr}(A_i X B_i X^t) - \text{tr}(C^t X)$$

Initially we are going to verify that to solve matricial equation (1) is equivalent to find a critical point of  $V$ . Next we will see what are the conditions for  $V(X) = \frac{1}{2}Q(X, X) - P(X) + k$  where  $Q$  is a positive definite symmetric bilinear form, which implies that this critical point is a minimal point of  $V$ .

To find a critical point of  $V$  we compute the derivative of  $V$  in the direction  $K$

$$\text{Der}[V(X)]K = 0.5 \sum_{i=1}^p \text{tr}(A_i K B_i X^t) + 0.5 \text{tr}(A_i X B_i K^t) - \text{tr}(C^t K)$$

Since  $\text{tr}(MN) = \text{tr}(NM)$  as long as  $MN$  and  $NM$  are well defined we have

$$\begin{aligned} \text{Der}[V(X)]K &= 0.5 \sum_{i=1}^p \text{tr}(K B_i X^t A_i) + 0.5 \sum_{i=1}^p \text{tr}(A_i X B_i K^t) - \text{tr}(K C^t) = \\ &= 0.5 \sum_{i=1}^p \text{tr}(A_i^t X B_i^t K^t) + 0.5 \sum_{i=1}^p \text{tr}(A_i X B_i K^t) - \text{tr}(C K^t) \end{aligned}$$

If we suppose that  $A_i$  and  $B_i$  are symmetric for every  $i$  we have

$$\text{Der}[V(X)]K = \left[ \sum_{i=1}^p \text{tr}(A_i X B_i) - \text{tr}(C) \right] K^t = \text{tr} \left[ \left( \sum_{i=1}^p (A_i X B_i - C) \right) K^t \right].$$

for every  $K$  in  $Z$ .

Defining  $\langle P, Q \rangle_V = \text{tr}(P Q^t)$

$$\text{Der}[V(X)]K = \left\langle \sum_{i=1}^p A_i X B_i - C, K \right\rangle_V$$

Then  $X$  is a critical point of  $V$  if and only if

$$\text{Der}[V(X)]K = 0 \quad \forall K \in Z, \text{ that is iff } \sum_{i=1}^p A_i X B_i = C.$$

Furthermore by Riez's theorem

$$(3) \quad \text{grad } V(X) = \sum_{i=1}^p A_i X B_i - C.$$

Defining  $Q : Z \times Z \rightarrow \mathbb{R}$  by

$$(4) \quad Q(X, Y) = \text{tr}\left(\sum_{i=1}^p A_i X B_i Y^t\right)$$

and

$$P : Z \rightarrow \mathbb{R}$$

by

$$P(X) = \text{tr}(C^t X), \quad C \in Z$$

we have

$$V(X) = \frac{1}{2} Q(X, X) - P(X).$$

If  $Q$  is positive definite, symmetric, bilinear then the critical point of  $V$  will be a minimal point and the conjugate gradient method (Algorithm 2.1) can be applied.

It is not difficult to prove that  $Q(X, Y)$  is bilinear and symmetric.

The theorem below set the conditions on  $A_i$  and  $B_i$  for the symmetric and bilinear form  $Q$  to be positive-definite.

**Theorem:** Let  $X_k$ ,  $1 \leq k \leq nm$  be an element of the canonical basis of  $\mathbb{R}^{n \cdot m}$ , that is  $X_k$  is a  $n \times m$  matrix with all elements zero except for  $x_{r,s} = 1$ , where

$$r = \begin{cases} \text{quotient of the division of } k \text{ by } m \text{ if the rest is zero} \\ \text{or} \\ (\text{quotient of the division of } k \text{ by } m) + 1 \text{ if the rest is not zero} \end{cases}$$

$$s = k - (r - 1)m.$$

Then the symmetric, bilinear form  $Q$  defined by (4) is positive definite if and only if

$$D = d_{\ell j} = Q(X_\ell, X_j) \quad n \cdot m \times n \cdot m$$

is positive definite

**Proof:**  $d_{\ell j} = Q(X_{\ell}, Y_j) = \sum_{i=1}^p \text{tr}(A_i X_{\ell} B_i Y_j^t)$

Then for every  $X \in Z$ ,  $X = \sum_{s=1}^{n \times m} v_s X_s$  and  $v = (v_s)_{n \times m \times 1}$  is a vector of  $\mathbb{R}^{n \times m}$ .

$$Q(X, X) = Q\left(\sum_{s=1}^{n \times m} v_s X_s, \sum_{s=1}^{n \times m} v_s X_s\right) = \sum_{i,j=1}^{n \times m} v_i v_j Q(X_i, X_j) = v^t D v.$$

To prove that  $D$  is positive definite we can use Sylvester's theorem stated below.

**Theorem:** A matrix  $A = a_{ij}$ ,  $n \times n$  is positive definite if and only if  $\Delta_{\ell} > 0$ ,  $\ell = 1, 2, \dots, n$  where  $\Delta_{\ell}$  is the principal minor of order  $\ell$  of matrix  $A$ .

If the dimensions  $n$  and  $m$  are big this verification is cumbersome even if  $p$  is small.

In next Section we will present classes of matrices where it is easy to see that  $Q(X, Y)$  is positive definite and a practical example where we apply our method.

#### 4. EXAMPLES

##### 4.1. THE EQUATION $C^t C X + C X D^t + C^t X D + X D D^t = E$

If  $p = 4\lambda$  and  $\sum_{i=1}^p A_i X B_i$  can be grouped as  $\sum_{j=1}^{\lambda} P_j$  where

$$P_j = A_{1j} X B_{1j} + A_{2j} X B_{2j} + A_{3j} X B_{3j} + A_{4j} X B_{4j}$$

with

$$\begin{aligned} A_{1j} &= C_j^t C_j & B_{1j} &= I \\ A_{2j} &= C_j & B_{2j} &= D_j^t \\ A_{3j} &= C_j^t & B_{3j} &= D_j \\ A_{4j} &= I & B_{4j} &= D_j D_j^t \end{aligned}$$

then

$$Q(X, X) = \sum_{j=1}^{\lambda} \text{tr}(S_j)$$

where

$$\begin{aligned} S_j &= A_{1j} X B_{1j} X^t + A_{2j} X B_{2j} X^t + A_{3j} X B_{3j} X^t + A_{4j} X B_{4j} X^t \\ &= (C_j X + X D_j)(C_j X + X D_j)^t = F_j F_j^t \end{aligned}$$

and  $Q(X, X) \geq 0 \forall X$ .

An example of equations of this type for which  $Q(X, X) = 0$  if and only if  $X = 0$  are equations ( $\lambda = 1, C_1 = C, D_1 = D$ ) with  $C$  and  $D$   $n \times n$  such that  $\det A \neq 0$  where  $A$  is the  $n^2 \times n^2$  matrix

$$A = \begin{bmatrix} M_1 & P_{12} & P_{13} & \cdots & P_{1n} \\ P_{21} & M_2 & & & \\ \vdots & & \ddots & & \\ P_{n1} & & & & M_n \end{bmatrix}$$

with  $M_i = c_{ii}I + D^t$  and  $P_{ij} = c_{ij}I$ .

#### 4.2. THE EQUATION $\sum_{i=1}^p A_i X A_i = C$

$$p \geq n^2, \quad n = \dim A_i \text{ for every } i$$

For equations of this type we see that

$$Q(X, X) = \sum_{i=1}^p \text{tr}(A_i X A_i X^t) = \sum_{i=1}^p [\text{tr}(A_i X)]^2$$

Then  $Q(X, X)$  is positive-definite if  $\text{rank}(M) = n^2$  where  $M$  is the matrix  $p \times n^2$  with the  $n^2$  elements of line  $j$  being the elements of  $A_j$ , that is, line  $j$  of  $M$  is  $(a_{11}^j a_{12}^j \cdots a_{1n}^j a_{12}^j a_{22}^j \cdots a_{2n}^j a_{13}^j a_{23}^j a_{33}^j \cdots a_{3n}^j \cdots a_{nn}^j)$ .

#### 4.3. THE EQUATION $BXC + DXE = G$

Equations of this type arise from the discretization by Finite Elements of the variational formulation of

$$(5) \quad \begin{cases} \Delta u = f & \text{in } \Omega = [a, b] \times [c, d] \subset \mathbb{R}^2 \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

In the variational formulation solving (5) is equivalent to solving

$$(6) \quad \text{Min } g(u) = \text{Min} \int_{\Omega} 0.5 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 - fu \right] dx dy$$

Taking

$$\Delta x = \frac{b-a}{n+1}, \quad \Delta y = \frac{d-c}{m+1}, \quad \varphi_i(x), \quad \psi_j(y)$$

as finite elements basis such that

$$\begin{aligned} \varphi_i(a) = \varphi_i(b) = 0, \quad 1 \leq i \leq n \\ \psi_j(c) = \psi_j(d) = 0, \quad 1 \leq j \leq m \end{aligned}$$

then an approximate solution is

$$(7) \quad u(x, y) \approx \sum_{i=1}^n \sum_{j=1}^m a_{ij} \varphi_i(x) \psi_j(y).$$

Taking (7) into (6) we have

$$g(u) \approx V(X) = 0.5 \operatorname{tr}(BXCX^t) + 0.5 \operatorname{tr}(DXEX^t) - \operatorname{tr}(G^t X) \quad \text{with}$$

$X$  being a  $n \times m$  matrix with elements  $a_{ij}$

$$B \text{ a } n \times n \text{ matrix } b_{ij} = \int_a^b \varphi_i'(x) \varphi_j'(x) dx = b_{ji}$$

$$C \text{ a } m \times m \text{ matrix } c_{ij} = \int_a^d \varphi_i(y) \psi_j(y) dy = c_{ji}$$

$$D \text{ a } n \times m \text{ matrix } d_{ij} = \int_a^b \varphi_i(x) \varphi_j(x) dx = d_{ji}$$

$$E \text{ a } m \times m \text{ matrix } \ell_{ij} = \int_c^d \psi_i'(y) \psi_j'(y) dy = \ell_{ji}$$

$$G \text{ a } n \times m \text{ matrix } g_{ij} = \int_a^b \int_c^d \varphi_i(x) \psi_j(y) f(x, y) dx dy$$

It is not difficult to see that

$$Q(X, X) = 0.5 \operatorname{tr}(BXCX^t) + 0.5 \operatorname{tr}(DXEX^t) = 0 \quad \text{if and only if } X = 0.$$

We can extend the method to equations

$$(8) \quad \begin{cases} \sum_{r=1}^2 \sum_{s=1}^2 \frac{\partial}{\partial x_r} \left( \alpha_{rs} \frac{\partial u}{\partial x_s} \right) = f \text{ in } \Omega \text{ with } \bar{\alpha} = (\alpha_{rs}) \text{ positive definite} \\ u = 0 \text{ in } \partial\Omega \end{cases}$$

whose variational formulation is

$$\begin{cases} \operatorname{Min} \int_{\Omega} \{ 0.5 \sum_{r=1}^2 \sum_{s=1}^2 \alpha_{rs}(x_1, x_2) \frac{\partial u}{\partial x_r} \frac{\partial u}{\partial x_s} - uf \} dx_1 dx_2 = \operatorname{Min} g(u) \\ u = 0 \text{ in } \partial\Omega. \end{cases}$$

We have

$$g(u) = 0.5 \sum_{i=1}^2 \sum_{j=1}^2 \operatorname{tr}(B_{ij}XCX^t) - \operatorname{tr}(G^t X)$$

with

$$B_{ij} = \int_a^b \int_c^d \alpha_{ij}(x_1, x_2) \varphi_1'(x_1) \varphi_j(x_1) dx_1 dx_2$$



a  $n \times n$  matrix and  $C, X, G$  as in the previous case.

Since  $\bar{\alpha}$  is positive definite,  $Q(X, X) = 0$  if and only if  $X = 0$ .

#### 4.4. AN APPROXIMATE SOLUTION OF THE HEAT EQUATION.

Lopes and Zago [5], present a numerical method for the approximate solution of heat equation

$$(9) \quad \begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ u(0, t) = u(1, t) = 0 \\ u(x, 0) = u_0(x) \quad 0 \leq t \leq T \quad j \leq x \leq 1 \end{cases}$$

Using the dual extremum principles of Noble and Sewell [6], we can transform (9) into a pair of restricted minimization and maximization problems.

Approximating these extremum principles by linear finite element basis both in time and space we got unrestricted maximum and minimum problems. The discretized problem consists of finding  $\max V(X)$  where

$$V : R^{(n-1) \times (m+1)} \rightarrow R \text{ is}$$

$$V(X) = 0.5 \text{tr}(CXR X^t S) + 0.5 \text{tr}(EXFX^t)$$

$$+ 0.5 \text{tr}(CXY_T X^t) + 0.5 \text{tr}(CXY_0 X^t) - \text{tr}(\bar{D}X\bar{1})$$

with

$$x_0 = 0, \quad x_1 = \Delta x, \dots, x_n = n\Delta x$$

$$t_0 = 0, \quad t_1 = \Delta t, \dots, t_m = m\Delta t = T$$

$X$  is the matrix of unknowns (coefficients of the linear combination of the basis functions).

$C, R, S, E, F$  are symmetric matrices whose elements are the integrals of products of basis functions and derivatives of basis functions.

$\bar{D}$  is a  $(n-1) \times (n-1)$  matrix with the initial condition of the problem

$$\bar{I} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & \dots & \dots & 0 \end{bmatrix}_{(m+1) \times (n-1)}$$

$$Y_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & & 0 \\ \vdots & \ddots & & \\ 0 & & & 0 \end{bmatrix}_{(m+1) \times (m+1)}$$

$$Y_T = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & & & 1 \end{bmatrix}_{(m+1) \times (m+1)}$$

They proved that the bilinear form associated to  $V$  is continuous, symmetric and positive definite and that

$$\text{grad } V(X) = SCXR + EXF + CX(Y_T + Y_0) - (\bar{I}\bar{D})^t$$

The functional formulation of the Conjugate Gradient Method here presented was used to solve the problem with good numerical results.

We solved problem (9) with

$$u_0(x) = \sin \pi x.$$

The exact solution is  $u(x, t) = \sin \pi x e^{-\pi^2 t}$ . In Table 4.1 we present our results.

i)  $\|\text{error}\|_\infty$  - is the maximum norm computed in the matrix  $ERT$  with

$$ERT_{ij} = u(x_i, t_j) - \tilde{u}(x_i, t_j)$$

and  $\tilde{u}(x, j)$  is our approximate solution.

ii)  $\Delta x = \frac{1}{n}$

iii)  $\Delta t = \frac{1}{m}$  ( $T = 1$ )

iv) Dim -  $X$  matrix dimension:  $(n-1) \times (m+1)$ .

v) Iter -number of iterations made by Conjugate Gradient until  $\|\text{grad } V(X)\|_\infty \leq 5 * 10^{-6}$ .

$\Delta x$	$\Delta t$	$\ error\ _{\infty}$	Dim	Iter
0.5	0.25	0.1965079	5	5
0.25	0.125	0.0625	27	11
0.125	0.0625	0.01812518	119	31
0.0625	0.03125	0.004675627	495	63
0.03125	0.015625	0.001188576	2015	111

Table 4.1 - Results obtained applying the proposed method to equation (9) with  $u_0(x) = \sin \pi x$

#### References

- [1] - Gantmacher, F.R. - Theory of Matrices, Chelsea, 1959.
- [2] - Golub, G. H. and Van Loan, C. V - Matrix Computations - The Johns Hopknis University Press, 1985.
- [3] - Hestenes, M. R and Stiefel, E. - Methods of Conjugate Gradients for Solving Linear Systems - J. Res. Nat. Bur Stand. 49, 409-36, 1952
- [4] - Hestenes, M. R. - Conjugate Direction Method in Optimization, Springer - Verlag, Berlin, 1980.
- [5] - Lopes, V. L. R. and Zago, J.V. - Dual Extremum Principles for the Heat Equation Solved by Finite Element Methods II - Technical Repport 13/89 IMECC-UNICAMP , 1989.
- [6] - Noble, B. and Sewell, M. J. - On Dual Extremum Principles in Applied Mathematics - J. Inst. Maths. and its Applics. vol. 9, number 2, April 1972.

1.1. Introduction 1

1.2. Normed Linear Spaces 1

1.3. Inner Product Spaces 1

1.4. Bounded Linear Operators 1

1.5. Adjoint Operators 1

1.6. Self-Adjoint Operators 1

1.7. Normal Operators 1

1.8. Isometries and Unitary Operators 1

1.9. The Spectral Theorem for Self-Adjoint Operators 1

1.10. The Spectral Theorem for Normal Operators 1

1.11. The Spectral Theorem for Compact Self-Adjoint Operators 1

1.12. The Spectral Theorem for Compact Normal Operators 1

1.13. The Spectral Theorem for Compact Operators 1

1.14. The Spectral Theorem for Compact Operators II 1

1.15. The Spectral Theorem for Compact Operators III 1

1.16. The Spectral Theorem for Compact Operators IV 1

1.17. The Spectral Theorem for Compact Operators V 1

1.18. The Spectral Theorem for Compact Operators VI 1

1.19. The Spectral Theorem for Compact Operators VII 1

1.20. The Spectral Theorem for Compact Operators VIII 1

1.21. The Spectral Theorem for Compact Operators IX 1

1.22. The Spectral Theorem for Compact Operators X 1

1.23. The Spectral Theorem for Compact Operators XI 1

1.24. The Spectral Theorem for Compact Operators XII 1

1.25. The Spectral Theorem for Compact Operators XIII 1

1.26. The Spectral Theorem for Compact Operators XIV 1

1.27. The Spectral Theorem for Compact Operators XV 1

1.28. The Spectral Theorem for Compact Operators XVI 1

1.29. The Spectral Theorem for Compact Operators XVII 1

1.30. The Spectral Theorem for Compact Operators XVIII 1

1.31. The Spectral Theorem for Compact Operators XIX 1

1.32. The Spectral Theorem for Compact Operators XX 1

1.33. The Spectral Theorem for Compact Operators XXI 1

1.34. The Spectral Theorem for Compact Operators XXII 1

1.35. The Spectral Theorem for Compact Operators XXIII 1

1.36. The Spectral Theorem for Compact Operators XXIV 1

1.37. The Spectral Theorem for Compact Operators XXV 1

1.38. The Spectral Theorem for Compact Operators XXVI 1

1.39. The Spectral Theorem for Compact Operators XXVII 1

1.40. The Spectral Theorem for Compact Operators XXVIII 1

1.41. The Spectral Theorem for Compact Operators XXIX 1

1.42. The Spectral Theorem for Compact Operators XXX 1

1.43. The Spectral Theorem for Compact Operators XXXI 1

1.44. The Spectral Theorem for Compact Operators XXXII 1

1.45. The Spectral Theorem for Compact Operators XXXIII 1

1.46. The Spectral Theorem for Compact Operators XXXIV 1

1.47. The Spectral Theorem for Compact Operators XXXV 1

1.48. The Spectral Theorem for Compact Operators XXXVI 1

1.49. The Spectral Theorem for Compact Operators XXXVII 1

1.50. The Spectral Theorem for Compact Operators XXXVIII 1

1.51. The Spectral Theorem for Compact Operators XXXIX 1

1.52. The Spectral Theorem for Compact Operators XL 1

1.53. The Spectral Theorem for Compact Operators XLI 1

1.54. The Spectral Theorem for Compact Operators XLII 1

1.55. The Spectral Theorem for Compact Operators XLIII 1

1.56. The Spectral Theorem for Compact Operators XLIV 1

1.57. The Spectral Theorem for Compact Operators XLV 1

1.58. The Spectral Theorem for Compact Operators XLVI 1

1.59. The Spectral Theorem for Compact Operators XLVII 1

1.60. The Spectral Theorem for Compact Operators XLVIII 1

1.61. The Spectral Theorem for Compact Operators XLIX 1

1.62. The Spectral Theorem for Compact Operators L 1

1.63. The Spectral Theorem for Compact Operators LI 1

1.64. The Spectral Theorem for Compact Operators LII 1

1.65. The Spectral Theorem for Compact Operators LIII 1

1.66. The Spectral Theorem for Compact Operators LIV 1

1.67. The Spectral Theorem for Compact Operators LV 1

1.68. The Spectral Theorem for Compact Operators LVI 1

1.69. The Spectral Theorem for Compact Operators LVII 1

1.70. The Spectral Theorem for Compact Operators LVIII 1

1.71. The Spectral Theorem for Compact Operators LIX 1

1.72. The Spectral Theorem for Compact Operators LX 1

1.73. The Spectral Theorem for Compact Operators LXI 1

1.74. The Spectral Theorem for Compact Operators LXII 1

1.75. The Spectral Theorem for Compact Operators LXIII 1

1.76. The Spectral Theorem for Compact Operators LXIV 1

1.77. The Spectral Theorem for Compact Operators LXV 1

1.78. The Spectral Theorem for Compact Operators LXVI 1

1.79. The Spectral Theorem for Compact Operators LXVII 1

1.80. The Spectral Theorem for Compact Operators LXVIII 1

1.81. The Spectral Theorem for Compact Operators LXIX 1

1.82. The Spectral Theorem for Compact Operators LXX 1

1.83. The Spectral Theorem for Compact Operators LXXI 1

1.84. The Spectral Theorem for Compact Operators LXXII 1

1.85. The Spectral Theorem for Compact Operators LXXIII 1

1.86. The Spectral Theorem for Compact Operators LXXIV 1

1.87. The Spectral Theorem for Compact Operators LXXV 1

1.88. The Spectral Theorem for Compact Operators LXXVI 1

1.89. The Spectral Theorem for Compact Operators LXXVII 1

1.90. The Spectral Theorem for Compact Operators LXXVIII 1

1.91. The Spectral Theorem for Compact Operators LXXIX 1

1.92. The Spectral Theorem for Compact Operators LXXX 1

1.93. The Spectral Theorem for Compact Operators LXXXI 1

1.94. The Spectral Theorem for Compact Operators LXXXII 1

1.95. The Spectral Theorem for Compact Operators LXXXIII 1

1.96. The Spectral Theorem for Compact Operators LXXXIV 1

1.97. The Spectral Theorem for Compact Operators LXXXV 1

1.98. The Spectral Theorem for Compact Operators LXXXVI 1

1.99. The Spectral Theorem for Compact Operators LXXXVII 1

2.00. The Spectral Theorem for Compact Operators LXXXVIII 1

## RELATÓRIOS TÉCNICOS — 1990

- 01/90 Harmonic Maps Into Periodic Flag Manifolds and Into Loop Groups — Caio J. C. Negreiros.
- 02/90 On Jacobi Expansions — E. Capelas de Oliveira.
- 03/90 On a Superlinear Sturm-Liouville Equation and a Related Bouncing Problem — D. G. Figueiredo and B. Ruf.
- 04/90  $F$ -Quotients and Envelope of  $F$ -Holomorphy — Luiza A. Moraes, Otília W. Paques and M. Carmelina F. Zaine.
- 05/90  $S$ -Rationally Convex Domains and The Approximation of Silva-Holomorphic Functions by  $S$ -Rational Functions — Otília W. Paques and M. Carmelina F. Zaine.
- 06/90 Linearization of Holomorphic Mappings On Locally Convex Spaces — Jorge Mujica and Leopoldo Nachbin.
- 07/90 On Kummer Expansions — E. Capelas de Oliveira.
- 08/90 On the Convergence of SOR and JOR Type Methods for Convex Linear Complementarity Problems — Alvaro R. De Pierro and Alfredo N. Iusem.
- 09/90 A Curvilinear Search Using Tridiagonal Secant Updates for Unconstrained Optimization — J. E. Dennis Jr., N. Echebest, M. T. Guardarucci, J. M. Martínez, H. D. Scolnik and C. Vacchino.
- 10/90 The Hypebolic Model of the Mean  $\times$  Standard Deviation "Plane" — Sueli I. R. Costa and Sandra A. Santos.
- 11/90 A Condition for Positivity of Curvature — A. Derdzinski and A. Rigas.
- 12/90 On Generating Functions — E. Capelas de Oliveira.
- 13/90 An Introduction to the Conceptual Difficulties in the Foundations of Quantum Mechanics a Personal View — V. Buonomano.
- 14/90 Quasi-Invariance of product measures Under Lie Group Perturbations: Fisher Information And  $L^2$ -Differentiability — Mauro S. de F. Marques and Luiz San Martin.
- 15/90 On Cyclic Quartic Extensions with Normal Basis — Miguel Ferrero, Antonio Paques and Andrzej Solecki.
- 16/90 Semilinear Elliptic Equations with the Primitive of the Nonlinearity Away from the Spectrum — Djairo G. de Figueiredo and Olimpio H. Miyagaki.
- 17/90 On a Conjugate Orbit of  $G_2$  — Lucas M. Chaves and A. Rigas.
- 18/90 Convergence Properties of Iterative Methods for Symmetric Positive Semidefnite Linear Complementarity Problems — Álvaro R. de Pierro and Alfredo N. Iusem.
- 19/90 The Status of the Principle of Relativity — W. A. Rodrigues Jr. and Q. A. Gomes de Souza.
- 20/90 Geração de Gerenciadores de Sistemas Reativos — Antonio G. Figueiredo Filho e Hans K. E. Liesenberg.
- 21/90 Um Modelo Linear Geral Multivariado Não-Paramétrico — Belmer Garcia Negrillo.