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Abstract: In this note we establish maximum principles for weakly coupled elliptic systems of the form (1) below. The maximum principles here relate to the existence of a sort of barrier function for the homogeneous system. In the case when A has constant entries and the differential operators are all equal to a same self adjoint operator then a necessary and sufficient condition for a maximum principle to hold relate to the characteristic polynomials of all principal minors of A .

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Résumé - Dans cette note, nous établissons des principes du maximum pour des systèmes elliptiques faiblement couplés, de la forme (1) ci-dessous. Ces principes du maximum sont liés à l'existence d'une sorte de fonction barrière pour le système homogène. Dans le cas où A a des coefficients constants et les opérateurs différentiels sont tous égaux à un opérateur auto-adjoint, alors une condition nécessaire et suffisante pour qu'un principe du maximum ait lieu, est donné en terme des polynomes caractéristiques des mineurs principaux de A .

Abstract - In this note we establish maximum principles for weakly coupled elliptic systems of the form (1) below. The maximum principles here relate to the existence of a sort of barrier function for the homogeneous system. In the case when A has constant entries and the differential operators are all equal to a same self adjoint operator then a necessary and sufficient condition for a maximum principle to hold relate to the characteristic polynomials of all principal minors of A .

Version française abrégée - Considérons n opérateurs elliptiques du second ordre à coefficients réels sur un ouvert borné Ω de \mathbb{R}^N :

$$L_k(D) := - \sum_{i,j} b_{ij}^k(x) D_i D_j + \sum_i b_i^k(x) D_i, \quad k = 1, \dots, n.$$

Nous supposons que les coefficients b_{ij}^k et b_i^k sont tels que chacun de ces opérateurs satisfait séparément le principe du maximum (cf. [3], [4]). Soit $A(x) = (a_{ij}(x))$ une matrice $n \times n$ coopérative (c.à.d. $a_{ij}(x) \geq 0$ dans Ω pour $i \neq j$). Nous considérons le système

$$(1) \quad \begin{cases} L_k(D) u_k = \sum_j a_{kj} u_j + f_k & \text{dans } \Omega \\ u_k = 0 & \text{sur } \partial \Omega \quad k = 1 \dots n \end{cases}$$

où $(f_i(x)) = F(x)$ est une fonction vectorielle réelle sur Ω . En notations abrégées:

$$\begin{cases} \mathcal{L}(D) U = A(x) U + F(x) & \text{dans } \Omega \\ U = 0 & \text{sur } \partial \Omega \end{cases}$$

où $U(x) = (u_i(x))$ représente une solution, supposée exister et appartenir à $C^0(\bar{\Omega}) \cap C^2(\Omega)$. On dit que le système (1) satisfait le principe du maximum si $F \geq 0$ (c.à.d. $f_k(x) \geq 0$ dans Ω pour tout k implique $U \geq 0$). Il est bien connu que le système coopératif (1) satisfait ce principe du maximum si, pour chaque $k = 1, \dots, n$,

$$(2) \quad a_k(x) := \sum_j a_{kj}(x) \leq 0 \quad \text{dans } \Omega$$

(cf. [4]). Remarquons que ce résultat, dans le cas d'une seule équation, est plus restrictif que le principe de maximum classique (où, avec des notations évidentes, on suppose seulement $a(x) \leq \hat{\lambda} < \lambda_1$, λ_1 étant la première valeur propre de l'opérateur différentiel, supposée positive).

Dans cette Note nous donnons des extensions du principe du maximum pour le système (1), extensions qui, dans le cas d'une seule équation, contiennent le principe du maximum classique.

Nous dirons que (\mathcal{L}, A) vérifie la propriété (ψ) s'il existe une fonction vectorielle $\psi(x) = (\psi_k(x))$ sur $\bar{\Omega}$ telle que

- (i) $\psi_k(x) > 0$ dans Ω pour tout k ,
- (ii) $\mathcal{L}(D) \psi \geq A \psi$ dans Ω .

Théorème 1. Supposons que (\mathcal{L}, A) vérifie la propriété (ψ) et qu'il existe une constante $C > 0$ et des nombres $0 < \alpha_k < 1$ tels que

$$\psi_k(x) \geq C [\text{dist}(x, \partial\Omega)]^{\alpha_k} \quad \text{pour } x \in \Omega.$$

Alors le système (1) satisfait le principe du maximum pour toute solution U qui est Lipschitzienne sur $\bar{\Omega}$.

Théorème 2. Supposons que (\mathcal{L}, A) vérifie la propriété (ψ) et que Ω joint de la propriété de la sphère intérieure. Alors le système (1) satisfait

le principe du maximum pour toute solution U qui est Lipschitzienne sur $\bar{\Omega}$.

Remarquons que si chaque ψ_i est > 0 dans $\bar{\Omega}$, alors il n'est pas nécessaire de supposer dans les théorèmes précédents U Lipschitzien sur $\bar{\Omega}$ ou la propriété de la sphère intérieure.

Théorème 3. Supposons la matrice coopérative A constante (c.à.d. $a_{ij}(x) = a_{ij}$) et chacun des opérateurs $L_k(D)$ self-adjoint, avec une première valeur λ_1^k positive. Alors le système (1) satisfait le principe du maximum si tous les déterminants des mineurs principaux de la matrice $\Lambda - A$ sont positifs, où Λ représente la matrice diagonale $[\lambda_1^1, \dots, \lambda_1^n]$.

Maximum principles. Consider the following set of second order elliptic operators with real coefficients defined in some bounded domain Ω in \mathbb{R}^N

$$L_k(D) = - \sum_{i,j} b_{ij}^k D_i D_j + \sum_i b_i^k D_i, \quad k = 1, \dots, n.$$

No regularity on the coefficients is assumed. However we require the conditions that will provide a maximum principle for each L_k separately, cf. [3], [4]. Let $A(x) := (a_{ij}(x))$ be a $n \times n$ cooperative matrix, that is, $a_{ij}(x) \geq 0$ in Ω for $i \neq j$.

In this note we establish maximum principles for the system

$$(1) \quad \begin{cases} L_k(D) u_k = \sum_j a_{kj} u_j + f_k & \text{in } \Omega \quad k = 1 \dots n \\ u_k = 0 & \text{on } \partial \Omega \end{cases}$$

or, in a short notation

$$\mathcal{L}(D) U = A U + F, \quad U = 0 \quad \text{on } \partial \Omega$$

where $U(x) = (u_1(x), \dots, u_n(x))$ is a solution of (1) in the classical sense, i.e. $u_k \in C^0(\bar{\Omega}) \cap C^2(\Omega)$, and $F(x) = (f_1(x), \dots, f_n(x))$ is a given vector whose components are real valued functions in Ω .

By a maximum principle we mean the statement that $F \geq 0$ (i.e., $f_k(x) \geq 0$ in Ω for all k) implies $U \geq 0$.

It is well known that the cooperative system (1) satisfies a maximum principle if

$$(2) \quad a_k(x) := \sum_j a_{kj}(x) \leq 0 \text{ in } \Omega$$

for all k . cf. [4]. Observe that such a result does not yield the classical maximum principle for a single equation, $L u = \lambda(x) u + f$, which requires only that $a(x) \leq \hat{\lambda} < \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of $L \phi = \lambda a \phi$ in Ω , $\phi = 0$ on $\partial \Omega$, which is supposed to be positive.

We say that (\mathcal{L}, A) satisfies Property (ψ) if there exists a vector function $\psi(x) = (\psi_1(x), \dots, \psi_n(x))$ with $\psi_k \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ such that (i) $\psi(x) > 0$ in Ω (i.e. $\psi_k(x) > 0$, for all k), and (ii) $\mathcal{L}(D) \psi \geq A \psi$ in Ω . Observe that we are not assuming that each $\psi_k(x) > 0$ on $\partial \Omega$.

Theorem 1. Suppose that (\mathcal{L}, A) satisfies Property (ψ) and that there exist a constant $C > 0$ and numbers α_k , with $0 < \alpha_k < 1$ for all k , such that

$$(3) \quad \psi_k(x) \geq C [\text{dist}(x, \partial \Omega)]^{\alpha_k}$$

for all $x \in \Omega$. Then (1) has a maximum principle for all solutions which are Lipschitz continuous in $\bar{\Omega}$, and correspond to $F > 0$ in Ω . [If $\mathcal{L}(D) > A \psi$ then one could take any $F \geq 0$ in Ω].

Sketch of the proof. We reduce our system (with the hypotheses of the present theorem) to a system satisfying condition (2) for the variables $v_k(x)$ defined by $v_k = u_k \cdot \psi_k^{-1}$.

Remark 1. If some is positive $\psi_k(x)$ is positive in the whole of $\bar{\Omega}$, then (3) is trivially satisfied for that k , and in this case the corresponding u_k does not have to be assumed Lipschitz continuous in $\bar{\Omega}$. Observe that such an assumption is made only to guarantee that v_k is continuous up to the boundary in the case where ψ_k vanishes there. In case of $\psi > 0$ in $\bar{\Omega}$ we

have the following strengthening of Theorem 1: system (1) has a maximum principle for all $F \geq 0$ and all solutions V (not necessarily Lipschitz continuous).

Remark 2. If $L_1(D) = \dots = L_n(D) = L(D)$ where $L(D)$ is self-adjoint with positive first eigenvalue $\lambda_1(\Omega)$, and $a_k(x)$ (defined above) is an L^∞ function defined in a open set $\Omega' \supset \bar{\Omega}$, with

$$(4) \quad a_k(x) \leq \hat{\lambda} < \lambda_k(\Omega) \quad \text{for all } x \in \Omega,$$

where $\hat{\lambda}$ is a constant, then (\mathcal{L}, A) satisfies Property (ψ) with $\psi_k(x) > 0$ in $\bar{\Omega}$.

So in this case, under hypothesis (4) a maximum principle holds for system (1). The proof consists simply in taking $\psi = (\phi, \dots, \phi)$ where $\phi > 0$ in Ω'' is a first eigenfunction for the eigenvalue problem $L(D)\phi = \lambda\phi$ in Ω'' , $\phi = 0$ on $\partial\Omega''$, where $\Omega' \supset \Omega'' \supset \Omega$ is chosen in such a way that $\hat{\lambda} \leq \lambda_1(\Omega'') < \lambda_1(\Omega)$.

Theorem 2. Suppose that (\mathcal{L}, A) satisfies Property (ψ) , and that the domain Ω satisfies the interior sphere condition. Then (1) has a maximum principle for all solutions which are Lipschitz continuous in $\bar{\Omega}$, and correspond to $F \geq 0$ and $F \neq 0$ in Ω .

Sketch of the proof. It follows from the fact that A is cooperative and from Lemma 1 in [5] that there exists a constant $C > 0$ such that $\psi_k(x) \geq C \text{ dist}(x, \partial\Omega)$ for all x in Ω .

Let us denote by

$$\mu_k = \text{Min} \{ \mu \in \mathbb{R} : u_k + \mu \psi_k(x) \geq 0 \text{ in } \Omega \}$$

with the provisos that we take $\mu_k = 0$ if $u_k \geq 0$. If there is some u_k which is negative somewhere in Ω , then the corresponding μ_k is positive. We may assume without loss of generality that $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$. And let us suppose, by contradiction, that $\mu_1 > 0$. Let M be the diagonal matrix $[\mu_1, \mu_2, \dots, \mu_n]$. Then the vector $V = U + M\psi$ has nonnegative components and it satisfies the system of inequalities

$$L v \geq A v + F + (M A - A M) \psi.$$

Writing down explicitly the first one we have

$$L_1 v_1 \geq \sum_j a_{1j} v_j + f_1 + \sum_j (\mu_1 - \mu_j) a_{1j}$$

which implies $v_1 \equiv 0$. Then it follows $L_1 v_1 \geq a_{11} v_1$, which gives via the strong maximum principle that $v_1 > 0$ in Ω . By Lemma 1 in [5] we see that there is $\varepsilon > 0$ s.t. $v_1 - \varepsilon \psi_1 \geq 0$, which contradicts the minimality of μ_1 .

Theorem 3. Suppose that the cooperative matrix A has constant entries, and that the operators $L_k(D)$ are selfadjoint with a positive first eigenvalue λ_1^k . Then (1) has a maximum principle provided the principal minors of the matrix $\Lambda - A$ have positive determinants, where Λ is the diagonal matrix $[\lambda_1^1, \dots, \lambda_1^n]$.

Remark 3. The matrix $\Lambda - A$ is of a type well studied in the literature [1], the so-called M-matrices. In view of known results, one has only to assume that the n first principal minors of $\Lambda - A$ have positive determinants, cf. [1].

Remark 4. Suppose that the differential operators $L_k(D)$ are all equal to the same operator $L(D)$, whose first eigenvalue λ_1 is positive. In this case, a maximum principle holds for (1) provided $p_j(\lambda_1) > 0$ $j = 1, \dots, n$ where $p_j(\lambda)$ is the characteristic polynomial of the first $j \times j$ principal of A . As a matter of fact, in this case those conditions are not only sufficient but they are also necessary provided $f_j \geq 0$ and $f_j \equiv 0$ in Ω , of [2].

Sketch of the proof. The case $n = 2$ is proved readily by inverting $L_k - a_{kk}$, $k = 1, 2$, and rewriting (1) as

$$u_1 - a_{12} B_1 u_2 = B_1 f_1$$

$$-a_{21} B_2 u_1 + u_2 = B_2 f_2$$

where B_k is the inverse of $L_k - a_{kk}$ under Dirichlet boundary conditions, which is a positive operator. Then

$$u_1 = (I - a_{12} a_{21} B_1 B_2)^{-1} (B_1 f_1 + a_{12} B_1 B_2 f_2).$$

From our assumption the norm of the operator $a_{12} a_{21} B_1 B_2$ is strictly less than one. So the result follows. The rest of the proof goes by induction.

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