A CONDITION FOR POSITIVITY
OF CURVATURE

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Abstract. In this note we describe a condition , necessary and sufficient, in order that a procedure of the type described in $[D-R]$ yields metrics of positive sectional curvature in the total space of a principal fiber bundle.

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A condition for positivity of curvature

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In this note we describe a condition , necessary and sufficient, in order that a procedure of the type described in $[D-R]$ yields metrics of positive sectional curvature in the total space of a principal fiber bundle.

Alan Weinstein has replaced the term "Unflat" used here by the term "Fat" (see$[W_2]$).

The note was written in 1979 as an addendum to $[D-R]$ and was never submitted for publication. Due to some revival of interest in the existence of metrics of nonnegative sectional curvature on vector bundles in recent years ($[G], [S-W], [W_1]$), we thought its publication might be of some help.

Let $\pi : P \to M$ be a principlal G - bundle (G a compact Lie group) over a compact manifold M of dimension $n \geq 2$. Given a connection form ω in P and a metric h on M, and a bi–invariant metric Q in G, one can define a family of metrics g_t in P, $t > 0$, by

$$g_t(X,Y) = h(d\pi(X), d\pi(Y)) + tQ(\omega(X), \omega(Y)).$$

One can ask when there are metrics of positive sectional curvature among the g_t. By obvious reasons (c.f. $[D-R]$), it is necessary that the curvature form is unflat, (M,h) is positively curved and so is (G,Q) unless $G = S^1$. Thus, we have $G = S^1$ or $G = S^3$ or $G = S0(3)$, Q being in the latter two cases a multiple of the Killing form.

Given a connection in a principal G – bundle P over a Riemannian manifold M and a bi – invariant metric in G, the curvature form Ω canbe viewed as a 2–form on M valued in the adjoint bundle $AdP = P \times_{Ad} \widehat{G}$, where $\widehat{G}$ is the Lie algebra of G, the latter having a natural fiber metric compatible with a natural connection. Therefore expressions like $\langle (\nabla_X \Omega)(Y,Z), u \rangle$ make sense for X, Y, Z tangent to M and $u \in AdP$.

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We can now formulate the condition for positivity of sectional curvatures of g_t, t close to zero. For the case of S^1-bundles, this condition was communicated to us by L. Bérard Bergery.

Theorem: Let $P \to M$ be a principal G–bundle with a connection ω over a compact manifold $M, dim M = n \geq 2$, where $G = S^3$ or $G = S0(3)$. Fix a Riemmanian metric h in M and a bi–invariant metric Q in G. Then the following conditions are equivalent: (i) g_t has positive sectional curvature for all sufficiently small $t > 0$.
(ii) The connection ω is $\{1\}$–unflat and for any point $x \in M$, mutually orthogonal unit vectors $X, Y \in T_xM$ and any non–zero element u of AdP over x, we have

$$(1) \qquad \overset{h}{R}(X,Y,X,Y)\sum_{k=1}^{n}(u,\Omega(X,X_k))^2 \geq (u,(\nabla_X\Omega)(X,Y))^2,$$

$X_1, \ldots, X_n$ being an arbitrary orthonormal basis of T_xM, while $\overset{h}{R}$ denotes the curvature tensor of (M,h).
Proof. First, we need

LEMMA. Given real parameters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \lambda$, with $\beta > 0$, define the family of functions $\varphi_t : \mathbb{R}^4 \to \mathbb{R}(t > 0)$ by

$$\begin{aligned}\varphi_t(A,B,C,D) &= \alpha A^2C^2 + t(\beta A^2C^2 + \gamma A^2CD + \delta AC^2B \\ &+ \varepsilon ABCD + \lambda B^2D^2) + t^2(\theta D^2A^2 + \eta B^2C^2 + \zeta ABCD).\end{aligned}$$

Then the following conditions are equivalent:
(i) There exists a positive real unumber t_0, depending continuously on $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \lambda$ and such that

$$(2) \qquad \varphi_t(A,B,C,D) > 0$$

whenever $0 < t \leq t_0$ and

$$(3) \qquad A^2 + B^2 = 1 = C^2 + D^2.$$

(ii) The parameters satisfy

$$(4) \qquad \begin{cases} \alpha \geq 0,\ \lambda > 0,\ \theta > 0,\ \eta > 0, \\ 4\alpha\theta \geq \gamma^2,\ 4\alpha\eta \geq \delta^2. \end{cases}$$

Proof of Lemma . Assume (i). Then (ii) follows immediately by setting in (2) first $B = D = 1$, $A = C = 0$, next $A = 1$, $B = 0$ and, finally, $C = 1$, $D = 0$. Assume now (ii). Since (4) and (3) together with $BD = 0$ imply (2) for every small

$t > 0$, we may consider only the case $BD \neq 0$. Dividing $\varphi_t(A, B, C, D)$ by B^2D^2 and then setting $x = A/B, y = C/D$, we see that our assertion is equivalent to

$$x^2[(\alpha + t\beta)y^2 + t\gamma y + t^2\theta] + x(t\delta y^2 + t\varepsilon y + t^2\zeta y) + t\lambda + t^2\eta y^2 > 0 \tag{5}$$

for all real x, y, provided that $t > 0$ is small enough. The expression $(\alpha+t\beta)y^2+t\gamma y+t^2\theta$ is positive for small $t > 0$ and all real y, since $\alpha + t\beta > 0$ and the discriminant (with respect to y), $t^2(\gamma^2 - 4\alpha\theta - 4t\beta\theta) < 0$ for t close to zero. Therefore, for $t > 0$ small enough the left hand side of (5) can be viewed as a binomial in the variable x with positive leading coefficient. Condition (5) is then equivalent to the negativity of the corresponding discriminant, i. e., to

$$\begin{aligned} &y^2[y^2(\delta^2 - 4\alpha\eta - 4t\beta\eta) + y(2\delta\varepsilon + 2\delta\zeta t - 4t\gamma\eta) + (\varepsilon + \zeta t)^2 - 4t^2\theta\eta] \\ &< \lambda[(\frac{4\alpha}{t} + 4\beta)y^2 + 4\gamma y + 4t\theta]. \end{aligned} \tag{6}$$

The left–hand side of (6) is a family of functions $F_t(y)$ of the variable y, depending on the small parameter $t > 0$, satisfying $F_t(0) = 0$, $\frac{d}{dy}F_t(y)|_{y=0} = 0$, $\lim_{|y|\to\infty} F_t(y) = -\infty$ and uniformly bounded in a fixed neighborhood of $y = 0$ (Fig. 1). For t small enough, F_t is almost independent of t. On the other hand, the right–hand side of (6) is equal to

$$\lambda(\frac{4\alpha}{t} + 4\beta)[(y + \frac{t\gamma}{2(\alpha + t\beta)})^2 + (\frac{t}{2\alpha + 2t\beta})^2(4\alpha\theta - \gamma^2 + 4t\beta\theta)]$$

which is a family $G_t(y)$ of binomials satisfying $\lim_{t\to 0} G_t'' = \infty$. The minimum of G_t taken at $m_t = -\dfrac{t\gamma}{2(\alpha + t\beta)}$ is equal to $\dfrac{\lambda t}{\alpha + t\beta}(4\alpha\theta - \gamma^2 + 4t\beta\theta)$. The graphs of G_t form a 1-parameter family of parabolas, for which the curve of vertices $t \mapsto L(t) = (m_t,\ G_t(m_t))$ satisfies $L(0) = (0, 0)$ and $L'(0) = (-\frac{\gamma}{2\alpha},\ \frac{\lambda}{\alpha}(4\alpha\theta - \gamma^2))$ with second component > 0 in view of (4). Therefore, L is not tangent to the horizontal axis at $(0, 0)$. The parabolas G_t behave now as in Fig. 1, which completes the proof.

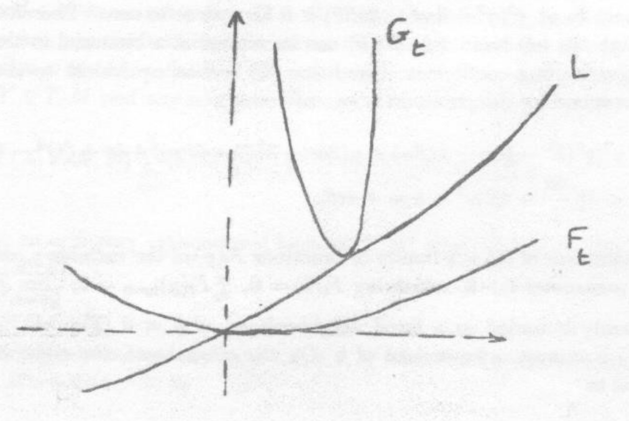

Fig. 1

Proof of Theorem. Fix an open subset $U \subset M$ with a field $e_1, \ldots, e_n$ of h– orthonormal frames in U and choose a Q–orthonormal basis u_1, u_2, u_3 of the lie algebra $\widehat{G}$ of G. From now on we assume that $i, j, k, l \in \{1, \ldots, n\}$, $\alpha, \beta, \gamma \in \{1, 2, 3\}$. Moreover, for a vector field X on M and $u \in \widehat{G}$ we denote by $\overline{X}$ the horizontal lift of X and by $u*$ the fundamental vector field in P corresponding to u. Finally, $\overset{t}{\nabla}$ and $\overset{t}{R}$ (resp. $\overset{h}{\nabla}$ and $\overset{h}{R}$ or $\overset{Q}{\nabla}$ and $\overset{Q}{R}$ stand for the covariant derivative and the curvature tensor of (P, g_t) (resp. of (M, h) or (G, Q)). The frame field $\overline{e_1}, \ldots, \overline{e_n}$, u_1^*, u_2^*, u_3^* satisfies now the relations $g_t(\overline{e_i}, \overline{e_j}) = \delta_{ij}$, $g_t(\overline{e_i}, u_\alpha^*) = 0$, $g_t(u_\alpha^*, u_\beta^*) = t\delta_{\alpha\beta}$, $\overset{t}{\nabla}_{\overline{e_i}} \overline{e_j} = (\overset{h}{\nabla}_{e_i} e_j) - \frac{1}{2}(\Omega(e_i, e_j))^*$, $\overset{t}{\nabla}_{\overline{e_i}} u_\alpha^* = \overset{t}{\nabla}_{u_\alpha^*} \overline{e_i} = \frac{1}{2}t \sum_k \Omega_{ik}^\alpha \overline{e_k}$, $\overset{t}{\nabla}_{u_\alpha^*} u_\beta^* = (\overset{Q}{\nabla}_{u_\alpha} u_\beta)^*$, where $\Omega = d\omega + [\omega, \omega]$ is the curvature form and $\Omega(\overline{e_i}, \overline{e_j}) = \sum_\alpha \Omega_{ij}^\alpha u_\alpha$. Using now the notations $\overset{t}{R}_{ijk\alpha} = \overset{t}{R}(\overline{e_i}, \overline{e_j}, \overline{e_k}, u_\alpha^*)$, $\overset{h}{R}_{ijkl} = \overset{h}{R}(e_i, e_j, e_k, e_l)$ etc., we have $\overset{t}{R}_{ijkl} = \overset{h}{R}_{ijkl} + tA_{ijkl}$, $\overset{t}{R}_{ijk\alpha} = tB_{ijk\alpha}$, $\overset{t}{R}_{i\alpha j\beta} = tC_{i\alpha j\beta} + \frac{1}{4}t^2 \sum_k \Omega_{jk}^\alpha \Omega_{ik}^\beta$, $\overset{t}{R}_{ij\alpha\beta} =$

$$2tC_{i\alpha j\beta} + \frac{1}{4} t^2 \sum_k (\Omega^\alpha_{jk} \Omega^\beta_{ik} - \Omega^\alpha_{ik} \Omega^\beta_{jk}), \ \overset{t}{R}_{i\alpha\beta\gamma} = 0, \ R^t_{\alpha\beta\gamma\delta} = t \overset{Q}{R}_{\alpha\beta\gamma\delta},$$

where A_{ijkl}, $B_{ijk\alpha}$, $C_{i\alpha j\beta}$ are certain expressions, independent of t. It is important to observe that $C_{i\alpha j\beta} + C_{j\alpha i\beta} = 0$ and $B_{ijk\alpha} = \frac{1}{2}\langle(\nabla_{e_i}\Omega)(e_j, e_k), u_\alpha\rangle - \frac{1}{2}\langle(\nabla_{e_j}\Omega)(e_i, e_k), u_\alpha\rangle$, u_α being viewed (in a non–canonical way) as an element of AdP. Consider now a 2–plane tangent to P. It is easy to verify that it always has a g_1–orthonormal basis of the form $X = A\overline{e_i} + Bu^*_\alpha$, $Y = C\overline{e_j} + Du^*_\beta$ with $i \neq j$, $\alpha \neq \beta$ and $A^2 + B^2 = C^2 + D^2 = 1$. From the above formulas, we have $\overset{t}{R}(X,Y,X,Y) = \varphi_t(A,B,C,D)$, φ_t being defined as in the Lemma with $\alpha = \overset{h}{R}_{ijij}$, $\beta = A_{ijij}$, $\gamma = 2B_{iji\beta}$, $\delta = -2B_{ijj\alpha}$, $\varepsilon = 6C_{i\alpha j\beta}$, $\zeta = \sum_k (\frac{1}{2} \Omega^\alpha_{jk} \Omega^\beta_{ik} - \Omega^\alpha_{ik} \Omega^\beta_{jk}), \eta = \frac{1}{4} \sum_k (\Omega^\alpha_{jk})^2$, $\theta = \frac{1}{4} \sum_k (\Omega^\beta_{ik})^2$, $\lambda = \overset{Q}{R}_{\alpha\beta\alpha\beta}$.

Our assertion follows now immediately from the Lemma.

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