ON KUMMER EXPANSIONS

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INTRODUCTION.

Many types of expansions for the special functions are known in the literature involving generating functions(1) or sum rules(2) and addition theorems(3).

There are two classical methods which are used to obtain expansions for the special functions: the algebraic method(4) and the group theory method(5). We note that into both cases these expansions involve always functions of the first kind. We call function of the first kind, the first solution of the differential equation satisfied by the special function.

In a recent paper(6) we presented a way to obtain many general expansions involving a product of two hypergeometric functions with different arguments.

In this paper we present a methodology to obtain new general expansions for the confluent hypergeometric confluent with different arguments using an expansion for the generalized Laguerre polynomials.

The paper is organized as follows: in the first section we obtain the expansion for two confluent hypergeometric functions, namely Kummer functions with different arguments. In the second section we present particular cases and many others applications are discussed.

I. KUMMER FUNCTIONS.

In this section we use an expansion for the generalized Laguerre polynomials to obtain an expansion for the Kummer functions with different arguments.

We consider the following expansion[7]

$$\sum_{\lambda=0}^{\infty} \frac{1}{\lambda!} \frac{\Gamma(\sigma-\alpha+\lambda)}{\Gamma(\sigma-\alpha)} s^{\lambda} \; {}_1F_1(-\lambda;\sigma-\beta;z) = \tag{1}$$

$$= (1-s)^{-\sigma+\alpha} \; {}_1F_1(\sigma-\alpha;\sigma-\beta;-\frac{sz}{1-s})$$

where z is arbitrary and $|s| < 1$. In this equation the Kummer function ${}_1F_1(-\lambda;\sigma-\beta;t)$ is related to the generalized Laguerre polynomials by means of the following relation[8].

$$L_m^a(x) = \frac{\Gamma(a+m+1)}{m!\Gamma(a+1)} \; {}_1F_1(-m;a+1;x). \tag{2}$$

Introducing eq.(1) in the following integral representation[9]

$$\int_0^1 x^{\beta-1}(1-x)^{\sigma-\beta-1} \; {}_1F_1(\alpha;\beta;\xi x) \; {}_1F_1(\sigma-\alpha;\sigma-\beta;\mu(1-x))dx = \tag{3}$$

$$= \frac{\Gamma(\beta)\Gamma(\sigma-\beta)}{\Gamma(\sigma)} e^{\xi} \; {}_1F_1(\alpha;\sigma;\mu-\xi)$$

where $\mathrm{Re}\,\sigma > \mathrm{Re}\,\beta > 0$, we obtain

$$(4)\qquad \begin{aligned} &(-1)^{\sigma-\beta} \int_0^\infty ds\, s^{\sigma-\beta-1}(1-s)^{-\sigma}\ {}_1F_1(\alpha;\beta;\frac{\xi}{1-s})(1-s)^{\sigma-\alpha}\cdot \\ &\cdot\sum_{\lambda=0}^{\infty} \frac{1}{\lambda!}\frac{\Gamma(\sigma-\alpha+\lambda)}{\Gamma(\sigma-\alpha)}\, s^\lambda\ {}_1F_1(-\lambda;\sigma-\beta;z) = \\ &= \frac{\Gamma(\beta)\Gamma(\sigma-\beta)}{\Gamma(\sigma)}\, e^\xi\ {}_1F_1(\alpha;\sigma;z-\xi) \end{aligned}$$

where we have defined the parameter which appear in eq.(3) by $\mu = z$.

Introducing a new variable in the above equation we have:

$$(5)\qquad \begin{aligned} &\sum_{\lambda=0}^{\infty}(-1)^\lambda \frac{1}{\lambda!}\frac{\Gamma(\sigma-\alpha+\lambda)}{\Gamma(\sigma-\alpha)}\ {}_1F_1(-\lambda;\sigma-\beta;z)\cdot \\ &\cdot\int_0^1 dt\, t^{-\sigma-\lambda+\alpha+\beta-1}(1-t)^{\sigma-\beta+\lambda-1}\ {}_1F_1(\alpha;\beta;\xi t) = \\ &= \frac{\Gamma(\beta)\Gamma(\sigma-\beta)}{\Gamma(\sigma)}\, e^\xi\ {}_1F_1(\alpha;\sigma;z-\xi) \end{aligned}$$

where $\mathrm{Re}\,\sigma > \mathrm{Re}\,\beta > 0$.

The integral which appears in the above equation can be evaluated in terms of the generalized confluent hypergeometric function[10]

$${}_2F_2(a_1,a_2\,;\,b_1,b_2\,;x) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n}{(b_1)_n(b_2)_n}\frac{x^n}{n!}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$, then we have

$$
\begin{aligned}
&\sum_{\lambda=0}^{\infty}(-1)^{\lambda}\,\frac{1}{\lambda!}\,\frac{\Gamma(\sigma-\alpha+\lambda)}{\Gamma(\sigma-\alpha)}\;{}_1F_1(-\lambda;\sigma-\beta;z)\,\frac{\Gamma(-\sigma-\lambda+\alpha+\beta)}{\Gamma(\alpha)}\,\cdot \\
(6)\qquad &\cdot\,\Gamma(\sigma-\beta+\lambda)\;{}_2F_2(-\sigma-\lambda+\alpha+\beta\,,\,\alpha;\alpha,\beta;\xi)= \\
&=\frac{\Gamma(\beta)\Gamma(\sigma-\beta)}{\Gamma(\sigma)}\,e^{\xi}\;{}_1F_1(\alpha;\sigma;z-\xi)
\end{aligned}
$$

where $\mathrm{Re}(-\sigma-\lambda+\alpha+\beta)>0$ and $\mathrm{Re}\,\sigma>\ \mathrm{Re}\,\beta>0$.

Introducing the Pochhammer symbol $(a)_n$ in eq.(6) and using a relation for the Kummer function[8] we obtain

$$
\begin{aligned}
&\sum_{\lambda=0}^{\infty}(-1)^{\lambda}\,\frac{1}{\lambda!}(\sigma-\alpha)_{\lambda}(\sigma-\beta)_{\lambda}\;\Gamma(-\sigma-\lambda+\alpha+\beta)\,\cdot \\
(7)\qquad &\cdot\,{}_1F_1(-\lambda,\sigma-\beta;z)\;{}_1F_1(\lambda+\sigma-\alpha;\beta;-x)= \\
&=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\sigma)}\;{}_1F_1(\alpha;\sigma;z-x)
\end{aligned}
$$

where $\mathrm{Re}(-\sigma-\lambda+\alpha+\beta)>0$, $\mathrm{Re}\,\sigma>\mathrm{Re}\,\beta>0$ and $\mathrm{Re}\,\sigma>\mathrm{Re}\,\alpha>0$. We note that, in this equation we have used the following relation ${}_2F_2(a_1,a_2;b_1,b_2;x)={}_1F_1(a_1;b_2;x)$.

The above expression is a new expansion for the product of two confluent hypergeometric function, namely Kummer functions, with different arguments.

II. PARTICULAR CASES AND APPLICATIONS.

In this section we present a particular case of eq.(7), namely a new expansion for the product of even Hermite polynomials which a confluent hypergeometric function. Making the parameter $\sigma - \beta = 1/2$, we have

$$\sum_{\lambda=0}^{\infty}(2\lambda)!(\beta-\alpha+1/2)_\lambda\,(1/2)_\lambda\,\Gamma(-\lambda+\alpha-1/2)\,H_{2\lambda}(z)\cdot$$

$$\cdot\,{}_1F_1(\lambda+\beta-\alpha+1/2;\beta;-x)=$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\beta+1/2)}\;{}_1F_1(\alpha;\beta+1/2;z-x)$$

where $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}\,\beta>0$ and $\mathrm{Re}(-\lambda+\alpha-1/2)>0$

In the same way we can obtain a new expansion for the product of odd Hermite polynomials with a confluent hypergeometric function making the parameter $\sigma-\beta=\frac{3}{2}$.

Another particular case of eq.(7) is obtained making $x=0$ and using another relation[9] for the Kummer functions we obtain a new sum for the Kummer function,

$$\sum_{\lambda=0}^{\infty}(-1)^\lambda(\sigma-\alpha)_\lambda(\sigma-\beta)_\lambda\,\Gamma(-\sigma-\lambda+\alpha+\beta)\,{}_1F_1(-\lambda;\sigma-\beta;z)=$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\sigma)}\;{}_1F_1(\alpha;\sigma;z)$$

where $\mathrm{Re}(-\sigma-\lambda+\alpha+\beta)>0$, $\mathrm{Re}(\sigma)>\mathrm{Re}\,\beta>0$ and $\mathrm{Re}\,\sigma>\mathrm{Re}\,\alpha>0$.

Many other cases can be obtained by means of a convenient choise of the parameters. We must remember that we can write eq.(7) in terms of the Whittaker function or in terms of the generalized Laguerre polynomials.

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