ON GENERATING FUNCTIONS

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ABSTRACT: We present an algebraic method to obtain expansions and generating functions for the product of two hypergeometric functions. Applications are discussed.

INTRODUCTION

Some time ago Brafman [1] published a series of papers involving hypergeometric functions and confluent hypergeometric functions. Using algebraic methods Brafman discussed generating functions and expansions for the hypergeometric functions and confluent hypergeometric functions.

At the same time Ossicini [2] and Toscano [3] published papers involving products of ultraspherical polynomials and product of Laguerre polynomials with Jacobi polynomials.

In the sixties Miller [4] published a book involving special functions and group theory where he discussed many addition theorems, many generating functions and many sum rules.

A few years ago Ciocci et al [5] obtained expansions and sum rules involving Bessel functions. Montaldi and Zucchelli [6] rederived a series of sum rules involving special functions without solving the respective differencial equations.

We note that the methodology of group theory and the algebraic methods involve only functions of the first kind, i.e., the first linearly independent solution of the respective differential equation.

Recently [7] we discussed sum rules for the product of two functions of different kinds using the methodology of the Green function. This methodology is a global methodology because it permits to obtain sum rules for the special functions and for the polynomials.

When we solved the differential equation for the spherical symmetrical top [8] parametrized in terms of Euler angles we obtained a sum rule for the product of two Legendre functions of different kinds and different arguments, in terms of Appel functions.

More recently [9] we obtained several sum rules and several generating functions for the products of two hypergeometric functions and confluent hypergeometric function using a wellknown expansion for the Whittaker function.

In the present paper we present an algebraic method to obtain expansions for the product of two hypergeometric functions and expansions for the product of hypergeometric functions with confluent hypergeometric functions by means of an expansion for the confluent hypergeometric function.

These expansions are important because contain many free parameters - without formal relations. In consequence of this freedom in the parameters we can obtain sum rules, generating functions and many expansions for the hypergeometric functions. Using a limit process we rederive an expansion for the confluent hypergeometric function.

The paper is organized as follow: in the first section we present an expansion involving a product of two hypergeometric functions: in the second section we discuss two particular case and applications of this expansion. In the third section we return to a confluent hypergeometric functions using a limit process and in the fourth section we present two particular cases of this expansion and finally we present our comments.

I. HYPERGEOMETRIC FUNCTIONS

In this section we present an algebraic derivation of an expansion for a product of two hypergeometric functions by means of an expansion for the confluent hypergeometric function.

We consider the following expansion [1]

$$
{}_1F_1\left[\beta; 1+\alpha; \frac{vxz}{(1-z)(1-z+vz)}\right] =
$$

$$
= (1-z)^{-\beta+\alpha+1}(1-z+vz)^{\beta}\ \exp\left(\frac{xz}{1-z}\right)\cdot \tag{1}
$$

$$
\sum_{n=0}^{\infty} \frac{z^n}{n!}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\ {}_2F_1(-n,\beta;1+\alpha;v)\ {}_1F_1(-n;1+\alpha;x)
$$

where $|z| < 1$ and ${}_2F_1(a, b; c; x)$ is a hypergeometric function and ${}_1F_1(a; c; x)$ is a confluent hypergeometric function.

Introducing the above expansion in the following integral representation. for the hypergeometric function [10]

$$\Gamma(b)s^{-b}\ {}_2F_1(a, b; c; k/s) = \tag{2}$$
$$= \int_0^\infty e^{-st} t^{b-1}\ {}_1F_1(a; c; kt)dt$$

which is valid for $|s| > |k|$, with $Reb > 0$, we have

$$(1-z)^{\alpha+1-\beta}(1-z+vz)^\beta \sum_{n=0}^\infty \frac{z^n}{n!}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\ {}_2F_1(-n, \beta; 1+\alpha; v)\cdot$$
$$\cdot \int_0^\infty e^{-t(s-\frac{z}{1-z})}\ t^{b-1}\ {}_1F_1(-n; 1+\alpha; t)dt =$$
$$= \Gamma(b)s^{-b}\ {}_2F_1\Big[\beta, b; 1+\alpha; \frac{vz}{s(1-z)(1-z+vz)}\Big]$$

The integral which appears in the left side of the above expression is the Laplace transform of the confluent hypergeometric function when $s > z/(1-z)$. Then, using eq. (2) we have,

$$(s - \frac{z}{1-z})^{-b}(1-z)^{\alpha+1-\beta}(1-z+vz)^\beta\cdot$$
$$\sum_{n=0}^\infty \frac{z^n}{n!}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\ {}_2F_1(-n, \beta; 1+\alpha; v)\ {}_2F_1\Big(-n, b; 1+\alpha; \frac{1-z}{s-sz-z}\Big) =$$
$$= s^{-b}\ {}_2F_1\Big[\beta, b; 1+\alpha; \frac{vz}{s(1-z)(1-z+vz)}\Big]$$

Introducing the variable u defined by $\frac{1}{u} = s - \frac{z}{1-z}$ in the above equation we have,

$$(1-z)^{1+\alpha-\beta-b}(1-z+vz)^{\beta}(1-z+uz)^{b}\cdot$$

$$\cdot\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\;{}_2F_1(-n,\beta+1;1+\alpha;v)\;{}_2F_1(-n,b;1+\alpha;u) = \tag{3}$$

$$= {}_2F_1\left[\beta,b;1+\alpha;\frac{zvu}{(1-z+vz)(1-z+uz)}\right]$$

where $|u| < 1$, $|v| < 1$ and $|z| < 1$.

This expression is an expansion for the product of two hypergeometric functions with different arguments, which can be interpreted as a generating function for the product of two hypergeometric functions with different arguments. This expansion can also be interpreted as an expansion for a hypergeometric function where the argument is a double product. To see this it is enough to define $v/x = (1-z)/(1-zx)$ and $u/y = (1-z)/(1-zy)$.

We note that this expansion is a symmetrical expansion; i.e., the expansion does not change when we interchange $\beta \rightleftharpoons b$ and $v \rightleftharpoons u$.

We also note that it does not exist a formal relation among the parameters. This fact is important because permit us to obtain a series of interesting results. This expression has been obtained firstly by Weiser [11] exploring techniques of group theory.

II. PARTICULAR CASES

In consequence of the freedom of the parameters we can obtain generating functions for the hypergeometric functions using a convenient choise of the parameters.

Firstly, we take the argument $u = 0$ (or $v = 0$) and using the fact

${}_2F_1(a,b;c;0)=1$ we obtain a generating function for a hypergeometric function

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\;{}_2F_1(-n,b;1+\alpha;u)= \tag{4}$$

$$=(1-z)^{b-\alpha-1}(1-z+uz)^{-b}$$

where $|z|<1$ and $|u|<1$. We note that when $u=0$ we have the expansion of the usual geometric series. This expansión has been obtained by Feldhein [12] studing relations between Jacobi, Laguerre and Hermite polynomials.

Another particular case of eq. (3) is the case where the parameters are related by $1+\alpha=2\beta=2b=2\mu$. We have,

$$(1-z+vz)^{\mu}(1-z+uz)^{\mu}\cdot$$

$$\cdot\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{\Gamma(n+2\mu)}{\Gamma(2\mu)}\;{}_2F_1(-n,\mu;2\mu;v)\;{}_2F_1(-n,\mu;2\mu;u)= \tag{5}$$

$$=\;{}_2F_1\left[\mu,\mu;2\mu;\frac{vzu}{(1-z+vz)(1-z+uz)}\right]$$

Now, using the following relation [10]

$${}_2F_1(-n,\mu;2\mu;v)=(1-v)^{-\frac{n}{2}}\;{}_2F_1\left[-n,n+2\mu;\mu+1/2;-\frac{(1-\sqrt{1-v})^2}{4\sqrt{1-v}}\right]$$

and expressing the hypergeometric functions which appear in eq. (5) in terms of Gegenbauer (ultraspherical) polynomials $C_{\mu}^{\nu}(x)$, we obtain,

$$\sum_{n=0}^{\infty} z^n\, n! \frac{\Gamma(2\mu)}{\Gamma(n+2\mu)} C_n^{\mu}\Big[\frac{2-v}{2\sqrt{1-v}}\Big] C_n^{\mu}\Big(\frac{2-u}{2\sqrt{1-u}}\Big) =$$

$$\Big[\frac{\sqrt{1-u} - z\sqrt{1-v}}{\sqrt{1-u}}\Big]^{-\mu}\Big[\frac{\sqrt{1-v} - z\sqrt{1-u}}{\sqrt{1-v}}\Big]^{-\mu} \cdot$$

$$\cdot\ {}_2F_1\Big[\mu, \mu; 2\mu; \frac{vzu}{(\sqrt{1-v} - z\sqrt{1-u})(\sqrt{1-u} - z\sqrt{1-v})}\Big]$$

where $|z| < 1$.

This expansion is a generating function for the product of two Gegenbauer polynomials. This generating function has been obtained firstly by Ossicini [2] using an integral representation for the Legendre function of second kind where he write the second member of this expansion in terms of Legendre functions. The case $\mu = 1/2$, Legendre polynomials, have been discussed by Watson [13] where the second member is express in terms of elliptical integrals.

III. CONFLUENT HYPERGEOMETRIC FUNCTIONS

In this section we discuss the confluent hypergeometric function by means of limit process.

The confluent hypergeometric function is obtained from the hypergeometric function using the following relation

$$\lim_{b\to\infty} {}_2F_1(a, b; c; \frac{x}{b}) = {}_1F_1(a; c; x) \tag{6}$$

Using eq. (3) we can write

$$(1-z)^{1+\alpha-\beta-b}(1 - z + z\frac{v}{\beta})^{\beta}(1 - z + uz)^{b} \cdot$$

$$\cdot \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\ {}_2F_1(-n, \beta; 1+\alpha; \frac{v}{\beta})\ {}_2F_1(-n, b; 1+\alpha; u) =$$

$$= {}_2F_1\Big[b, \beta; 1+\alpha; \frac{zuv/\beta}{(1-z+uz)(1-z+zv/\beta)}\Big]$$

Now taking the limit when $\beta \to \infty$ in both members of the above equation and using eq. (6) we obtain

$$(1-z)^{1+\alpha-b}(1-z+uz)^{b}\ \exp\left(\frac{vz}{1-z}\right)\cdot$$

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\ \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\ {}_1F_1(-n;1+\alpha;v)\ {}_2F_1(-n,b;1+\alpha;u)= \tag{7}$$

$$= {}_1F_1\left[b;1+\alpha;\frac{zvu}{(1-z)(1-z+uz)}\right]$$

where $|z|<1$. This is exactly eq. (1) and has been obtained firstly by Brafman [1].

IV. PARTICULAR CASES

Firstly we obtain a generating function for the product of Laguerre polynomials with Jacobi polynomials.

The Jacobi polynomials are related with hypergeometric function by mean of the relation

$$P_n^{(\alpha,\beta)}(x)=\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}\ {}_2F_1(-n,\alpha+\beta+n+1;\alpha+1;\frac{1-x}{2})$$

and the Laguerre polynomials are related with the confluent hypergeometric function by

$$L_n^{\alpha}(x)=\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}\ {}_1F_1(-n;\alpha+1;x)$$

Then, identifying the above relations with eq. (7) we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} z^n \frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}\, L_n^{\alpha}(v) P_n^{(\alpha,b-\alpha-n-1)}(1-2u) = \\
(8) \qquad &= (1-z)^{b-\alpha-1}(1-z+uz)^{-b} \exp\left(-\frac{vz}{1-z}\right)\cdot \\
&\cdot {}_1F_1\left[b; 1+\alpha; \frac{zvu}{(1-z)(1-z+uz)}\right]
\end{aligned}$$

This expression is a generating function for the product of Laguerre polynomials with Jacobi polynomials. We note that this generating function is only commented in the paper by Toscano [3].

Secondly, we take $u = 0$ in eq. (8) and using the normalization of Jacobi polynomials

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}$$

we obtain the following expression

$$\sum_{n=0}^{\infty} z^n L_n^{\alpha}(v) = (1-z)^{-\alpha-1} \exp\left(-\frac{vz}{1-z}\right)$$

it is the wellknown generating function for the generalized Laguerre polynomials. We note that when $v = 1$ in eq. (8) we obtain a generating function for the Jacobi polynomials.

V. COMMENTARY

In this paper we present a very simple way to obtain generating functions for the product of two hypergeometric functions. Several particular cases are discussed.

By the fact that the parameters and the arguments are free many others expansions can be obtained, involving hypergeometric functions.

We belive that this methodology for the generating functions can be generalized for the hypergeometric functions of two variables.

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