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TOURNAMENTS THAT ADMIT EXACTLY
ONE HAMILTONIAN CYCLE

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On the Simple Quotients of Tournaments that Admit Exactly one Hamiltonian Cycle

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Abstract: In this paper we describe the tournaments that admit exactly one hamiltonian cycle, by considering their simple quotients as normal tournaments.

Introduction

In 1960, P. Camion [3] gave some characterizations for the tournaments admitting a unique hamiltonian cycle. In 1969, R.J. Douglas [5] described a structural characterization of these tournaments. In this paper we give a presentation of these tournaments, which we call *Douglas tournaments*, by relating them to the class of the normal tournaments by means of the following result (Theorems 3.10 and 3.11):

Theorem: Let H_n be a hamiltonian tournament with $cc(H_n) = k \geq 3$. H_n is a Douglas tournament if and only if:

1.1) H_n has as simple quotient a tournament Q_m ($m \geq 5$) such that:

- a) Q_m is normal;
- b) the subtournament of the poles in Q_m is transitive;
- c) the poles of Q_m are all of class 1;
- d) between two poles x_i and x'_j of Q_m of class 1, the following rules of adjacencies

hold $x_i \rightarrow x'_j$ implies $j \leq i + 1$.

1.2) H_n can be constructed from Q_m by replacing all the vertices of Q_m but the vertices a_2, \dots, a_{k-1} of its characteristic cycle A_k , by some transitive tournament.

2) H_n is the composition of a singleton and two transitive tournaments with a 3-cycle.

1. Notations and Preliminary Results

A tournament T_n is a digraph in which every pair of vertices is joined by exactly one arc. We will often identify T_n or a cycle C with their vertex sets $V(T_n)$ or $V(C)$.

If the arc joining the vertices v_i and v_j is directed from v_i to v_j , v_i is said to be a predecessor of v_j and v_j a successor of v_i and we prefer write $v_i \rightarrow v_j$ instead of (v_i, v_j) . Moreover $A \rightarrow B$ denotes that the vertices of the subtournament A are all predecessors of all the vertices of the subtournament B .

Let T_n be a tournament of order n . A cycle C of T_n is coned by a vertex v (i.e. v cones C) if there exists a vertex $v \in V(T_n) - V(C)$ such that either $v \rightarrow V(C)$ or $V(C) \rightarrow v$.

Let H_n be a hamiltonian tournament. If no vertex of $H_n - C$ cones C , C is said to be non-coned in H_n and the vertices of $H_n - C$ are called poles associated to C .

A non-coned cycle C of H_n is minimal if each cycle C' such that $V(C') \subset V(C)$ is coned by at least one vertex of H_n . A minimal cycle is characteristic if it possesses the shortest length of the minimal cycles and its length is the cyclic characteristic $cc(H_n)$ of H_n . Then the difference $n - cc(H_n)$ is called cyclic difference $cd(H_n)$ of H_n .

A vertex v of H_n is called a neutral vertex of H_n if $H_n - v$ is hamiltonian, and the number of neutral vertices of H_n is denoted by $\nu(H_n)$. We recall that between $\nu(H_n)$ and $cd(H_n)$, the inequality $\nu(H_n) \geq cd(H_n)$ holds (see [2]).

Definition 1.1: The tournament A_n ($n \geq 4$), with vertex set $V(A_n) = \{a_1, a_2, \dots, a_n\}$ and arc set $E(A_n) = \{a_i \rightarrow a_j / j < i - 1 \text{ or } j = i + 1\}$ is called the bineutral tournament of order n .

Remark: It is known that A_n is the unique hamiltonian tournament that has exactly 2 neutral vertices. Moreover, for $n \geq 5$, $\{a_1, a_2, a_{n-1}, a_n\}$ is its maximal transitive subtournament.

The vertices of a subtournament S of T are equivalent if each vertex, not in $V(S)$, cones $V(S)$. If the vertices of T_n can be partitioned into disjoint subtournaments $S^{(1)}, S^{(2)}, \dots, S^{(m)}$ of equivalent vertices and R_m denotes the tournament of the m vertices w_1, w_2, \dots, w_m in which $w_i \rightarrow w_j$ if and only if $S^{(i)} \rightarrow S^{(j)}$, then $T_n = R_m(S^{(1)}, S^{(2)}, \dots, S^{(m)})$ is the composition of the m components $S^{(1)}, S^{(2)}, \dots, S^{(m)}$ with the quotient R_m .

A tournament T_n is simple if $T_n = R_m(S^{(1)}, S^{(2)}, \dots, S^{(m)})$ implies that either $m = 1$

or $m = n$.

We also recall the following results about the quotient tournaments:

Proposition 1.2: The quotient tournament R_m is isomorphic to a subtournament of T_n .

Proposition 1.3: Every tournament T_n , with $n \geq 2$, admits exactly one simple quotient tournament.

Proposition 1.4: A tournament H_n is hamiltonian if and only if every one of its quotient tournament is hamiltonian.

Proposition 1.5: Let H_n be a hamiltonian tournament and Q_m one of its quotient tournaments, then $cc(H_n) = cc(Q_m)$.

Now we recall the definitions and the structural characterization of normal tournaments (see Nos. 3 and 6 of [4]).

Definition 1.6: A hamiltonian tournament H_n is said to be *normal* if it contains only one minimal cycle (the characteristic one) or equivalently if $cd(H_n) = \nu(H_n)$.

Proposition 1.7: Let H_n be a normal tournament, then its characteristic cycle C_k is either the 3-cycle A_3 ($k = 3$) or a bineutral tournament A_k ($k \geq 4$).

Proposition 1.8: Let H_n be a normal tournament with cyclic characteristic k ($k \geq 3$) and let A_k be its characteristic cycle. A pole z , associated to A_k , must have the following adjacencies with respect to A_k :

- 1) $(a_{i+1}, a_{i+2}, \dots, a_k) \rightarrow z \rightarrow (a_1, a_2, \dots, a_i)$ $i = 1, 2, \dots, k-1$;
- 2) $(a_i, a_{i+2}, a_{i+3}, \dots, a_k) \rightarrow z \rightarrow (a_1, a_2, \dots, a_{i-1}, a_{i+1})$ $i = 1, 2, \dots, k-1$.

Definition 1.9: The pole z is called *pole of kind i and class 1 or class 2* and denoted by x_i or y_j if its adjacencies are given by the previous conditions 1) or 2) respectively.

Theorem 1.10: A tournament H_n is normal if, and only if, the following conditions are true:

- a) H_n allows a bineutral tournament A_k ($k \geq 4$) or the 3-cycle A_3 as a minimal tournament;
- b) the poles associated to A_k are of the $(k-1)$ kinds of class 1 (x_1, x_2, \dots, x_{k-1}) and of the $(k-1)$ kinds of class 2 (y_1, y_2, \dots, y_{k-1}), considered in Definition 1.9;
- c) the subtournament P_{n-k} of the poles associated to A_k is not hamiltonian;

d) P_{n-k} allows the following transitive composition:

$$P_{n-k} = Tr_{k+1}^*(T^{(0)}, T^{(1)}, \dots, T^{(k)}) \text{ (that is, } T^{(i)} \rightarrow T^{(j)} \iff i > j)$$

where:

$T^{(0)}$	is formed by poles of type	x_1
$T^{(1)}$	is formed by poles of type	x_1, x_2, y_1
$T^{(2)}$	is formed by poles of type	x_1, x_2, x_3, y_2
...		
$T^{(i)}$	is formed by poles of type	$x_{i-1}, x_i, x_{i+1}, y_i$
...		
$T^{(k-2)}$	is formed by poles of type	$x_{k-3}, x_{k-2}, x_{k-1}, y_{k-2}$
$T^{(k-1)}$	is formed by poles of type	$x_{k-2}, x_{k-1}, y_{k-1}$
$T^{(k)}$	is formed by poles of type	x_{k-1} ,

and each component different from $T^{(0)}$ and $T^{(k)}$ can be empty or not.

2. The \mathcal{D} Tournaments

Definition 2.1: We call a *tournament of Douglas* every tournament that admits exactly one hamiltonian cycle. We denote by D_n one such tournament of order n , and by \mathcal{D} the class of all these tournaments. We denote by $v_1 \implies v_2 \implies \dots \implies v_n \implies v_1$ the hamiltonian cycle of D_n .

We now prove the following propositions:

Proposition 2.2: Every component $S^{(i)}$ of a D_n tournament is transitive.

Proof: In fact, if it were not transitive then we would have at least two distinct hamiltonian paths in $S^{(i)}$, hence two hamiltonian cycles in D_n . \square

Proposition 2.3: Every hamiltonian subtournament H_m of a tournament D_n is a D_m tournament.

Proof: If $m = n - 1$ this is true, for otherwise every hamiltonian cycle of D_m would be extended to a hamiltonian cycle of D_n .

If $m < n - 1$, two different possibilities must be considered:

1) There exists a chain of Hamiltonian subtournaments $H'_{m+1}, H'_{m+2}, \dots, H'_{n-1}$ such that $V(H_m) \subset V(H'_{m+1}) \subset \dots \subset V(H'_{n-1}) \subset V(H_n)$. Then step by step, it follows that $H'_{n-1} \in \mathcal{D}, H'_{n-2} \in \mathcal{D}, \dots, H_m \in \mathcal{D}$.

2) There exists a hamiltonian subtournament H'_s ($m \leq s < n - 1$), such that between

H_m and H'_s there is a chain of hamiltonian subtournaments as in 1), whereas, for each subtournament T'_{s+1} such that $V(H'_s) \subset V(T'_{s+1})$, T'_{s+1} is not hamiltonian, i.e. H'_s is coned by all the vertices $V(H_n) - V(H'_s) = \{v_1, v_2, \dots, v_{n-s}\}$. Then H_n must be the composition $H_n = H'_{n-s+1}(\{v_1\}, \{v_2\}, \dots, \{v_{n-s}\}, H'_s)$, which is a contradiction to Proposition 2.2. \square

Corollary 2.4: If $D_n \in \mathcal{D}$ is the composition $D_n = R_m(S^{(1)}, S^{(2)}, \dots, S^{(m)})$, then $R_m \in \mathcal{D}$.

Proof: Since R_m is isomorphic to a subtournament of D_n , R_m belongs to \mathcal{D} by Proposition 2.3. \square

Corollary 2.5: A hamiltonian tournament T_n ($n \geq 4$) is a D_n tournament if and only if for every quadruple of different vertices a, b, c, d such that the pairs (a, b) and (c, d) are separated in a cycle of T_n , then the same pairs are separated in every other cycle of T_n containing them.

Proof: If T_n is a D_n tournament then the result is obvious, since the induced ordering is the same in all the hamiltonian subtournaments of D_n .

On the other hand, we consider a hamiltonian tournament T_n which is not a D_n tournament, and let C and C' be two of its hamiltonian cycles. There exist two consecutive vertices a and c in C and non consecutive in C' . Hence there exist a vertex b in the path in C' joining a to c , and a vertex d in the path in C' from c to a . Therefore the pairs (a, b) and (c, d) are separated in C' and not in C . \square

3. Relations between \mathcal{D} and Normal Tournaments

We now characterize the tournaments of Douglas in terms of normal tournaments.

Theorem 3.1: Every characteristic cycle C_k of a D_n tournament (with $cc(D_n) = k \geq 4$) is bineutral.

Proof: We shall prove by absurd. So let's assume there exist 3 neutral vertices in C_k , namely x, y, z , then there exist three poles called x', y', z' coning $C_k - x, C_k - y, C_k - z$ (see Lemma 15, [2]), respectively. If, for example, $x' \rightarrow y', x \rightarrow x'$ and $y \rightarrow y'$, we obtain $x \rightarrow x' \rightarrow (C_k - x), y \rightarrow y' \rightarrow (C_k - y)$.

We have to consider two different cases:

(i) The hamiltonian path from x to y in C_k contains some other vertices, let's say u

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