ON THE CONVERGENCE OF SOR AND JOR TYPE METHODS FOR CONVEX LINEAR COMPLEMENTARITY PROBLEMS

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Abstract. We consider SOR and JOR type iterative methods for solving linear complementarity problems. If the solution set is not discrete, weak convergence proofs are usually obtained for these methods; i.e., every accumulation point of the generated sequence is a solution. We prove that, for the convex case, the whole sequence converges and, if the limit point is nondegenerate, convergence is linear.

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Running title: **Iterative Methods for LCP.**

1. Introduction

We analyze in this paper the behaviour of the sequence generated by some well known methods for solving the linear complementarity problem

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(*LCP*)

$$\begin{aligned} z^t(Mz+b) &= 0, \\ z &\geq 0, \\ Mz+b &\geq 0, \end{aligned} \qquad (1)$$

where M is an $n \times n$ symmetric positive semidefinite matrix with positive diagonal, z and b are n–vectors (z^t denotes the transpose of z). When M is large and sparse, iterative methods like SOR [4] and JOR [6] are suitable for solving (1). These methods act on (1) handling one (or a few) row(s) per iteration in a sequential (SOR) or parallel (JOR) manner. If $\{z^k\}$ is the sequence generated by one of those methods, standard convergence results are derived after proving two main properties:

(i) $\{z^k\}$ is bounded,

(ii) $z^{k+1} - z^k \underset{k \to \infty}{\longrightarrow} 0.$

In order to prove (i) a Slater condition is required for (1); i.e.; $Mx+b>0$ or $Mx>0$ for some $x \in I\!R^n$ (Theorem 2.2, [4]). Using (i) and (ii) it is possible to deduce that every limit point is a solution of the LCP and this behaviour is calle and JOR–type methods when the matrix M is symmetric positive semidefinite but possibly degenerate without using any other hypothesis on (1) except for the solution set to be non empty. As far as we know, this is the first strong convergence result not based on a discrete structure of the solution set, for this type of methods. In order to achieve this result we show first in the following section that there is a dual problem associated with (1). In the same way, associated with each method for solving (1), there is a dual method for solving the dual problem. Recently [2], we proved that the corresponding methods for the dual are strongly convergent at a linear rate. This fact is the key result to deduce strong convergence for the primal variables in Section 3. Moreover, if there are no degenerate limit points (z_j and $(Mz+b)_j$ simultaneously zero for some j), we show that the methods behave like the equivalent methods for solving linear systems of equations and the linear convergence rate follows.

2. Dual Problems and Algorithms

It is well known from elementary linear algebra that for every symmetric positive semidefinite matrix M with positive diagonal there exists a matrix A with non–zero rows such that $M = AA^t$; A can be taken, for example, as the $n \times n$ matrix arising from the eigenvalue decomposition of M; i.e., if $M = Q\Lambda Q^t$, where Q is the orthogonal eigenvectors matrix and Λ is a diagonal matrix of eigenvalues, then $A = Q\,\Lambda^{1/2}$. Using this decomposition, problem (1) becomes

$$\begin{aligned} z^t(AA^tz + b) &= 0, \\ z &\geq 0, \\ AA^tz + b &\geq 0. \end{aligned} \tag{2}$$

If (2) has a solution z^*, $x^* = -A^tz^*$ is the solution of the optimization problem

$$\text{minimize } \frac{||x||^2}{2} \tag{3}$$

$$\text{subject to } Ax \leq b. \tag{3'}$$

The converse is also true and this can be easily proven noting that (2) together with $x = -A^tz$ are the Kuhn–Tucker conditions for (3), z being the multiplier vector corresponding to the constraints (3'). In other words (3)–(3') is a dual problem for (1).

The Algorithms

Before presenting the algorithms we introduce some notation and definitions.

Definition 2.1. A given sequence $\{i(k)\}_{k\geq 0}$ is called almost cyclic for the integer set $I = \{1, 2, \ldots, n\}$ if there exists an integer C, positive, such that $\{i(k),\, i(k+1), \ldots, i(k+C)\} \supseteq I$. (See [3]).

In the following, m^i and a^i will denote the i-th row of M and A respectively, b_i the i-th component of vector b, and e^i the i-th vector of the canonic basis ($e^i_j = \delta_{ij}$ the Kronecker's delta). $\langle \cdot \rangle$ will denote the standard inner product and $\| \cdot \|$ the 2–norm.

The Sequential Algorithm (almost cyclic SOR).

For a given starting point $z^0 \in I\!R^n_+$ (nonnegative orthant), define

$$z^{k+1} = z^k - \beta_k e^{i(k)}, \tag{4}$$

where

$$\beta_k = \min\left\{ z_{i(k)},\ \alpha_k \frac{\langle m_{i(k)}, z^k \rangle + b_{i(k)}}{m_{i(k),i(k)}} \right\}, \tag{5}$$

and $\alpha_k \in [\varepsilon, 2-\varepsilon]$, ε a given positive number.

Taking into account (2), we can multiply (4) by $-A^t$ and, defining the new variable $x^k = -A^t z^k$, we obtain

$$x^{k+1} = x^k + \beta_k a^{i(k)}, \tag{6}$$

and

$$\beta_k = \min\left\{ z^k_{i(k)},\ \alpha_k \frac{b_{i(k)} - \langle a^{i(k)}, x^k \rangle}{\|a^{i(k)}\|^2} \right\}. \tag{7}$$

(4)–(5) is the SOR method with an almost cyclic control; for $C = n$ we get the standard SOR and other choices of the sequence give other well known methods for the LCP, for example SSOR. (6)–(7) is the generalization of Hildreth's method presented in [3] for the dual problem.

The Parallel Algorithm (Jacobi–type)

For a given starting point $z^0 \in I\!R^n_+$, define

$$z^{k+1} = z^k - \Lambda c^k, \tag{8}$$

where c^k is a vector with components, for $i = 1, \ldots, n$,

$$c^k_i = \min\left\{ z^k_i / \lambda_i,\ \alpha_k \frac{\langle m^i, z^k \rangle + b_i}{\|m^i\|^2} \right\}, \tag{9}$$

λ_i are real numbers such that $\sum_{i=1}^{n} \lambda_i = 1,\ 0 < \lambda_i < 1$, for $i = 1, \ldots, n$, and Λ is the diagonal matrix of the λ_i's.

As before, taking $x^k = -A^t z^k$ (multiplying (8) by $-A^t$) we obtain the dual algorithm

$$x^{k+1} = \sum_{i=1}^{n} \lambda_i x^{i,k}, \tag{10}$$

where

$$x^{i,k} = x^k + c_i^k a^i, \tag{10'}$$

and

$$c_i^k = \min\Big\{ z_i^k \,/\, \lambda_i,\ \alpha_k \frac{b_i - \langle a^i, x^k \rangle}{||a^i||^2} \Big\}. \tag{11}$$

Convergence results for the sequential algorithm (6)–(7) can be found in [3] and in [1] for the parallel version (10)–(11). For the latter, it is proven in [1] that it converges even if the feasible set defined by (3') is empty, but in that case it is easy to verify that (2) has no solution and (8) is divergent. In [2] we prove that convergence is linear for both algorithms.

3. Convergence Results

As mentioned before, we know the following facts about the sequences $\{x^k\}$ generated by (6)–(7) or (10)–(11):

1) If the system (3') is consistent, then

(a) $x^k \underset{k \to \infty}{\longrightarrow} x^*$ such that $Ax^* \leq b$

(b) there exist $\rho \in (0,1)$, $K \in \mathbb{Z}_{\geq 0}$ such that for $k > K$

$$||x^{k+r} - x^*|| \leq \rho ||x^k - x^*||,$$

where $r = C$ (the almost cyclicallity constant) for the sequential algorithm and $r = 1$ for the parallel algorithm.

2) (a) $z^k \geq 0$ for all $k \geq 0$,

(b) $x^k = -A^t z^k$ for all $k \geq 0$.

The proofs of 1(a), 1(b) and 2 for algorithm (6)–(7) can be found in Theorem 3.1 of [3], Theorem 1 of [2] and Lemma 3.5 of [3] respectively. The proofs of these statements in the case of algorithm (10)–(11) can be found in Theorem 4 of [1], Theorem 2 of [2] and Propositions 1 and 4 of [1].

We will prove convergence of the sequence $\{z^k\}$. Proofs will refer always to both sequential and parallel methods. To avoid repetitions, algorithm (6)–(7) will be called SA and algorithm (10)–(11) PA.

Let now

$$\hat{I} = \{i : \langle a_i, x^* \rangle < b_i\}, \tag{12}$$

where x^* is the unique solution of (3),

$$L = \{k \in \mathbb{Z}_+ : i(k) \in \hat{I}\}, \tag{13}$$

$$\gamma_k = \alpha_k \frac{b_{i(k)} - \langle a^{i(k)}, x^k \rangle}{||a^{i(k)}||^2} \quad (\text{for } \; SA) \tag{14}$$

and

$$\gamma_k^i = \alpha_k \frac{b_i - \langle a^i, x^k \rangle}{||a^{i(k)}||^2} \quad (\text{for } \; PA), \;\; i = 1, \ldots, n. \tag{15}$$

Proposition 3.1. There exists K' such that for all $k \geq K'$ and for all $i \in \hat{I}$, $z_i^k = 0$.

Proof. Since $\{x^k\}$ converges, $\lim_{k\to\infty} ||x^{k+1} - x^k|| = 0$, which implies

$$\lim_{k\to\infty} \beta_k = 0, \tag{16}$$

for SA and

$$\lim_{k\to\infty} c_i^k = 0, \qquad i = 1, \ldots, m, \tag{17}$$

for PA.

Let

$$\eta = \frac{\varepsilon}{2} \min_{i \in \hat{I}} \left\{ \frac{b_i - \langle a^i, x^* \rangle}{||a^i||^2} \right\}.$$

Take K' such that for $k \geq K'$

$$\beta_k < \eta \quad \text{and} \quad \gamma_k = \eta, \quad \text{for} \quad k \in L, \quad (SA)$$

or

$$c_i^k < \eta \quad (i = 1, \ldots, n) \quad \text{and} \quad \gamma_k^i \geq \eta \quad \text{for} \quad i \in \hat{I} \qquad (PA);$$

since $\beta_k = \min\{z_{i(k)}^k, \gamma_k\}$ and $c_i^k = \min\{z_i^k, \gamma_k^i\}$, conclude that for $k \geq K'$

$$\beta_k = z_{i(k)}^k \Longrightarrow z_{i(k)}^{k+1} = 0 \quad \text{for} \quad k \in L \qquad (SA)$$

or

$$c_k = z_i^k \Longrightarrow z_i^{k+1} = 0 \quad \text{for} \quad i \in \hat{I} \qquad (PA).$$

and this will remain valid for $k \geq K'$, because $\eta > 0$.

□

Corollary 3.1. For $k \geq K'$ it holds that

$$\beta_k = 0, \qquad k \in L \qquad (SA), \tag{18}$$

or

$$c_i^k = 0 \qquad i \in \hat{I} \qquad (PA). \tag{19}$$

Proof. Imediate from Proposition 3.1 and (5) or (9).

□

Let

$$H_i = \{x \in I\!\!R^n : \langle a^i, x \rangle = b_i\}.$$

Proposition 3.2. If $u \in H_{i(k)}$, then $||x^{k+1} - u|| \leq ||x^k - u||$ (SA) or, if $u \in H_i$, then $||x^{i,k} - u|| \leq ||x^k - u||$ (PA).

Proof. Let $y^k = x^k + \gamma_k a^{i(k)}$, then

$$\begin{aligned} ||y^k - u||^2 - ||x^k - u||^2 &= \gamma_k^2 ||a^{i(k)}||^2 + 2\gamma_k \langle x^k - u, a^{i(k)} \rangle \\ &= \gamma_k^2 ||a^{i(k)}||^2 - 2\gamma_k (\langle a^{i(k)}, x^k \rangle - b_{i(k)}) \\ &= ||a^{i(k)}||^2 \gamma_k^2 \Big(1 - \frac{2}{\alpha_k}\Big) \leq 0, \end{aligned} \tag{20}$$

using the bounds on α_k. From (20) we have that

$$\|y^k - u\|^2 \leq \|x^k - u\|. \tag{21}$$

If $\gamma_k < 0$ then $x^{k+1} = y^k$ because of property 2(a) and the result follows from (21). If $\gamma_k \geq 0$ then x^{k+1} is in the segment between x^k and y^k; so, using (21)

$$\|x^{k+1} - u\| \leq \max\{\|y^k - u\|, \|x^k - u\|\} = \|x^k - u\|. \tag{22}$$

For the PA the same argument hold, substituting γ_k by γ_k^i, $a^{i(k)}$ by a^i and x^{k+1} by $x^{i,k}$,

□

Proposition 3.3. For $k \geq K'$, the sequence $\{\|x^k - x^*\|\}$ is nonincreasing.

Proof. If $k \notin L$ (SA) then $x^* \in H_{i(k)}$ and the result follows from Proposition 3.2; if $k \in L$ then $\beta_k = 0$ by Corollary 3.1, so $x^{k+1} = x^k$ and $\|x^{k+1} - x^*\| = \|x^k - x^*\|$.

For PA, if $i \notin \widehat{I}$, by the preceding arguments $\|x^{i,k} - x^*\| \leq \|x^k - x^*\|$, if $i \in \widehat{I}$, then $c_i^k = 0$ and $\|x^{i,k} - x^*\| = \|x^k - x^*\|$, so

$$\|x^{k+1} - x^*\| = \left\|\sum_{i=1}^n \lambda_i x^{i,k} - x^*\right\| \leq \sum_{i=1}^n \lambda_i \|x^{i,k} - x^*\| = \|x^k - x^*\|.$$

□

Let $\eta' = \min_{i\in\widehat{I}}\{\|a^i\|\}$.

Proposition 3.4. For $k \geq K'$, $|\beta_k| \leq \frac{2}{\eta'}\|x^k - x^*\|$ (SA) and $|c_i^k| \leq \frac{2}{\eta'}\|x^k - x^*\|$ (PA), $i = 1, \ldots, n$.

Proof. If $k \in L$ (SA) then $\beta_k = 0$ by Corollary 3.1. If $k \notin L$, then $i(k) \notin \widehat{I}$, so that $\langle a^{i(k)}, x^* \rangle = b_{i(k)}$. In view of 2(a), if $\gamma_k < 0$ then $\beta_k = \gamma_k$ and otherwise $0 \le \beta_k \le \gamma_k$. Therefore

$$|\beta_k| \le |\gamma_k|, \tag{23}$$

and

$$|\gamma_k| \le \alpha_k \frac{|b_{i(k)} - \langle a^{i(k)}, x^k \rangle|}{\eta' \|a^{i(k)}\|} = \alpha_k \frac{|\langle a^{i(k)}, x^* - x^k \rangle|}{\eta' \|a^{i(k)}\|} \le \alpha_k \frac{\|x^k - x^*\|}{\eta'}, \tag{24}$$

using Schwarz inequality. From (23), (24) and the fact that $\alpha_k \in (0, 2)$,

$$|\beta_k| \le \frac{\alpha_k \|x^k - x^*\|}{\eta'} \le \frac{2}{\eta'} \|x^k - x^*\|.$$

For PA, the same argument holds.

□

Define

$$\overline{K} = \max\{K, K'\},$$

$$L_1 = \{k > \overline{K} : i(k) = i\}.$$

Theorem 3.1. The sequence $\{z^k\}$ generated by algorithms (4)–(5) (SA) and (8)–(9) converges to a solution of problem (1).

Proof. Consider first SA. If $i \in \widehat{I}$, $\lim_{k \to \infty} z_i^k = 0$ (in fact z_i^k eventually reachs zero and remains zero thereafter) by Proposition 3.1.

Take $i \notin \widehat{I}$, $k > \overline{K}$ then:

$$z_i^k = z_i^{\overline{K}} - \sum_{\substack{j \in L_i \\ \overline{K} \le j \le k-1}} \beta_j, \tag{24}$$

$$\sum_{\substack{j\in L_i\\ \overline{K}\le j\le k-1}} |\beta_j| \le \sum_{j=\overline{K}}^{k-1} |\beta_j| \le \sum_{j=\overline{K}}^{\infty} |\beta_j| \le \frac{2}{\eta'} \sum_{j=\overline{K}}^{\infty} \|x^j - x^*\|$$

$$= \frac{2}{\eta'} \sum_{j=0}^{\infty}\sum_{s=1}^{r} \|x^{\overline{K}+jC+S} - x^*\| \le \frac{2C}{\eta} \sum_{j=0}^{\infty} \|x^{\overline{K}+jC} - x^*\|$$

$$\le \frac{2C}{\eta'} \|x^{\overline{K}} - x^*\| \sum_{j=o}^{\infty} \rho^j = \frac{2C\|x^{\overline{K}} - x^*\|}{\eta'(1-\rho)} < \infty \qquad (25)$$

using Proposition 3.4 in the third inequality, Proposition 3.3 in the fourth one and 1(b) in the fifth one. It follows that, when k goes to infinity, the summation in the right hand side of (24) becomes an absolutely convergent series, hence convergent.

For PA we consider the inequalities

$$\sum_{\overline{K}\le j\le k} |c_i^j| \le \sum_{j=\overline{K}}^{\infty} |c_i^j| \le \frac{2}{\eta'} \sum_{j=\overline{K}}^{\infty} \|x^j - x^*\|$$

$$\le \frac{2}{\eta'} \|x^{\overline{K}} - x^*\| \sum_{j=\overline{K}}^{\infty} \rho^j \le \frac{2\|x^{\overline{K}} - x^*\|}{\eta'(1-\rho)} < \infty.$$

$z^* \ge 0$ and $AA^t z^* + b \ge 0$ are imediate consequences of 1(a), 2(a), and 2(b).

□

From the strong convergence of the sequence z^k and Proposition 3.1 we deduce linear convergence for nondegenerate points. Recall the following

Definition 3.1. z is a nondegenerate solution of (1) iff $z + Mz + b > 0$.

Theorem 3.2. If the sequence $\{z^k\}$ generated by (4)–(5) or (8)–(9) converges to a nondegenerate point z^*, convergence is linear.

Proof. By Proposition 3.1 for k large enough and $i \in \widehat{I}$, $z_i^k = 0 = z_i^*$. Therefore we need only to consider $z_{i(k)}^k$, $k \notin L$. If $i \notin \widehat{I}$ $\langle a^i, x^* \rangle = b_i$ or $\langle m^i, z^* \rangle + b_i = 0$ and $z_i^* > 0$ (nondegeneracy assumption). But $\lim_{k\to\infty} \beta_k = 0$ implies that for k large enough the method (4)–(5) will be defined for

$k \in L$, by

$$\beta_k = \alpha_k \frac{\langle m^{i(k)}, z^k \rangle + b_{i(k)}}{m_{i(k),i(k)}}. \tag{26}$$

Let $\overline{M}$ be the matrix extracted from M with elements m_{ij}, $i, j \in \widehat{I}'$, where $\widehat{I}'$ denotes the complement index set of $\widehat{I}$ and $\overline{b}$ the vector extracted from b with elements b_i, $i \in \widehat{I}'$. Then (26) means that for $z^k_{i(k)}$ such that $k \in L$, the method becomes SOR (with an almost cyclic control) applied to the linear system

$$\overline{M}\,\overline{z} + \overline{b} = 0. \tag{27}$$

For the PA, replace (26) by

$$c_i^k = \alpha_k \frac{\langle m^i, z^k \rangle + b_i}{m_{ii}} \tag{28}$$

and the method (8)–(9) becomes JOR applied to a linear system like (27). SOR and JOR, applied to linear systems, converge with a linear rate [7]

□

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