ON JACOBI EXPANSIONS

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ABSTRACT: We present a new expansion for the product of two hypergeometric functions with different arguments using the expansion for the confluent hypergeometric function. Applications are discussed.

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1. INTRODUCTION.

Many types of expansions including special functions are known in the literature involving generating functions, or sum rules and addition theorems. The Legendre functions and Legendre polynomials are the most exaustively studied cases.

There are two classical methods for obtaining expansions for the special functions: the algebraic method[1] and the group theory method[2] we note that into both cases these expansions involve always functions of the first kind. We call function of the first kind, the first solution of the differential equation.

In a recent paper[3] we presented a methodology to obtain new expansions for a product of two Jacobi functions of different kinds and with different arguments. We note that this methodology is a global methodology because it is valid for functions and for polynomials. In the polynomial case we generalize the well know addition theorem for the Legendre polynomials.

We also note that the generating functions for the product of two ultrasferical polynomials[4] and the product of Laguerre polynomials with ultraspherical polynomials[5] are given only for the functions of first kind.

In this paper we present a way to obtain more general expansions involving a product of two hypergeometric functions with different arguments.

The methodology is to make use of the expansion for the confluent hypergeometric function, namely the Whittaker function, in terms of a series involving the product of a hypergeometric function and a Whittaker function. We discuss some applications of this expansion.

The paper is organized as follows: in the second section we obtain the expansion for two hypergeometric functions with different arguments. In the third section we discuss an expansion for Jacobi functions of different kinds with different arguments and in the last section we present a particular case and discuss many other cases.

2. HYPERGEOMETRIC FUNCTIONS.

We consider the following expansion[6] for the Whittaker function, $M_{\mu,\nu}(z)$ in terms of the product of a hypergeometric function ${}_2F_1(a,b;c;z)$ and another Whittaker function

$$
\begin{aligned}
M_{\xi+\frac{\mu-\nu}{2},\frac{\nu}{2}}(zk^2) &= z^{\frac{\nu-\mu}{2}}(k^2)^{\frac{1+\nu}{2}}e^{-\frac{z}{2}(1-k^2)}\,. \\
&\cdot\sum_{n=0}^{\infty}\frac{1}{n!}(\mu+2n)\frac{\Gamma(\mu+n)}{\Gamma(\mu+2n+1)}(\xi+\frac{1+\mu}{2})_n\;{}_2F_1(-n,\mu+n;\nu+1;k^2)M_{\xi,\frac{\mu}{2}+n}(z)
\end{aligned}
\tag{1}
$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$. This expansion is valid for every k and z, with $\mu,\nu \neq -1,-2,-3\ldots$

The hypergeometric function has the following integral representation in term of the Whittaker function[7]:

$$
\begin{aligned}
&\gamma^{c+1/2}\Gamma(c+a+1/2)\;{}_2F_1(c+a+1/2,c-b+1/2;1+2c;\frac{2\gamma}{2\rho+\gamma}) = \\
&= (s+\gamma/2)^{c+a+1/2}\int_0^{\infty}t^{a-1}M_{b,c}(\gamma t)e^{-st}dt
\end{aligned}
\tag{2}
$$

where $\mathrm{Re}(c+a+1/2)>0$ and $\mathrm{Re}(s)>1/2|\mathrm{Re}(\gamma)|$.

Introducing, in the above expression the following parameters,

$$b=\xi+\frac{\mu-\nu}{2} \quad \text{and} \quad c=\frac{\nu}{2}$$

and using eq.(1) we have,

$$\Gamma(a+\frac{\nu+1}{2})(s+\frac{k^2}{2})^{-a-\frac{\nu+1}{2}}\ {}_2F_1(a+\frac{\nu+1}{2},\nu-\xi-\frac{\mu-1}{2};1+\nu;\frac{2k^2}{2s+k^2})=$$

$$(3)\quad =\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(\mu+n)}{\Gamma(\mu+2n)}\frac{\Gamma(\xi+n+\frac{\mu+1}{2})}{\Gamma(\xi+\frac{\mu+1}{2})}\ {}_2F_1(-n,\mu+n;\nu+1;k^2)\cdot$$

$$\cdot\int_0^{\infty}dt\ e^{-\beta t}t^{a+\frac{\nu-\mu}{2}-1}M_{\xi,\frac{\mu}{2}+n}(t)$$

where $\beta=s+\frac{1-k^2}{2}$, $\mathrm{Re}(s)>\frac{1}{2}\,\mathrm{Re}(k^2)$, $\mathrm{Re}(a+\frac{\nu+1}{2})>0$ and $\mu,\nu\neq-1,-2,\ldots$. In this equation we exchange the integral and sum signals because the serie in eq.(1) is uniform and absolutely convergent.

The integral in the second member of the above equation is calculated by using again eq.(2), then we have,

$$\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(\mu+n)}{\Gamma(\mu+2n)}\frac{\Gamma(\xi+\frac{\mu+1}{2}+n)}{\Gamma(\xi+\frac{\mu+1}{2})}\Gamma(a+n+\frac{\nu+1}{2})\ {}_2F_1(-n,\mu+n;\nu+1;x)\cdot$$

$$(4)\qquad \cdot\ y^{-a-n-\frac{\nu+1}{2}}\ {}_2F_1(a+n+\frac{\nu+1}{2};\mu+2n+1;\frac{1}{y})=$$

$$=\Gamma(a+\frac{\nu+1}{2})(y+x-1)^{-a-\frac{\nu+1}{2}}\ {}_2F_1(a+\frac{\nu+1}{2},\nu-\xi-\frac{\mu-1}{2};\nu+1;\frac{x}{x+y-1})$$

where $\mathrm{Re}(x) > 0$, $\mathrm{Re}(y) > 1$, $Re(a + n + \frac{\nu+1}{2}) > 0$ and $\mu, \nu \neq -1, -2, \ldots$. In this equation we have defined $x = k^2$ and $y = \beta + 1/2$.

Using the linear transformations for the hypergeometric functions[8] which appear in the above expansion we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(\mu+n)}{\Gamma(\mu+2n)} \frac{\Gamma(\xi+n+\frac{\mu+1}{2})}{\Gamma(\xi+\frac{\mu+1}{2})} \Gamma(a+n+\frac{\nu+1}{2}) z^{-n} \cdot$$

$$(5) \quad \cdot\, {}_2F_1(-n, \mu+n; \nu+1; x)\ {}_2F_1(a+n+\frac{\nu+1}{2}, \xi+n+\frac{\mu+1}{2}; \mu+2n+1; \frac{1}{z}) =$$

$$= \Gamma(a+\frac{\nu+1}{2})\ {}_2F_1(a+\frac{\nu+1}{2}, \xi+\frac{\mu+1}{2}; \nu+1; \frac{x}{z})$$

where the parameter z is $z = 1 - y$, and $\mathrm{Re}(a + n + \frac{\nu+1}{2}) > 0$ and $\mu, \nu \neq -1, -2, \ldots$. Finally we introduce the parameter $\mu \to 2\mu + 1$ and making $a = \frac{\nu+1}{2}$ we have,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(n+1+2\mu)}{\Gamma(2n+1+2\mu)} \frac{\Gamma(\xi+n+\mu+1)}{\Gamma(\xi+\mu+1)} \Gamma(n+\nu+1) z^{-n} \cdot$$

$$(6) \qquad \cdot\, {}_2F_1(-n, \mu+n; \nu+1; x)\ {}_2F_1(\nu+n+1, \xi+n+\mu+1; 2n+2\mu+2; \frac{1}{z}) =$$

$$= \Gamma(\nu+1)\ {}_2F_1(\nu+1, \xi+\mu+1; \nu+1; \frac{x}{z})$$

where $\mathrm{Re}(\nu + 1) > 0$ and $\mu \neq -1, -3/2, -2, \ldots$

This equation is a new expansion for the product of two hypergeometric functions with different arguments.

3. JACOBI FUNCTION.

In this section we consider the expansion given by eq.(4) writting the hypergeometric function in terms of the Jacobi functions.

The Jacobi function are given by the following expressions[8]

$$P_m^{(\alpha,\beta)}(x) = \frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)}\ {}_2F_1(-m, \alpha+\beta+m+1; \alpha+1; \frac{1-x}{2})$$

and

$$Q_m^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+m+1)\Gamma(\beta+m+1)}{\Gamma(\alpha+\beta+2m+2)} 2^{\alpha+\beta+m}(x+1)^{-\beta}(x-1)^{-\alpha-m-1}$$

$$\cdot\ {}_2F_1(m+1, \alpha+m+1; \alpha+\beta+2m+2; \frac{2}{1-x})$$

where $P_m^{(\alpha,\beta)}(x)$ is the Jacobi function of first kind and $Q_m^{(\alpha,\beta)}(x)$ is the Jacobi function of second kind. We note that $P_m^{(\alpha,\beta)}(x)$ and $Q_m^{(\alpha,\beta)}(x)$ are linearly independents.

We identify the above expression with the hypergeometric functions in eq.(4) and obtain

$$\sum_{n=0}^{\infty}(\mu+2n)\frac{\Gamma(\mu+n)}{\Gamma(-\xi+n+\frac{\mu+1}{2})} P_n^{(\nu,\mu-\nu-1)}(1-2x)\ Q_{n+\nu}^{(\xi-\nu+\frac{\mu-1}{2}, -\xi-\nu+\frac{\mu-1}{2})}(2y-1) =$$

(7)

$$= \frac{\Gamma(\nu+1)}{\Gamma(-\xi+\nu-\frac{\mu-1}{2})} y^{\xi+\nu-\frac{\mu-1}{2}}(y+x-1)^{-\xi-\frac{\mu+1}{2}} Q_\nu^{(\xi-\nu+\frac{\mu-1}{2}, -\xi-\frac{\mu+1}{2})}(1+2\frac{y-1}{x})$$

where $\mathrm{Re}(x) > 0, \mathrm{Re}(y) > 1, \mathrm{Re}(\nu+1) > 0$ and $\mu \neq -1, -2 \ldots$

We note that in this expression we used again $a = \frac{\mu+1}{2}$.

Now, introducing the following parameters $1-2x = t$ and $2y-1 = t'$ we obtain,

$$
\sum_{n=0}^{\infty}(2n+2\mu+1)\frac{\Gamma(n+2\mu+1)}{\Gamma(-\xi+n+\mu+1)}P_n^{(\nu,2\mu-\nu)}(t)\,Q_{n+\nu}^{(\xi-\nu+\mu,-\xi-\nu+\mu)}(t') =
$$

(8)

$$
= 2^{2\mu-\nu+1}\frac{\Gamma(\nu+1)}{\Gamma(-\xi+\nu-\mu)}(t'+1)^{\xi+\nu-\mu}(t'-t)^{-\xi-\mu-1}\,Q_{\nu}^{(\xi-\nu+\mu,-\xi-\mu-1)}(1+2\frac{t'-1}{t-1})
$$

where $\mathrm{Re}(t) < 1, \mathrm{Re}(t') > 1, \mathrm{Re}(\nu) > -1$ and $\mu \neq -1, -\frac{3}{2}, -2, \ldots$

The above equation is a new expansion for the product of two Jacobi functions of different kinds and different arguments.

4. APPLICATIONS.

In this section we present an application of eq.(8). Let be $\nu = \mu = \lambda - 1/2$ with $\lambda > -1/2$ and consider $\xi = -m (m = 1, 2, \ldots)$, then we have,

$$
\sum_{n=0}^{\infty}(2\lambda+2n)\frac{\Gamma(n+2\lambda)}{\Gamma(m+n+\lambda+1/2)}\,\frac{\Gamma(2\lambda)\Gamma(n+\lambda+1/2)}{\Gamma(\lambda+1/2)\Gamma(n+2\lambda)}P_n^{\lambda}(t)\,Q_{n+\lambda-1/2}^{(-m,m)}(t') =
$$

(9)

$$
= 2^{\lambda+1/2}\frac{\Gamma(\lambda+1/2)}{\Gamma(m)}(t'+1)^{-m}(t'-t)^{m-\lambda-1/2}Q_{\lambda-1/2}^{(-m,m-\lambda-1/2)}(1+2\frac{t'-1}{t-1})
$$

where $P_n^{\lambda}(t)$ is the associated Legendre polynomial, which is related with the Jacobi function by the following expression

$$
P_n^{\lambda}(x) = \frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)}\,\frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1/2)}\,P_n^{(\lambda-1/2,\lambda-1/2)}(x).
$$

where for $\lambda = 0$ the polynomial $P_n^{\lambda}(x)$ vanishes identically for $m \geq 1$. Using the normalization for the Legendre polynomials[9] we obtain the following expansion for the associated Legendre function (second kind associated Legendre function).

$$\sum_{n=0}^{\infty} \frac{1}{n!}(\lambda+n)\frac{\Gamma(n+2\lambda)}{\Gamma(m+n+\lambda+1/2)}(\lambda+1/2)_n\, Q_{n+\lambda-1/2}^{(-m,m)}(t) =$$

$$= 2^{\lambda-1/2}\frac{\Gamma(\lambda+1/2)}{\Gamma(m)}(t+1)^{-m}(t-1)^{m-\lambda-1/2}\, Q_{\lambda-1/2}^{(-m,m-\lambda-1/2)}(2t-1)$$

where $m = 1, 2, \ldots$ and $\lambda > -1/2$ and $t > 1$.

Many other expressions can be obtained using a convenient choise of parameters. E.g., considering $\nu = 0$ and $\mu = 1$ in eq.(7) we have a new sum for the product of a Legendre polynomial with an associated Legendre function of second kind; for $\nu = -1/2$ and $\mu = 0$ we obtain a new sum for the product of a Tchebichef polynomial with an associated Legendre function of second kind, etc.

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