ON THE COMPLETE DIGRAPHS WHICH ARE SIMPLY DISCONNECTED

Davide C. Demaria

and

J. Carlos S. Kiihl

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Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
IMECC – UNICAMP
Caixa Postal 6065
13.081 – Campinas – SP
BRASIL

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On the Complete Digraphs Which Are Simply Disconnected*

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1. Introduction

In [5], Burzio and Demaria obtained an application of the regular homotopy of digraphs, introduced in [2], [3] and [4], by giving a structural characterization of tournaments T, called *simply disconnected tournaments*, whose fundamental group $Q_1(T)$ is non trivial. In [4], Burzio and Demaria have obtained a new characterization for the simply disconnected tournaments by using coned 3-cycles.

In this paper we extend those results to the case of digraphs D which are complete, and we get analogous results if $Q_1(D) \neq 0$. First of all, we must generalize the concept of simple quotient for every type of digraph and we prove the following theorem:

Theorem 4.4. Every (non trivial) digraph has a unique simple quotient.

In this way we obtain the following theorems:

Theorem 5.3. A complete digraph D is simply disconnected if and only if its simple quotient is a highly regular tournament.

Theorem 5.10. A complete digraph D_n is simply disconnected if and only if:

a) there exists in D_n a non coned 3-cycle;

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b) every symmetric pair and every 3–cycle in D_n are shrinkable in D_n.

2. Some Definitions and Notations

Definition 2.1. Let V be a finite non–empty set and E a set of ordered pairs $(u, v) \in V \times V$, such that $u \neq v$. We call the pair $D = (V, E)$ a *directed graph* or *digraph*. The elements of V are called the *vertices* of D, the cardinality of V the *order* of D and the elements of E the *arcs* of D. Moreover, we write $u \to v$ instead of (u, v), and we call u a *predecessor* of v and v a *successor* of u.

Remark 1. Given two distinct vertices u and v we have *a priori* four possibilities, and then four types of arcs:

(1) There is no oriented arc between u and v, and then we shall denote $u|v$ the *null arc*;

(2) there is the oriented arc (u, v), but not the arc (v, u), and then we shall denote the *simple arc* $u \to v$;

(3) there is the oriented arc (v, u), but not the arc (u, v), and then we shall denote the *simple arc* $u \leftarrow v$;

(4) there are both oriented arcs (u, v) and (v, u), and then we shall denote the *double arc* $u \leftrightarrow v$. A double arc is also called a *symmetric pair*.

Definition 2.2. A digraph is called *oriented* if between two distinct vertices there is at most one ordered arc, that is, the possible arcs are either simple arcs or null arcs. A digraph is called a *non oriented graph* if between two distinct vertices there is either a double arc or a null arc. A digraph is called *complete* if between two distinct vertices there is at least one ordered arc, the possible arcs in this case are either simple or double arcs.

Definition 2.3. A digraph T is called a *tournament* if between every pair of distinct vertices there is one and only one arc. A tournament T is called *hamiltonian* if it contains a spanning cycle, i.e., a cycle passing through all the vertices of T.

Definition 2.4. Let $D = (V, E)$ and $D' = (V', E')$ be digraphs. A function $f : V \to V'$ is said to be a *homomorphism* between D and D' if, for every $u, v \in V$, $u \to v$ implies either $f(u) \to f(v)$ or $f(u) = f(v)$.

Remark 2. We can consider two kinds of dualities for a given digraph D:

(a) The first one is the dually oriented digraph $\overline{D}$, which is obtained by changing the orientation of the arcs. In this case, both digraphs are of the same type.

(b) The second kind is the digraph $\overline{\overline{D}}$; which is obtained by mantaining the simple arcs and by changing the null arcs into double arcs, and vice–versa. In this case oriented digraphs become complete digraphs, and vice–versa. On the other hand tournaments and non oriented graphs don't change.

It is known that given a tournament T we can associate an algebraic structure to T in a natural way. In fact we have (see [7]):

Proposition 2.5. A tournament T becomes the commutative groupoid $A(T)$ if we define the following binary operation $*$:

$$(\forall u, v \in T)\ u * v = v * u = \begin{cases} u, & \text{if } u \to v \text{ or } u = v; \\ v, & \text{if } v \to u. \end{cases}$$

Remark 3. Similarly we can associate with T the dual commutative groupoid $A'(T)$, defining:

$$(\forall u, v \in T)\ u * v = v * u = \begin{cases} u, & \text{if } v \to u \text{ or } u = v; \\ v, & \text{if } u \to v. \end{cases}$$

Remark 4. Every homomorphism between two tournaments T and T' is also an algebraic homomorphism between the commutative groupoids $A(T)$ and $A(T')$ ($A'(T)$ and $A'(T')$), and vice–versa.

The same definitions can be applied to the case of a digraph D of any type, and then we have the associated groupoids $A(D)$ and $A'(D)$, which are dual between themselves. In this case we set for $A(D)$:

$$(\forall u, v \in D)\ u * v = \begin{cases} u, & \text{if } u \to v \text{ or } u = v; \\ v, & \text{if } v \not\to u; \end{cases}$$

and for $A'(D)$:

$$(\forall u, v \in D)\ u * v = \begin{cases} u, & \text{if } u \not\to v \text{ or } u = v; \\ v, & \text{if } u \to v. \end{cases}$$

Remark 5. In general the two groupoids are not commutative. In fact, for example, in the first case from $u \leftrightarrow v$ we get

$$u * v = u \quad \text{and} \quad v * u = v;$$

on the other hand from $u|v$ we get:

$$u * v = v \quad \text{and} \quad v * u = u.$$

Remark 6. While the homomorphisms between two tournaments coincide with the algebraic homomorphisms between the associated groupoids, this is not true in the general case, for there are homomorphisms between digraphs which are not algebraic homomorphisms between the associated groupoids. In fact, for example, given the 3-cycle $C : u \to v \to w \to u$ and the symmetric pair $D : x \leftrightarrow y$, if we define $f : C \to D$ by $f(u) = x$; $f(v) = f(w) = y$; then in $A(C)$ and $A(D)$ we have

$$u * w = w, \quad f(u) * f(w) = x * y = x \quad \text{and} \quad f(w) = y.$$

Remark 7. We can still associate other two groupoids $\overline{A}(D)$ and $\overline{A}'(D)$ to the digraph D, in the following way:

$$(\forall u, v \in D)\ u * v = \begin{cases} u, & \text{if} \quad u = v \quad \text{or} \quad u \leftarrow v; \\ v, & \text{if} \quad u \not\leftarrow v. \end{cases}$$

and

$$(\forall u, v \in D)\ u * v = \begin{cases} u, & \text{if} \quad u = v \quad \text{or} \quad u \not\leftarrow v; \\ v, & \text{if} \quad u \leftarrow v. \end{cases}$$

We observe that if we change the orientation of the arcs, then $A(D)$ becomes $\overline{A}(D)$ and $A'(D)$ becomes $\overline{A}'(D)$; while in changing the double arcs into null arcs and the null arcs into double arcs, then $A(D)$ becomes $\overline{A}'(D)$, and $A'(D)$ becomes $\overline{A}(D)$.

3. Quotient Digraphs

We say that a subset X of vertices of a digraph D is a *set of equivalent vertices* if for any vertex u in $D - X$ the oriented arcs from u to any vertex v in X are all of the same type. Of course the type of oriented arc can change if we vary the vertex u in $D - X$.

If $p : A(D) \to A(Q)$ is a surjective algebraic homomorphisms between the groupoids which are associated to the two digraphs D_n and Q_m, of order n and m, respectively, then we have the groupoid $A(Q)$ is isomorphic to the quotient groupoid $A(D)/p$. For, if we consider the m pre-immages of the vertices $v_1, \dots, v_m$ in Q_m and we set $S^{(i)} = p^{-1}(v_i)$, $i = 1, 2, \dots, m$, we can subdivide the n vertices of D in m disjoint subdigraphs $S^{(1)}, S^{(2)}, \dots, S^{(m)}$ of equivalent vertices, because the type of arc which joins the vertex v_i to v_j is of the same type of the arcs which join every vertex in $S^{(i)}$ to every vertex in $S^{(j)}$.

If we are in the conditions described above then we write

$$D_n = Q_m(S^{(1)}, S^{(2)}, \dots, S^{(m)})$$

and we say the digraph D_n is the *composition* of the m digraphs $S^{(1)}, S^{(2)}, \ldots, S^{(m)}$. The subdigraphs $S^{(1)}, S^{(2)}, \ldots, S^{(n)}$ are called the *components* of the digraph D_n a Q_m is the *quotient* of the digraph D_n.

We say a digraph is *simple* if the composition

$$D_n = Q_m(S^{(1)}, S^{(2)}, \ldots, S^{(m)})$$

implies $m = 1$ or $m = n$; that is, if the quotient Q_m or the components $S^{(i)}$ concide with the trivial digraph of order 1.

4. Properties of the Quotient Digraphs

Proposition 4.1. Let D be a digraph, Q be one of the quotient digraphs of D and S be one of the components of D with respect to Q, then S is a set of equivalent vertices. Vice–versa, if X is a set of equivalent vertices, then X is a component of D.

PROOF. The first statement is obvious. For the second, we consider the partition of D in the subset X and the singular subsets of $D - X$. □

Proposition 4.2. Let X and Y be two sets of equivalent vertices of a digraph D. If $X \cap Y \neq \emptyset$ and $X \cup Y \neq D$, then $X \cup Y$ is a set of equivalent vertices.

PROOF. In fact, let u be a vertex in $X \cap Y$ and let v be a vertex in $D - (X \cup Y)$, for all vertex w in $X \cup Y$ the oriented arc between v and w is of the same type of the oriented arc between v and u. □

Proposition 4.3. Let D be a digraph and Q one of its quotient digraphs, then Q is isomorphic to a subdigraph E of D.

PROOF. In fact we can construct E by choosing one vertex in each components of D. □

Theorem 4.4. Every (non trivial) digraph has a unique simple quotient.

PROOF. By absurd, let $P = [S^{(1)}, S^{(2)}, \ldots, S^{(h)}]$ and $Q = [T^{(1)}, T^{(2)}, \ldots, T^{(k)}]$ be two different partitions of the digraph D in components of equivalent vertices, such that the quotient digraphs P_h and Q_k are not trivial and simple. Then we should have:

$$D = P_h(S^{(1)}, \ldots, S^{(h)}) = Q_k(T^{(1)}, \ldots, T^{(k)}).$$

Let's suppose $h > 2$. Since the two partitions are distinct, there exist necessarily two components S and T different between themselves and having non empty intersection. If we assume S is not contained in T and considering the other components T having non empty intersection with S, we have that at least one among these cannot be contained in S, for otherwise in the partition Q we can replace these particular components T by the unique component S, but this contradicts the simplicity of Q_k.

Therefore there exists at least a component S and a component T such that $S \cap T \neq \emptyset$, $S \not\subset T$ and $T \not\subset S$, and we choose them to be $S^{(1)}$ and $T^{(1)}$. We number the components S in such a way that $S^{(1)}, S^{(2)}, \ldots, S^{(r)}$ intersect $T^{(1)}$, while the rest $S^{(r+1)}, S^{(r+2)}, \ldots, S^{(h)}$ don't intersect $T^{(1)}$.

We must distinguish two cases:

(a) $1 < r < h$.

Since the components $S^{(1)}, S^{(2)}, \ldots, S^{(r)}$ intersect $T^{(1)}$, the union $U = S^{(1)} \cup \cdots \cup S^{(r)}$ is a subset of equivalent vertices, hence to the partition $P' = [U, S^{(r+1)}, S^{(r+2)}, \ldots, S^{(h)}]$ we can associate a new composition of D, which will induce a composition of P_h. But this contradicts the simplicity of P_h.

(b) $r = h$.

The oriented arcs between a vertex u in $S^{(1)} - T^{(1)}$ and any vertex in the union $U = S^{(2)} \cup S^{(3)} \cup \cdots \cup S^{(h)}$ are all of the same type, because every component $S^{(i)}$, $i = 2, 3, \ldots, h$ intersects the component $T^{(1)}$. On the other hand, since u is a vertex in $S^{(1)}$, the oriented arcs between any vertex in $S^{(1)}$ and any vertex in U are of the same type; but this means that the partition $[S^{(1)}, U]$ is a composition of D. Absurd, for we have supposed the quotient P_h is simple of order $h > 2$.

Finally, if $h = 2$, k must be also equal to 2, for otherwise it would be sufficient to interchange the two compositions between themselves and repeat the previous considerations. On the other hand the digraphs P_2 and Q_2 must be isomorphic, for if P_2 is the null arc, the digraph D is disconnected, if P_2 is the simple arc, the digraph D is weakly, but not strongly connected, if P_2 is the double arc, the digraph D is strongly connected. □

Remark 1. It follows from the proof that for $h > 2$ we have not only a unique simple quotient, but also a unique partition in components. On the other hand in general for $h = 2$ one may have more partitions. For example, if $D = [u, v, w;\ u \leftrightarrow v,\ u \leftrightarrow w,\ v \leftrightarrow w]$ we have the following partitions

$$P = [[u, v], w], \quad Q = [[u, w], v] \quad \text{and} \quad R = [[v, w], u].$$

Remark 2. We recall that a digraph is said to be *hamiltonian* if there is a cycle passing through all the vertices. So we shall consider *a symmetric pair as a hamil-*

tonian cycle.

Proposition 4.5. A complete digraph is hamiltonian if and only if every one of its (non trivial) quotients are hamiltonian.

PROOF. First of all from results due to Rado (1943), Roy (1958) and Camion (1959) we have that *a complete digraph D is hamiltonian if and only if the tournament T_2 (the simple oriented arc) is not the simple tournament related to D.* Now if we observe that either the initial digraph or any of its quotient digraphs have the same simple quotient, then we get the assertion. □

5. Complete Digraphs which are Simply Disconnected

Definition 5.1. A tournament T is called *regular* if for each vertex $v \in T$ the number of predecessors and successors of v is the same (hence the order of T is odd). A tournament T_{2m+1} is called *highly regular* if there exists a cyclical ordering $v_1, v_2, \ldots, v_{2m+1}, v_1$ on the vertices of T_{2m+1} such that $v_i \to v_j$ if and only if v_j is one of the first m successors of v_i in the cyclical ordering of T_{2m+1}.

Definition 5.2. A digraph D is called *simply connected* if its first homotopy group $Q_1(D)$ is trivial. A digraph D is called *simply disconnected* if $Q_1(D)$ is non trivial.

We have the following theorem.

Theorem 5.3. A complete digraph D is simply disconnected if and only if its simple quotient is a highly regular tournament.

This theorem is a generalization of the analogous theorem for tournaments (see Theorem 3.9, in [5]). For the proof we need the following lemmas.

Lemma 5.4. A complete digraph D is simply connected if and only if every one of its non trivial quotient digraphs D^* is simply connected.

PROOF. It is analogous to the proof of Proposition 2.1 in [5] for tournaments. □

Lemma 5.5. For every complete digraph D_n, of order n, there exists at least one tournaments T_n of order n which is a subdigraph of D_n.

PROOF. In fact it is sufficient to eliminate one oriented arc from every symmetric pair. □

Lemma 5.6. Let D_n and F_n be two given complete digraphs both of them having order n, such that F_n is a subdigraph of D_n. Then if F_n is simply connected, so is D_n.

PROOF. In fact every edge–loop in the polyhedron associated to the digraph D_n is null–homotopic, since it is null-homotopic in the sub–polyhedron associated to the digraph F_n. □

Proving now Theorem 5.3:

(a) In a complete digraph D, if its simply quotient is a highly regular tournament, then D is simply disconnected.

This follows directly from Lemma 5.4 and the property which says a highly regular tournament is simply disconnected.

(b) Let D_n be a complete digraph which is simply disconnected, then D_n has a highly regular tournament as simple quotient.

In fact, a tournament T_n which is a subdigraph of D_n, it is also simply disconnected, and therefore it is the composition of a highly regular tournament by the analogous theorem for tournaments.

It is now sufficient to prove that the simple arcs which ought to be replaced by double arcs in order to pass from the tournament T_n to the initial complete digraph D_n they all belong to subtournaments which are components of T_n.

First of all from Lemma 5.4 we have that the replacements of simple arcs by double arcs, which is done in the components of T_n, they don't change the first homotopy group of the digraphs which are obtained one by one.

On the other hand, if we assume by absurd there exists a double arc with vertices u and v, which belongs to two different components of T_n, then we have that:

1) if we construct a 3–cycle C in T_n using the vertices u an v, then the edge–loop determined by C is not null homotopic in the polyhedron associated to T_n, for it is a generator of $Q_1(T_n)$ (cfr. [5], Proposition 3.6);

2) on the other hand, if we replace the simple arc between u and v by a double arc, the same loop becomes null–homotopic in the polyhedron associated to the complete digraph obtained in such a manner from T_n.

Therefore the first homotopy group of such a digraph is trivial, and hence by Lemma 5.6 the group $Q_1(D_n)$ is also trivial.

Hence the result follows. □

Corollary 5.7. A simply disconnected complete digraph is hamiltonian .

PROOF. It follows easily from Proposition 4.5. □

Before we obtain a second characterization for the simply disconnected complete digraphs we need to introduce some definitions.

Definition 5.8. A subdigraph F of a digraph D is said to be *coned* if there is at least one vertex u in $D - F$, such that u is either a predecessor or a successor of the vertices in F; otherwise the subdigraph E is said to be *non–coned*.

Definition 5.9. A subdigraph F of a digraph D is said to be *shrinkable* if there exists a proper subset of D consisting of equivalent vertices, such that it contains F.

Theorem 5.10. A complete digraph D_n is simply disconnected if and only if:

(a) there exists in D_n a non–coned 3–cycle;

(b) every symmetric pair and every coned 3–cycle in D_n are shrinkable in D_n.

PROOF. By Theorem 5.3 it is sufficient to show that a complete digraph, whose simply quotient is a highly regular tournament, it is characterized by the conditions (a) and (b).

1) The proof the conditions (a) and (b) are necessary it is analogous to the one given in the case of tournaments (see [4], Th. 7).

2) If we suppose the conditions (a) and (b) hold for the complete digraph D_n, then by (b) the simply quotient digraph Q of D_n is a tournament, satisfying the condition that every one of its 3–cycle are non–coned. Hence by the analogous theorem for tournaments which was mentioned above and by the simplicity of Q, we get that the tournament Q is highly regular. And the result is proved. □

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Davide C. Demaria
Dipartimento di Matematica
Universitá di Torino
via Principe Amedeo 8
10123 Torino (Italia)

J. Carlos S. Kiihl
Departamento de Matemática
IMECC – UNICAMP
Caixa Postal 6065
13.081 – Campinas, SP, Brazil

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