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ABSTRACT – We give a thoughtful presentation of proper orthocronous Lorentz transformations using the space–time ($\mathbb{R}_{1,3}$) and the Pauli ($\mathbb{R}_{3,0}$) algebra. We succeeded in finding a closed finite form for a generic $M \in SL(2,\mathbb{C})$ together with a physical interpretation of the parameters. On the basis of our master equation (Eq(32)) we study several important topics of Special Relativity as e.g., the exact product of two boosts and the Thomas precession. We derive also other results necessary for a dynamical interpretation of Lorentz transformations.

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A Thoughtful Study of Lorentz Transformations by Clifford Algebras

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Abstract: We give a thoughtful presentation of proper orthocronous Lorentz transformations using the space-time ($\mathbb{R}_{1,3}$) and the Pauli ($\mathbb{R}_{3,0}$) algebra. We succeeded in finding a closed finite form for a generic $M \in SL(2,\mathbb{C})$ together with a physical interpretation of the parameters. On the basis of our master equation (Eq(32)) we study several important topics of Special Relativity as e.g., the exact product of two boosts and the Thomas precession. We derive also other results necessary for a dynamical interpretation of Lorentz transformations.

1. Introduction

It is the purpose of this paper to give a thoughtful presentation of proper and orthocronous Lorentz transformations ($L \in \mathcal{L}_+^\uparrow$) using Clifford algebras, namely the space-time (or Minkowski) algebra $\mathbb{R}_{1,3}$ and the Pauli algebra $\mathbb{R}_{3,0}$ (*) . In Section 2 these algebras are defined and the relation among them is given (eq.(5)).

Our main result (Section 3) is the the fact that we obtain $M \in SL(2,\mathbb{C})$ (the universal covering group of $\mathcal{L}_+^\uparrow$) as the exponential of a generator in *finite closed form*. Our master equation [eq.(32)] written in the Pauli algebra appears here for the first time, to the best of our knowledge, and can be said to be a generalization of the well known formulas for boosts and rotations.

Our master equation permits the study of several important topics of special relativity in an almost elementary way.

In Section 4 we give the decomposition of a generic $M \in SL(2,\mathbb{C})$ as the product of a boost (eqs.(55) and (56)) and a rotation (eq.(48)), thereby identifying the physical meaning of the parameters involved in the master equation.

In Section 5, to develope a feeling with our Clifford algebra approach, we study the transformation of vectors (elements of $\mathbb{R}^{1,3}$, eqs.(68), (69) and (71)) and antisymmetric tensors (elements of $\Lambda^2(\mathbb{R}^{1,3})$, eqs.(75) and (77)) under particular Lorentz transformations.

In Section 6 we give the transformation of vectors under a generic Lorentz transformation (eqs.(80), (81), (84) and (85)).

In Section 7 we give the exact product of two boosts deducing from it the law of composition of the velocities (eq.(87) and eq.(90)) and show how to apply the result for the Thomas precession (eq.(92) and eq.(95)).

Finally in Section 8 we present our conclusions.

We would like to emphasize that the subject of Lorentz transformations using Clifford algebras has been treated by several authors[3,4,5,6]. Nevertheless the presentation of a finite closed form for the exponential of a generic $M \in SL(2,\mathbb{C})$ (our master equation, eq.(32)) cannot be found elsewhere. Also the treatment of the product of two boosts using $\mathbb{R}_{1,3}$ [5] is very confusing and has produced some mistakes as pointed out in[6] (see also the Errata in[5]). Our treatment of this problem is almost straightforward.

If obtaining the master equation is by itself a gratifying result, the satisfaction is increased by finding out that this equation has an interesting dynamical meaning: namely for a convenient parametrization it is the integral solution for the motion of a charged particle in a constant electromagnetic field (and some other configurations). This topic has been treated "perturbatively" in [4] using the Campbel–Hausdorff–Backer equation. In a following paper [7] using our master equation we shall present an exact solution of it.

(*) In this paper we follow the notations of [1] with minor modifications. See [2] for a deep presentation of the relation of the $Spin(p,q)$ groups with the orthogonal groups $SO(p,q)$ of isometries of $\mathbb{R}^{p,q}$.

2. Minkowski ($\mathbb{R}_{1,3}$) and Pauli ($\mathbb{R}_{3,0}$) Algebras

$\mathbb{R}_{1,3}$, the Minkowski algebra, is the real Clifford algebra generated by 1 and e_μ , $\mu = 0,1,2,3$ such that

$$e_\mu e_\nu + e_\nu e_\mu = 2\eta_{\mu\nu} \tag{1}$$

where $\{e_\mu\}$, $\mu = 0,1,2,3$ is the canonical basis of $\mathbb{R}^{1,3}$ and $\eta_{\mu\nu} = g(e_\mu, e_\nu) = \text{diag}(+1,-1,-1,-1)$, g being the Lorentz metric. We remember that $\mathbb{R}_{1,3}$ is isomorphic as a vector space to the Grassmann algebra $\Lambda(\mathbb{R}^{1,3})$ and then is 2^4 dimensional.

Each $m \in \mathbb{R}_{1,3}$ can be written as

$$m = s + a^\mu e_\mu + \frac{1}{2}a^{\mu\nu}e_\mu e_\nu + \frac{1}{3!}a^{\mu\nu\rho}e_\mu e_\nu e_\rho + p e_0 e_1 e_2 e_3 \tag{2}$$

$$s, a^\mu, a^{\mu\nu}, a^{\mu\nu\rho}, p \in \mathbb{R}.$$

where we assume the summation over repeated indices, and we also consider the coefficients to be antisymmetric under indices exchange (e.g., $a^{\mu\nu} = -a^{\nu\mu}$).

Now let us recall some basic facts about Clifford algebras. The real Clifford algebra generated by a real linear space $\mathbb{R}^{p,q}$, where (p,q) is the metric's signature ($n = p+q$ is the dimension), will be denoted by $\mathbb{R}_{p,q}$.

The elements of the Clifford algebra $\mathbb{R}_{p,q}$, that can be expressed as a linear combination of products of two different basis vectors, like $e_\mu e_\nu$ ($\mu \neq \nu$), are called bivectors. [3,8] The importance of bivectors is that the algebra of their commutators close a subalgebra of the Clifford algebra $\mathbb{R}_{p,q}$, which is isomorphic to the Lie algebra $so(p,q)$ related to the linear space $\mathbb{R}^{p,q}$. [8] We shall see more about bivectors in Section 3 (see Theorem C).

Also of fundamental importance is that the numbers of the Clifford algebra $\mathbb{R}_{p,q}$, that are only linear combination of products involving an even number of distinct basis vectors (e.g., $1, e_\mu e_\nu$ ($\mu \neq \nu$) , $e_0 e_1 e_2 e_3$, in eq.(2)), close a subalgebra of the Clifford algebra, naturally named the even subalgebra and denoted by $\mathbb{R}^+_{p,q}$. [1,3]

Moreover, as the even subalgebra $\mathbb{R}^+_{p,q}$ has 2^{n-1} dimension, we can establish an isomorphism between $\mathbb{R}^+_{p,q}$ and some convenient Clifford algebras of identical dimension, more exactly $\mathbb{R}_{p,q-1}$ or $\mathbb{R}_{q,p-1}$. [2]. In this paper, we are only going to use the even subalgebra of $\mathbb{R}_{1,3}$, which is the well known Pauli algebra ($\mathbb{R}_{3,0} \simeq \mathbb{R}^+_{1,3}$).

$\mathbb{R}_{3,0}$, the Pauli algebra, is the real Clifford algebra generated by 1 and σ_i , $i = 1,2,3$ such that

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} \tag{3}$$

where $\delta_{ij} = \text{diag}(+1,+1,+1)$ and $\{\sigma_i\}$ is the canonical basis of the Euclidean vector space $\mathbb{R}^{3,0}$.

$\mathbb{R}_{3,0}$ is isomorphic as a vector space to the Grassmann algebra $\Lambda(\mathbb{R}^{3,0})$ and is then 2^3 dimensional. Each $P \in \mathbb{R}_{3,0}$ can be written as

$$P = s + a^i\sigma_i + \frac{1}{2}a^{ij}\sigma_i\sigma_j + p\sigma_1\sigma_2\sigma_3 \tag{4}$$

$$s, a^i, a^{ij}, p \in \mathbb{R}$$

As we said before, we have the isomorphism $\mathbb{R}_{3,0} \overset{f}{\simeq} \mathbb{R}^+_{1,3}$ where f is the linear extension of $f : \sigma_i \mapsto e_ie_0$ [3]. In what follows we suppose $\mathbb{R}_{3,0}$ canonically embedded in $\mathbb{R}_{1,3}$ through the identification

$$\sigma_i = e_ie_0 \ , \ i = 1, 2, 3. \tag{5}$$

From eq.(5) we can deduce all the others relations among the elements of $\mathbb{R}^+_{1,3}$ and $\mathbb{R}_{3,0}$

$$\sigma_i\sigma_j = -e_ie_j \quad (i \neq j) \tag{5a}$$

$$\sigma_1\sigma_2\sigma_3 = e_0e_1e_2e_3 \tag{5b}$$

Due to the isomorphism $\mathbb{R}^+_{1,3} \simeq \mathbb{R}_{3,0}$ it is indifferent to work with the even elements of $\mathbb{R}_{1,3}$ (as bivectors) or with the elements of $\mathbb{R}_{3,0}$. Hereafter we shall use the algebra $\mathbb{R}_{3,0}$ in making our calculations. However we prefer to work with a more compact (and more usual) representation of the Pauli algebra than that given by eq.(4). For this, we notice that the following isomorphism holds: [2,3,6] $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$, the algebra of 2×2 complex matrices(*). We can easily verify, using eq.(3), that $\mathrm{i} = \sigma_1\sigma_2\sigma_3$ represents the imaginary unit inside the Pauli algebra. Moreover, from eq.(3), $\sigma_i^2 = 1$, and from definition above, $\mathrm{i} = \sigma_1\sigma_2\sigma_3$, we get

$$\sigma_i\sigma_j = \mathrm{i}\varepsilon_{ijk}\sigma_k \tag{6}$$

which are the well known rules for the product of the Pauli matrices, althought it is important to emphasize here that we are not going to use matrices in the present paper.

It follows that each $P \in \mathbb{R}_{3,0}$ can be written in the basis $\{1, \sigma_1, \sigma_2, \sigma_3\}$ as

$$\begin{aligned} P &= w1 + H_1\sigma_1 + H_2\sigma_2 + H_3\sigma_3 = w + \vec{H} \\ & w, H_1, H_2, H_3 \in \mathbb{C} \end{aligned} \tag{7}$$

It is convenient for what follows to write each complex vector $\vec{H}$ as

$$\vec{H} = \vec{k} + \mathrm{i}\vec{\ell} \tag{8}$$

(*) There is an analogous relation for $\mathbb{R}_{1,3}$ i.e., $\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$, the algebra of 2×2 quaternionic matrices. [2]

where $\vec{k}$ and $\vec{\ell}$ are real vectors, i.e., $\vec{k}, \vec{\ell} \in \mathbb{R}^{3,0}$.

Let us notice that the even subalgebra of the Pauli algebra is isomorphic to the quaternion algebra. It is generated by the Pauli numbers (eq.(7)) that are the sum of a real scalar with an imaginary vector.

The product of two Pauli numbers can be directly obtained using the linearity of the Clifford product. If $P = w + \vec{H}$, $Q = z + \vec{F}$, $P, Q \in \mathbb{R}_{3,0}$ we have

$$PQ = (w + \vec{H})(z + \vec{F}) = wz + w\vec{F} + z\vec{H} + \vec{H}\vec{F}. \tag{9}$$

The terms $wz, w\vec{F}$ and $z\vec{H}$ involve only the product of complex numbers. To know the result of eq.(9) we need then to know the Clifford product of two complex vectors. This can be solved at once if we know the product of two basis vectors. From eq.(6) and from the fact that $\sigma_i^2 = 1$ it follows that

$$\vec{H}\vec{F} = \vec{H} \cdot \vec{F} + \mathrm{i}\vec{H} \times \vec{F} \tag{10}$$

where the dot product and the cross product of the complex vectors are the usual ones of vector algebra, i.e.,

$$\vec{H} \cdot \vec{F} = \sum_i H_i F_i \ ; \ \vec{H} \times \vec{F} = \varepsilon_{ijk} H_i F_j \sigma_k$$

$\vec{H} \cdot \vec{F}$ is called the scalar part of the Pauli product $\vec{H}\vec{F}$, whereas $\mathrm{i}\vec{H} \times \vec{F}$ is called the vector part.

We now introduce for the Pauli algebra the operations corresponding to the main automorphism and the reversion and conjugation antiautomorphisms of the theory of Clifford algebras[1,2].

The main automorphism $\square$ in $\mathbb{R}_{3,0}$ will be called spatial conjugation. If $P = w + \vec{H}$, we have,

$$P^{\square} = w^* - \vec{H}^* \tag{11}$$

where $*$ is the usual complex conjugation. Observe that $\{1^{\square}, \vec{\sigma}^{\square}\} = \{1, -\vec{\sigma}\}$ and that

$$(PQ)^{\square} = P^{\square} Q^{\square} \tag{12}$$

The reversion operation (the main antiautomorphism) in the Pauli algebra will be called *hermitian conjugation* and will be denoted by $+$. If $P = w + \vec{H}$, we have

$$P^+ = w^* + \vec{H}^*. \tag{13}$$

Observe that $\{1^+, \vec{\sigma}^+\} = \{1, \vec{\sigma}\}$ and that

$$(PQ)^+ = Q^+ P^+. \tag{14}$$

Finally, the conjugation-operation(an antiautomorphism) in the Pauli algebra will denoted by $\sim$. If $P = w + \vec{H}$, we have

$$\tilde{P} = (P^{+})^{\square} = w - \vec{H}. \tag{15}$$

Observe that $(\tilde{1}, \tilde{\vec{\sigma}}) = (1, -\vec{\sigma})$ and that

$$(PQ)^{\sim} = \tilde{Q}\tilde{P} \tag{16}$$

The above operations permit us to characterize the Pauli numbers; e.g., if $P^{+} = P$ then P has only real coefficients in its expansion. If $P^{\square} = P$, the scalar component of P is a real number and the vector component is imaginary. If $\tilde{P} = -P$, then P has only the (complex) vector component.

The conjugation is of fundamental importance, since it permit us to construct the *inverse* of a given Pauli number when such an inverse exists. To see that, observe that from eq.(9) it follows$^{(*)}$

$$P\tilde{P} = \tilde{P}P = w^2 - \vec{H}^2 = \text{ complex number} \tag{17}$$

since $\vec{H}^2 = \vec{H} \cdot \vec{H}$ is a complex number (eq.(10)).

Then if $P\tilde{P} \neq 0$ we can immediately write

$$P^{-1} = \frac{1}{(w^2 - \vec{H}^2)}\tilde{P} \tag{17a}$$

We notice for future reference that the Pauli numbers that satisfy $P\tilde{P} = 1$ close the $SL(2,\mathbb{C})$ group.

3. The Exponential Form in Finite Closed Form of the Transformations of $SL(2,\mathbb{C})$

Before we develop the finite form of the proper orthocronous Lorentz transformations, let us make a brief review of the orthogonal transformations that clarify the language of Clifford algebras. The orthogonal transformations of $\mathbb{R}^{p,q}$ are the elements of $\mathbb{R}^{p,q} \otimes {}^{*}\mathbb{R}^{p,q}$ (where ${}^{*}\mathbb{R}^{p,q}$ is the dual of $\mathbb{R}^{p,q}$) such that $g(Lu, Lv) = g(u, v)$, (where g is the metric tensor), for any $u, v \in \mathbb{R}^{p,q}$, $L \in \mathbb{R}^{p,q} \otimes {}^{*}\mathbb{R}^{p,q}$. The set of all orthogonal transformations define the group $O(p, q)$. The proper$^{(**)}$ orthogonal transformations define the group $SO(p, q)$. The orthogonal transformations that are connected with the

(*) Then, eq.(17) is identified with the determinant of the Pauli number in the matricial representation of the Pauli algebra.

(**) those transformations that are not involved with reflections or inversions.

identity form a subgroup of $SO(p,q)$, indicated by $SO_+(p,q)$. (*) Our interest rests in this latter group.

If $\varepsilon \in I\!R$, then $SO_+(p,q) \ni L = 1 + \varepsilon T$ is called an infinitesimal orthogonal transformation if $g(Lu, Lu) = g(u,u)$ is correct in first approximation. This means that $g(u, Tu) = 0$ and writing $u = x + y$ gives $g(y, Tx) = -g(x, Ty)$. A linear transformation as T is called skew–symmetric (or, for short, skew) relative to the metric g.

It is a well known result that every orthogonal transformation e^T generated by a skew symmetric transformation T belongs to the connected component of the identity. The important result that we want to quote here is that the converse of this statement is true only for Euclidian ($p = 0$ or $q = 0$) and Lorentzian spaces ($p = 1$ or $q = 1$). More precisely it holds[8]:

Theorem A: Every orthogonal transformation of an Euclidean or Lorentzian space connected with the identity can be generated by some skew symmetric transformation through exponentation.

The above result is indeed remarkable since it is not shared by any other real metric.

Let us relate the above results to the Clifford algebra $I\!R_{p,q}$. We describe the orthogonal transformations by Clifford algebras using the following form for the transformation of a vector $u \in I\!R^{p,q}$:

$$u' = MuM^{-1} \tag{18}$$

where $M \in I\!R_{p,q}$ is called the operator that describes the transformation.

Eq.(18) tell us that we are looking for the Clifford numbers, $M \in I\!R_{p,q}$, that generate the automorphisms of $I\!R^{p,q}$ (it is clear from eq.(18) that the metric $g(u,u)$ given by u^2 in the Clifford language is preserved). These numbers define the Clifford group. [1,2] Also, we are interested only in the orthogonal transformations that are proper. These are described by the elements of the Clifford group that belong to the even subalgebra, $I\!R^+_{p,q}$. [1,2] Moreover, among the orthogonal transformations we are only interested in those connected to the identity, which are represented in the Clifford algebra $I\!R_{p,q}$ by the $Spin_+(p,q)$ group given by the following theorem valid for low dimensional Clifford algebras. [1,2]

Theorem B: If $(p + q \leq 5)$, then

$$Spin_+(p,q) = \{M \in I\!R^+_{p,q} |\ MM^\dagger = 1\} \tag{19}$$

where $M^\dagger$ indicates the reverse of M († indicates the reversion or main antiautomorphism of the Clifford algebra $I\!R_{p,q}$).

(*) In Lorentz spaces the elements of $SO_+(p,q)$ are called proper orthocronous transformations and are indicated by $L^\uparrow_+$.

From the above comments it follows that $Spin_+(p,q)$ contains at least one representation of $SO_+(p,q)$, i.e., the elements of $Spin_+(p,q)$ can represent the transformations of $SO_+(p,q)$. In particular, $Spin_+(1,3) \simeq SL(2,\mathbb{C})$ [1,2], the universal covering group of $SO_+(1,3)$, which is in turn the proper orthocronous Lorentz transformations group (also indicated by $\mathcal{L}_+^\uparrow$). This latter result is shown below (see eq.(20)).

To accomplish the calculus involving $\mathcal{L}_+^\uparrow$ within the Pauli algebra, we note that the main antiautomorphism existent in the algebra $\mathbb{R}_{1,3}$ (used in Theorem B) induces the conjugation operation in the algebra $\mathbb{R}_{3,0}$, defined by eq.(15), through the isomorphism $\mathbb{R}_{1,3}^+ \simeq \mathbb{R}_{3,0}$. [3]. Then, from the above comments and theorem B we see that the proper orthocronous Lorentz transformations can be described by the element of the Pauli algebra satisfying

$$M\widetilde{M} = 1 \tag{20}$$

We now turn our attention to the exponential form of the operators $M \in SL(2,\mathbb{C})$. As we said in Section 2 the bivectors of the Clifford algebra $\mathbb{R}_{p,q}$ close through the commutator product a subalgebra isomorphic to the Lie algebra $so(p,q)$ whose elements are the skew symmetric transformations like the ones involved in Theorem A. Then we expect that the bivectors of $\mathbb{R}_{p,q}$ must appear as the generators of the $Spin_+(p,q)$ group, what is guaranteed by the following theorem.

Theorem C:[8] Let $\mathbb{R}^{p,q}$ be a real vector space of Euclidian or Lorentz signature and let $\mathbb{R}_{p,q}$ be the corresponding Clifford algebra. Then, for every orthogonal transformations of $\mathbb{R}^{p,q}$ connected to the identity ($L \in SO_+(p,q)$), there exists a bivector $f \in \Lambda^2(\mathbb{R}^{p,q}) \subset \mathbb{R}_{p,q}$ such that

$$Lx = e^f x e^{-f} \tag{21}$$

$\forall x \in \mathbb{R}_{p,q}$. This property does not belong to any other signature.

As we said before, L can always be written as the exponential e^T of a skew symmetric T. If $\{e_\mu\}$ is the canonical basis of $\mathbb{R}^{p,q}$ and $\{e^\mu\}$ is the "reciprocal basis", i.e., $e^\mu e_\nu = \delta^\mu_\nu$ then T can be expressed by relations of the form $Te_\nu = \alpha_{\mu\nu} e^\mu$. Writing $f = \frac{1}{2} f_{\mu\nu} e^\mu e^\nu$, we get from eq.(21)

$$Tx = fx - xf = [f,x] \quad ; \quad \alpha_{\mu\nu} = 2f_{\mu\nu} \tag{22}$$

For the special case $p = 1$, $q = 3$, which is the one we are interested in, the last theorem simply means that every element of $Spin_+(1,3) \simeq SL(2,\mathbb{C})$ can be written in the form e^f, where f is a bivector of $\mathbb{R}_{1,3}$. Now from eq.(5) to each bivector $f \in \Lambda^2(\mathbb{R}^{1,3}) \subset \mathbb{R}_{1,3}$ there corresponds a complex vector $\vec{F} \in \mathbb{R}_{3,0}$. Indeed, from $f = \frac{1}{2} f_{\mu\nu} e^\mu e^\nu$ and using the canonical identification $\sigma_i = e_i e_0$ (eq.(5)) and eqs.(6), we get

$$\mathbb{R}_{1,3}^+ \supset \Lambda^2(\mathbb{R}^{1,3}) \ni f \mapsto \vec{F} = a_i\sigma_i + \mathrm{i} b_i \sigma_i = \vec{E} + \mathrm{i}\vec{H} \in \mathbb{R}_{3,0} \tag{23}$$

$$a_i = f_{oi} \ ; \ b_k = -\frac{1}{2} f_{ij}\varepsilon_{ijk}$$

Basing ourselves on Theorem C that states the existence of the exponential form we are going now to write $M \in SL(2,\mathbb{C})$ in a finite form and show that this finite form corresponds to a unique exponential of a complex vector in the Pauli algebra. We start by writing $M = w + \vec{H}$. Imposing the constraint given by eq.(20) we have

$$M\widetilde{M} = w^2 - \vec{H}^2 = 1 \tag{24}$$

where w^2 and $\vec{H}^2$ are complex numbers which we parametrize as follows

$$w^2 = \cosh^2 z \quad , \quad \vec{H}^2 = \sinh^2 z \quad ; \quad z \in \mathbb{C} \tag{25}$$

We choose $w = \cosh z$ and parametrize the vector part of M as

$$\vec{H} = \sinh z \hat{F} \tag{26}$$

where $\hat{F}$ is a complex vector such that $\hat{F}^2 = 1$. The parametrization given by eq.(25) is always possible when $\vec{H}^2 \neq 0$ and $\hat{F}$ is given by

$$\hat{F} = \frac{\sinh z^*}{|\sinh z|^2}\vec{H} \tag{27}$$

With eq.(25) and eq.(26) we can write

$$M = \cosh z + \hat{F} \sinh z \tag{28}$$

Eq.(28) is the finite form of the transformation $M \in SL(2,\mathbb{C})$. There are, of course, six parameters involved, namely, the complex variable z and the complex vector $\hat{F}$ subject to the condition $\hat{F}^2 = 1$.

These parameters, as we shall show in what follows, are directly related to the parameters with a direct physical meaning. To see this, it is necessary to find the exponential form of $M \in SL(2,\mathbb{C})$ which Theorem C guarantees to exists.

We can define formally the exponential of a complex vector $\vec{F} \in \mathbb{R}_{3,0}$ through its series expansion, i.e., we write

$$e^{\vec{F}} = \sum_{n=0}^{\infty} \frac{\vec{F}^n}{n!} = 1 + \vec{F} + \frac{\vec{F}^2}{2!} + \cdots \tag{29}$$

Now, we know (cf. eq.(10)) that the square of a complex vector is a complex number and then the series of the even powers in eq.(29) is a series of complex numbers that defines the hyperbolic cosine:

$$\cosh z = 1 + \frac{\vec{F}^2}{2} + \frac{\vec{F}^4}{4!} + \cdots \tag{30}$$

where $z^2 = \vec{F}^2$ (supposing $\vec{F}^2 \neq 0$).

Analogously to eq.(26) we can write $\vec{F} = z\hat{F}$ with $\hat{F}^2 = 1$ and $\hat{F} = z^*\vec{F}/|z|^2$. In this case the series of the odd powers in eq.(29) is given by

$$\hat{F}\cosh z = \hat{F}(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots) = \vec{F} + \frac{\vec{F}^3}{3!} + \cdots \tag{31}$$

We then get the important result (our *master equation*) that the exponential of a complex vector can be written as a well defined finite form, showing the convergence of the series given by eq.(29). We have

$$e^{\vec{F}} = e^{z\hat{F}} = \cosh z + \hat{F}\sinh z \tag{32}$$

The reader certainly perceived that it had been intentional the use of the same variables (z and $\hat{F}$) to parametrize both the complex vector $\vec{F}$ and the operator $M \in SL(2,\mathbb{C}) \subset \mathbb{R}_{3,0}$, since in this way M is simply given by the exponential of the complex vector $\vec{F}$. By some abuse of language and due to eq.(21), $\vec{F}$ can be said the generator of the proper orthocronous transformation described by M.

The physical meaning of the parameters $\vec{F}$ of a generic $M \in SL(2,\mathbb{C})$ will be given in Section 4. Here we give the transformations corresponding to boosts and rotations.

Spatial Rotations: A spatial rotation (or rotation, for short) has as generator an imaginary vector

$$\vec{F} = \mathrm{i}\vec{n}/2 \tag{33}$$

where $\vec{n} \in \mathbb{R}^{3,0}$ is associated with the physical parameters of the rotation, defining the direction of the rotation, and its module, labelled $\theta = |\vec{n}|$, defining the rotation angle.

Putting $z = \mathrm{i}\theta/2$ and $\hat{F} = \hat{n} = \vec{n}/\theta$, the operator of rotation $\mathcal{R}$ is then given by (cf. eq.(32))

$$\mathcal{R} = e^{\mathrm{i}\vec{n}/2} = \cos(\theta/2) + \mathrm{i}\hat{n}\sin(\theta/2) \tag{34}$$

and is then the sum of a real number and an imaginary vector. The rotation operators close a subgroup of $Spin_+(1,3) \simeq SL(2,\mathbb{C})$, namely the group $Spin_+(3,0) \simeq SU(2)$. They determine also a subalgebra of the Pauli algebra, namely the algebra of unitary quaternions ($\mathcal{R}\widetilde{\mathcal{R}} = 1$). Note that $\mathcal{R}$ is unitary: $\mathcal{R}^+ = \widetilde{\mathcal{R}}$.

Boosts: A boost has as generator a real vector

$$\vec{F} = \vec{\nu}/2 \tag{35}$$

where $\vec{\nu} \in \mathbb{R}^{3,0}$ defines the direction of the relative velocity between two inertial reference frames. $\nu = |\vec{\nu}|$ is called the rapidity of the boost and is related to the relative velocity by the well known formula

$$v = tgh\nu \tag{36}$$

Putting $z = \nu/2$ and $\hat{F} = \hat{v} = \vec{v}/v$ in eq.(32) yields for the boost operator $B \in SL(2,\mathbb{C}) \subset \mathbb{R}_{3,0}$:

$$B = e^{\nu/2} = \cosh \nu/2 + \hat{v} \sinh \nu/2 \tag{37}$$

which is the sum of a real number with a real vector. Then boost operators are hermitian, i.e. $B^+ = B$.

Note that the square of a boost operator is simply related to the physical parameters

$$B^2 = e^{\vec{v}} = \cosh \nu + \hat{v} \sinh \nu = \gamma(1 + \vec{v}) \tag{36a}$$

where $\gamma = \cosh \nu$ is the Lorentz factor.

The boosts do not close a subgroup of $SL(2,\mathbb{C})$. When two boosts are composed we get a boost followed by a spatial rotation. This is related, as is well known, to the Thomas precession and we shall discuss this topic within the present methodology in Section 7.

Another important result that follows trivially from the Pauli algebra is the Lie algebra $s\ell(2,\mathbb{C})$ of $SL(2,\mathbb{C})$. As we said in Section 2, the commutators of the bivectors of $\mathbb{R}_{1,3}$ close an algebra isomorphic to the Lie algebra associated with $SO_+(1,3)$, which is the same Lie algebra as that associated with $SL(2,\mathbb{C})$. As the bivectors of $\mathbb{R}_{1,3}$ are represented in the Pauli algebra $\mathbb{R}_{3,0}$ by complex vectors (eq.(23)) through the isomorphism $\mathbb{R}^+_{1,3} \simeq \mathbb{R}_{3,0}$, these latter are the elements of the Lie algebra $s\ell(2,\mathbb{C}) \subset \mathbb{R}_{3,0}$.

As we said before, the boost generators are real vectors (eq.(35)) and the rotation generators are imaginary vectors (eq.(33)). Then, defining the "standard" boost generators K_i and the standard rotation generators L_i by

$$K_i = \frac{1}{2}\sigma_i \ , \quad L_i = -\frac{\mathrm{i}}{2}\sigma_i \ , \quad i = 1,2,3, \tag{38}$$

we immediately get from eqs.(6)

$$[L_i, L_j] = \mathrm{i}\varepsilon_{ijk} L_k \ ; \ [K_i, L_j] = \varepsilon_{ijk} K_k \ , \ [K_i, K_j] = -\varepsilon_{ijk} L_k \tag{39}$$

that are the usual commutation relations for $SL(2,\mathbb{C})$. [4].

We end this section by giving the solution for z and $\hat{F}$ in terms of the components of the generator $\vec{F}$ which defines the generic M in eq.(32).

We write

$$\vec{F} = \vec{E} + i\vec{B} \ , \ \vec{F} = z\hat{F} \ , \ z = x + iy \ , \ \hat{F} = \hat{e} + i\hat{b} \tag{40}$$

Then from the relation $z^2 = \vec{F}^2$ we get

$$\begin{cases} x^2 = \frac{1}{2}\{(\vec{E}^2 - \vec{B}^2) + |z|^2\} \\ y^2 = \frac{1}{2}\{|z|^2 - (\vec{E}^2 - \vec{B}^2)\} \\ |z|^2 = \sqrt{(\vec{E}^2 - \vec{B}^2)^2 + (2\vec{E}\cdot\vec{B})^2} \end{cases} \tag{41}$$

and from $\widehat{F} = \frac{z^* \vec{F}}{|z|^2}$ we get:

$$\begin{cases} \widehat{e} = \frac{1}{|z|^2}(x\vec{E} + y\vec{B}) \\ \widehat{b} = \frac{1}{|z|^2}(x\vec{B} - y\vec{E}) \end{cases} \tag{42}$$

We observe that the relation $\widehat{F}^2 = 1$ is verified, i.e., $\widehat{e}^2 - \widehat{b}^2 = 1$ and $\widehat{e} \cdot \widehat{b} = 0$.

4. The Decomposition of a Generic $M \in SL(2,\mathbb{C})$ in the Product of a Boost and a Rotation

In relativistic kinematics it is usual to look at the six parameters of the Lorentz group as defining the relations between two inertial frames as follows: three parameters define the orientation of spatial axis and three parameters define the relative velocity. We are going now to show how these parameters are related to the generator (eq.(40)) of a generic Lorentz transformation.

It is a well known result that each $M \in SL(2,\mathbb{C})$ can be written

$$M = \mathcal{B}\mathcal{R} \tag{43}$$

although the right hand side of eq.(43) is not unique[16].

We write

$$M = e^{\vec{F}} = e^{\vec{E} + i\vec{B}} \ , \ \ \mathcal{B} = e^{\vec{\nu}/2} \ , \ \ \mathcal{R} = e^{i\vec{n}/2} \tag{44}$$

Using the master equation (eq.(32)) and eq.(40) we can express M in terms of $x, y, \widehat{e}$ and $\widehat{b}$. We have:

$$\begin{aligned} M = \cosh z + \widehat{F} \sinh z \ &= \ \cosh x \cos y + \mathrm{i} \sinh x \sin y \\ &+ \ (\widehat{e} \sinh x \cos y - \widehat{b} \cosh x \sin y) \\ &+ \ \mathrm{i}(\widehat{b} \sinh x \cos y + \widehat{e} \cosh x \sin y) \end{aligned} \tag{45}$$

On the other hand:

$$\begin{aligned} M = \mathcal{B}\mathcal{R} = e^{\vec{\nu}/2} e^{i\vec{n}/2} \ &= \ \cosh(\nu/2)\cos(\theta/2) + \mathrm{i}\widehat{\nu} \cdot \widehat{n} \sinh(\nu/2)\sin(\theta/2) \\ &+ \ \widehat{\nu} \sinh(\nu/2)\cos(\theta/2) - \widehat{\nu} \times \widehat{n} \sinh(\nu/2)\sin(\theta/2) \\ &+ \ \mathrm{i}\widehat{n} \cosh(\nu/2)\sin(\theta/2) \end{aligned} \tag{46}$$

Comparing eqs.(45) and (46), we get

$$\widehat{n} \ tg(\theta/2) = \widehat{b} tghx + \widehat{e} tgy \tag{47}$$

Squaring eq.(47) we obtain for the rotation angle

$$tg^2\,(\theta/2) = \hat{b}^2 tgh^2 x + \hat{e}^2 tg^2 y$$
$$= \frac{1}{2} tg^2 y \left\{ \frac{\vec{E}^2 + \vec{B}^2}{|z|^2} + 1 \right\} + \frac{1}{2} tgh^2 x \left\{ \frac{\vec{E}^2 + \vec{B}^2}{|z^2|} - 1 \right\} \tag{48}$$

where we use the fact that $\hat{e}^2 - \hat{b}^2 = 1$ and $\hat{e}^2 + \hat{b}^2 = \frac{\vec{E}^2 + \vec{B}^2}{|z|^2}$, which follows from the relation $\hat{F}\hat{F}^* = \frac{\vec{F}\vec{F}^*}{|z|^2}$ (see also eq.(54)).

To determine the rapidity vector $\vec{\nu}$ we observe that

$$MM^+ = B\mathcal{R}\mathcal{R}^+B^+ = BB^+ = B^2 \tag{49}$$

since the hermitian conjugation does not change a boost operator, as it is trivial to verify. Now from $M^+ = \cosh z^* + \hat{F}^* \sinh z^*$, it follows

$$MM^+ = |\cosh z|^2 + \hat{F} \sinh z \cosh z^* + \hat{F}^* \sinh z^* \cosh z + \hat{F}\hat{F}^* |\sinh z|^2 \tag{50}$$

Observe that $(\hat{F} \sinh z \cosh z^*) = (\hat{F}^* \sinh z^* \cosh z)^*$ and the sum of the second and third terms in eq.(50) is equal to

$$2Re(\hat{F} \sinh z \cosh z^*) = \hat{e} \sinh 2x - \hat{b} \sin 2y \tag{51}$$

Using eq.(42) we can rewrite the above equation as follows

$$2Re(\hat{F} \sinh z \cosh z) = \frac{1}{|z|^2}(\vec{E} f(x,y) + \vec{B} g(x,y)) \tag{52}$$

where

$$f(x,y) = x \sinh 2x + y \sin 2y \tag{53a}$$

$$g(x,y) = y \sinh 2x - x \sin 2y \tag{53b}$$

Also, the last term in eq.(50) is given by

$$\hat{F}\hat{F}^* = \frac{\vec{F}\vec{F}^*}{|z|^2} = \frac{(\vec{E}^2 + \vec{B}^2) + 2\vec{E} \times \vec{B}}{|z|^2} \tag{54}$$

It then follows from eq.(49), eq.(50), eq.(52), eq.(54) and eq.(36a) that

$$\cosh \nu = \gamma = |\cosh z|^2 + \frac{\vec{E}^2 + \vec{B}^2}{|z|^2} |\sinh z|^2 \tag{55}$$

and

$$\hat{\nu}\sinh\nu = \gamma\vec{v} = 2\frac{\vec{E}\times\vec{B}}{|z|^2}|\sinh z|^2 + \frac{1}{|z|^2}(\vec{E}f(x,y) + \vec{B}g(x,y)) \tag{56}$$

5. Transformations of Vectors and Antisymmetric Tensors under Particular Lorentz Transformations.

Eq.(18) and theorem B tell us that each $p \in \mathbb{R}^{1,3}$ transforms under a proper orthocronous Lorentz transformation described by the operator $M \in Spin_+(1,3)$ as follows

$$p' = MpM^\dagger \tag{57}$$

where $\dagger$ is the main antiautomorphism in $\mathbb{R}_{1,3}$.

It is important to emphasize [2,9,10,17] that every element $a \in \mathbb{R}_{1,3}$ transforms as in eq.(57). More precisely we have

$$a' = MaM^\dagger \tag{58}$$

This result can be seen immediately, once we realize that each $a \in \mathbb{R}_{1,3}$ has the from given by eq.(2) and so each product of the basis vectors transforms as $Me_\mu M^\dagger Me_\nu M^\dagger \cdots Me_\rho M^\dagger$, where $MM^\dagger = 1$ (Theorem B).

We want now to know how eq.(58) can be described in the Pauli algebra $\mathbb{R}_{3,0}$. If $a \in \mathbb{R}^+_{1,3}$ we need only replace the operation $M^\dagger$ by $\widetilde{M}$, since, as we said in section 3, the main antiautomorphism of $\mathbb{R}_{1,3}$ ($\dagger$) induces the conjugation in $\mathbb{R}_{3,0}$ ($\sim$, see eq.(15)) through the isomorphism $\mathbb{R}^+_{1,3} \simeq \mathbb{R}_{3,0}$ [3,17]. Then, if $a \in \mathbb{R}^+_{1,3}$, we have in the Pauli algebra that

$$a' = Ma\widetilde{M} \tag{59}$$

For example, the above equation holds in the case of the electromagnetic field which is a bivector of $\mathbb{R}_{1,3}$ [3,4], and is represented by a complex vector in $\mathbb{R}_{3,0}$ (as is the case of the generators of $\mathcal{L}^\uparrow_+$, eq.(23)). In this case, the invariants of the field are given by the real and the imaginary parts of $\vec{F}^2$ [3,17]

$$\vec{F}^2 = (\vec{E}^2 - \vec{B}^2) + 2i(\vec{E}\cdot\vec{B})$$

We easily verify from eq.(59) that it holds $\vec{F}'^2 = \vec{F}^2$, since $M\widetilde{M} = 1$ (eq.(24)).

However, if $a \notin \mathbb{R}^+_{1,3}$, as a vector of $\mathbb{R}^{1,3}$, we need some other change in eq.(59).

In Section 2 we established that $\mathbb{R}^+_{1,3} \simeq \mathbb{R}_{3,0}$ and then we supposed that $\mathbb{R}_{3,0}$ was canonically embedded in $\mathbb{R}_{1,3}$ through the identification $\sigma_i = e_i e_0$. It follows that each vector $p = p^\mu e_\nu \in \mathbb{R}^{1,3} \subset \mathbb{R}_{1,3}$ can be represented in the Pauli algebra by $P = pe_0 \in \mathbb{R}^+_{1,3} \simeq \mathbb{R}_{3,0}$ [3,17]. We then write

$$P = p_0 + \vec{p} = p_0 + p^i\sigma_i \ , \ \ p_0, p^i \in \mathbb{R}. \tag{60}$$

We observe that it holds

$$g(p,p) = P\widetilde{P} = (p_0 + \vec{p})(p_0 - \vec{p}) = p_0^2 - \vec{p}^{\,2} \tag{61}$$

Then, to represent eq.(57) in the Pauli algebra we multiply it by e_0, and we insert $e_0^2 = 1$ between p and $M^\dagger$. We obtain

$$p' e_0 = M(p e_0)(e_0 M^\dagger e_0) \tag{62}$$

Now, all the elements that appear in the above equation are elements of the even subalgebra $\mathbb{R}_{1,3}^+$ and we can represent them in the Pauli algebra. We need only to know what happens to a number of $\mathbb{R}_{1,3}^+$ when we multiply it by e_0 on both sides. We can show that the result represented in the Pauli algebra is identical to the ones obtained by using the spatial conjugation of that number, [3,17], i.e., we get (see eq.(15))

$$e_0(M^\dagger)e_0 \Longrightarrow (\widetilde{M})^\Box = M^+ \tag{63}$$

Then, the transformation of a vector of $\mathbb{R}_{1,3}$ can be represented in Pauli algebra as follows:

$$P' = MPM^+. \tag{64}$$

To calculate explicitly the components of $P' = p_0' + \vec{p}\,'$ we observe first that

$$P' = p_0 MM^+ + M\vec{p}M^+ \tag{65}$$

Now, many simplifications occur if we can separate the components of the vector $\vec{p}$ that commutes or anticommutes with the generator of the transformation M. In general, the generator of the transformation is a complex vector and such a separation is not possible. However, when the complex vector denoted by $\vec{F}$, that represents the generator, can be written as a multiple of an Euclidian vector, i.e., $\vec{F} = z\hat{n}$, $\hat{n} \in \mathbb{R}^{3,0}(\hat{n}^2 = 1)$, $z \in \mathbb{C}$ (a condition that gives the important cases of boosts and spatial rotations), it is possible to decompose the vector $\vec{p}$ in components parallel or orthogonal to the direction of the generator. We write $\vec{p} = \vec{p}_{//} + \vec{p}_\perp$, where

$$\vec{p}_{//} = (\vec{p}\cdot\hat{n})\hat{n} \ ; \ \vec{p}_\perp = \hat{n} \times (\vec{p} \times \hat{n}) \tag{66}$$

We observe that, since $\vec{p}_{//} \times \hat{n} = 0$ and $\vec{p}_\perp \cdot \hat{n} = 0$, then, relative to the Clifford product in the Pauli algebra, eq.(10), $\vec{p}_{//}$ commutes with $\hat{n}$ and $\vec{p}_\perp$ anticommutes with $\hat{n}$. In this way the commutation relations with the operator $M = e^{\vec{F}} = \cosh z + \hat{F}\sinh z$ are

$$M\vec{p}_{//} = \vec{p}_{//}M \ \ ; \ \ M\vec{p}_\perp = \vec{p}_\perp\widetilde{M} \tag{67}$$

where $\widetilde{M}$ is given by eq.(15), and we can write eq.(65) as

$$P' = (p_0 + \vec{p}_{//})MM^+ + \vec{p}_\perp\widetilde{M}M^+ \tag{68}$$

We emphasize that eq.(68) is not valid in general, its validity being restricted only for the cases in which the generator is a multiple of an Euclidian vector, i..e, $\vec{F} = z\hat{n}$. We now study some particular cases in which eq.(68) can be applied.

(i) Rotations: The rotation operators are represented by the "unitary" quaternions, (eq.(34)) i.e., $\widetilde{\mathcal{R}} = \mathcal{R}^+ (\mathcal{R}\mathcal{R}^+ = 1)$. So we have in this case that eq.(68) becomes

$$P' = p_0 + \vec{p}_{//} + \vec{p}_\perp (\mathcal{R}^+)^2 \tag{69}$$

We observe that the scalar part of P and also the component $\vec{p}_{//}$ are invariant under the rotation $\mathcal{R}$ since the last term in eq.(69) does not contribute to these parts. The calculation of the relevant term is immediate. We get

$$\vec{p}_\perp (\mathcal{R}^+)^2 = \vec{p}_\perp \cos\theta + \vec{p} \times \vec{n} \sin\theta \tag{70}$$

(ii) Boosts: The boost operators, as well as the vectors of $\mathbb{R}^{1,3}$, are represented by hermitian Pauli numbers, eq.(37), i.e., $\mathcal{B}^+ = \mathcal{B}$; and, as every operator of $\mathcal{L}_+^\uparrow$, satisfies eq.(24), i.e. $\mathcal{B}\widetilde{\mathcal{B}} = 1$. So, we have from eq.(68)

$$\begin{aligned} P' &= (p_0 + \vec{p}_{//})\mathcal{B}^2 + \vec{p}_\perp \\ &= [p_0 \cosh\nu + \vec{p} \cdot \hat{\nu} \sinh\nu] + [p_0 \sinh\nu + (\vec{p} \cdot \hat{\nu}) \cosh\nu]\hat{\nu} + \vec{p}_\perp \end{aligned} \tag{71}$$

Using eq.(36a) we can rewrite eq.(71) as follows

$$p_0' = \gamma(p_0 + \vec{p} \cdot \vec{v}) \quad ; \quad \vec{p}_{//}' = \gamma(p_0 \vec{v} + \vec{p}_{//}) \ ; \ \vec{p}\,'_\perp = \vec{p}_\perp \tag{71'}$$

(iii) Parallel Generators: To end this section, we discuss now the general case of eq.(68) when z is a complex number. Then

$$\vec{F} = z\hat{n} = (\nu + i\theta)\hat{n} = \vec{\nu} + i\vec{n} \tag{72}$$

The Lorentz transformation $M(\vec{\nu}, \vec{n})$ generated by this $\vec{F}$ is not a boost nor a rotation. Nevertheless it is an easy task to obtain the explicit form of this transformation since we can write

$$M(\vec{\nu}, \vec{n}) = e^{\vec{F}} = e^{\vec{\nu} + i\vec{n}} = e^{\vec{\nu}} e^{i\vec{n}} = \mathcal{B}(\vec{\nu})\mathcal{R}(\vec{n}) \tag{73}$$

Then, from eq.(68) it follows that

$$\begin{aligned} p_0' + \vec{p}\,'_{//} &= (p_0 + \vec{p}_{//})[\mathcal{B}(\vec{\nu})]^2 \\ \vec{p}\,'_\perp &= \vec{p}_\perp [\mathcal{R}^+(\vec{n})]^2 \end{aligned} \tag{74}$$

and the final result can be written immediately, if needed, by using eqs.(70) and (71).

Transformation of the electromagnetic field under boosts: We already know that the electromagnetic field $f \in \Lambda^2(\mathbb{R}^{1,3}) \subset \mathbb{R}_{1,3}$ is represented in the Pauli algebra by a complex vector $\vec{F} = \vec{E} + i\vec{B}$. They transform under the action of $M \in SL(2,\mathbb{C})$ as given by eq.(59). To calculate eq.(59) we decompose the real and imaginary parts of $\vec{F}$ in components parallel and orthogonal to the relative velocity and analogously to eq.(68), we get

$$\vec{F}' = \vec{F}_{//} M\widetilde{M} + \vec{F}_{\perp}\widetilde{M}^2 \tag{75}$$

and, since $M\widetilde{M} = 1$ only, the orthogonal component of the electromagnetic field changes. Since, in the case of boosts, we have from eq.(36a)

$$\widetilde{M}^2 = e^{-\vec{\vartheta}} = \gamma(1 - \vec{v}), \tag{76}$$

then we have

$$\begin{aligned} \vec{E}'_{\perp} &= \gamma(\vec{E}_{\perp} + \vec{B} \times \vec{v}) \\ \vec{B}'_{\perp} &= \gamma(\vec{B}_{\perp} - \vec{E} \times \vec{v}) \end{aligned} \tag{77}$$

6. Transformation of a Vector ($p \in \mathbb{R}^{1,3}$) under a Generic Proper Orthocronous Lorentz Transformation.

In this section we discuss the transformation of vectors of $\mathbb{R}^{1,3}$ under a generic proper orthocronous Lorentz transformation $M = e^{\vec{F}} \in SL(2,\mathbb{C})$. In this case the generator $\vec{F} = \vec{E} + i\vec{B}$ is such that $\vec{E} \times \vec{B} \neq 0$.

One of the difficulties in treating such a generic transformation is that it does not exist any component of an euclidean vector $\vec{p}$ ($P = p_0 + \vec{p}$) which commutes with the generator.

To find an equation analogous to eq.(68), we must use the fact that the dot product is the commutative part of the Pauli product of two vectors, as eq.(10) shows, and then for any vectors $\vec{p}$ and $\vec{F}$ it holds that

$$\hat{F}\vec{p} = 2\vec{p}\cdot\hat{F} - \vec{p}\hat{F} \tag{78}$$

Then, for the operator $M = e^{\vec{F}} = \cosh z + \hat{F}\sinh z$, we have

$$M\vec{p} = \vec{p}\widetilde{M} + 2(\vec{p}\cdot\hat{F})\sinh z \tag{79}$$

It then follows that eq.(65) can be written as

$$P' = p_0 M M^+ + \vec{p}\widetilde{M}M^+ + 2(\vec{p}\cdot\hat{F})M^+\sinh z \tag{80}$$

Before we start the explicity calculations we recall that it is only necessary to calculate the real part of each term in eq.(80) since P, P' have only real components (see eq.(60)). We now calculate separately the three contributions to P'. The final result will be given in terms of $\vec{E}, \vec{B}$ and $z = x + iy$ (see eq.(40) and eq.(41)).

(i) The term $p_0 M M^+$. Observe that the product MM^+ had only a real part since it is hermitian. We already calculated this term in Section 4, through eq.(50), eq.(52) and eq.(54). In this way the term $p_0 M M^+$ gives the following contributions for P' :

scalar component: $$p_0(|\cosh z|^2 + (\vec{E}^2 + \vec{B}^2)\frac{|\sinh z|^2}{|z|^2}) \tag{81a}$$

vector component: $$\frac{p_0}{|z|^2}(\vec{E}f(x,y) + \vec{B}g(x,y) + 2\vec{E} \times \vec{B}|\sinh z|^2) \tag{81b}$$

(ii) The term $\vec{p}\widetilde{M}M^+$.
We have:

$$\widetilde{M}M^+ = |\cosh z|^2 + iIm\vec{\Omega} - \widehat{F}\widehat{F}^*|\sinh z|^2 \tag{82}$$

where $\vec{\Omega} = 2\widehat{F}^* \cosh z \sinh z^*$ is a complex vector (see eq.(51)). We have

$$Im\vec{\Omega} = \frac{1}{|z|^2}(\vec{E}g(x,y) - \vec{B}f(x,y)) \tag{83}$$

where $f(x,y)$ and $g(x,y)$ are given by eq.(53). Also, the product $\widehat{F}\widehat{F}^*$ is given by eq.(54). In this way $Re(\vec{p}\widetilde{M}M^+)$ has the following contributions:

scalar component: $$-2\vec{p}\cdot(\vec{E}\times\vec{B})\frac{|\sinh z|^2}{|z|^2} \tag{84a}$$

vector component: $$\vec{p}(|\cosh z|^2 - (\vec{E}^2 + \vec{B}^2)\frac{|\sinh z|^2}{|z|^2})$$

$$-\frac{\vec{p}}{|z|^2} \times (\vec{E}g(x,y) - \vec{B}f(x,y)) \tag{84b}$$

(iii) The term $2(\vec{p}\cdot\widehat{F})M^+ \sinh z$.

In this case we get the following contributions to the transformed vector P':

scalar component: $$\frac{1}{|z|^2}\{(\vec{p}\cdot\vec{E})f(x,y)+(\vec{p}\cdot\vec{B})g(x,y)\} \qquad (85a)$$

vector component: $$\frac{2|\sinh z|^2}{|z|^2}\{(\vec{p}\cdot\vec{E})\vec{E}+(\vec{p}\cdot\vec{B})\vec{B}\} \qquad (85b)$$

7. The Exact Composition of Two Boosts and the Thomas Precession.

The composition of two boosts has been largely discussed in the literature, either by using matrix algebra methods[11,12,13] or by using the Clifford algebras $\mathbb{R}_{1,3}$ [3,5] and $\mathbb{R}_{3,0}$ [6].

The calculations are all based on the fact that the composition of the boosts $\mathcal{B}_1$ and $\mathcal{B}_2$ can be written

$$\mathcal{B}_2\mathcal{B}_1 = \mathcal{B}\mathcal{R} \qquad (86)$$

To clarify the precise physical meaning of eq.(86) let us define $\mathcal{M}$ as being a Minkowski space–time. Reference frames in $\mathcal{M}$ are defined as vector fields $\mathbf{z}\in\tau\mathcal{M}$ (where τM is the tangent bundle) such that $g(\mathbf{z},\mathbf{z})=1$, where g is the Lorentz metric[14,15]. Inertial references frames are characterized by $D\mathbf{z}=0$, where D is the Levi–Civita connection of g in $\mathcal{M}$. Consider three inertial frames $\mathbf{z}_0,\mathbf{z}_1,\mathbf{z}_2$. Frame $\mathbf{z}_1$ moves with velocity $\vec{v}_1$ relative to $\mathbf{z}_0$ and $\mathbf{z}_0$ moves with velocity $\vec{v}_2$ relative to $\mathbf{z}_2$. The problem we want to solve is the following: which is the velocity of $\mathbf{z}_1$ relative to $\mathbf{z}_2$? Such a velocity is characteristic of the boost $\mathcal{B}$ in eq.(86). Taking into account that the rotation operators are unitary $\mathcal{R}\mathcal{R}^+=1$ and that the boost operators are hermitian ($\mathcal{B}=\mathcal{B}^+$) we get

$$\mathcal{B}_2\mathcal{B}_1^2\mathcal{B}_2 = \mathcal{B}^2 \qquad (87)$$

Eq.(36a) tell us that $\mathcal{B}_1^2$ is the representative in $\mathbb{R}_{3,0}$ of a vector: it is the vector velocity of $\mathbf{z}_1$ relative to $\mathbf{z}_0$. The left hand side of eq.(87) can thus be interpreted as a boosting of this vector by $\mathcal{B}_2$. Also $\mathcal{B}^2$ is the representative of the velocity vector of the reference frame $\mathbf{z}_1$ relative to $\mathbf{z}_2$.

To solve eq.(87) we start by splitting $\vec{v}_1$ in components, parallel and orthogonal to $\vec{v}_2$, respectively, as we made in eq.(66). Now, using eq.(67) we have

$$\mathcal{B}_2\vec{v}_1 = \vec{v}_{1//}\mathcal{B}_2 + \vec{v}_{1\perp}\tilde{\mathcal{B}}_2 \qquad (88)$$

Then,

$$\mathcal{B}_2\mathcal{B}_1^2\mathcal{B}_2 = \gamma_1(1+\vec{v}_{1//})\mathcal{B}_2^2 + \gamma_1\vec{v}_{1\perp} \qquad (89)$$

Being $\mathcal{B}^2 = \gamma(1+\vec{v})$ and taking into account that $\vec{v}_{1//}\vec{v}_2 = \vec{v}_1 \cdot \vec{v}_2$, the scalar and vector parts in eq.(87) are:

$$\gamma = \gamma_1\gamma_2(1+\vec{v}_1\cdot\vec{v}_2)$$
$$\gamma\vec{v}_\perp = \gamma_1\vec{v}_{1\perp} \tag{90}$$
$$\gamma\vec{v}_{//} = \gamma_1\gamma_2(\vec{v}_2+\vec{v}_{1//})$$

Let us now calculate the rotation resulting from the composition of $\mathcal{B}_1$ and $\mathcal{B}_2$. We write $\mathcal{B}_1 = e^{\vec{\nu}_1/2}$, $\mathcal{B}_2 = e^{\vec{\nu}_2/2}$. Then using eq.(37) we get

$$\begin{aligned} \mathcal{B}_2\mathcal{B}_1 = e^{\vec{\nu}_2/2}e^{\vec{\nu}_1/2} &= \cosh(\nu_1/2)\cosh(\nu_2/2) + (\hat{\nu}_1\cdot\hat{\nu}_2)\sinh(\nu_1/2)\sinh(\nu_2/2) \\ &+ \hat{\nu}_1\sinh(\nu_1/2)\cosh(\nu_2/2) + \hat{\nu}_2\sinh(\nu_2/2)\cosh(\nu_1/2) \\ &+ \mathrm{i}\hat{\nu}_2\times\hat{\nu}_1\sinh(\nu_1/2)\sinh(\nu_2/2) \end{aligned} \tag{91}$$

Comparing eq.(91) with eq.(46), that gives the product of a boost by a rotation, we see that for the validity of eq.(86), $\mathcal{B}_2\mathcal{B}_1 = \mathcal{BR}$, it is necessary that the rotation axis labelled by $\vec{n}$ be orthogonal to the direction of the resulting velocity $\vec{v}$, since the scalar part of $\mathcal{B}_2\mathcal{B}_1$ is a real number. Comparing the vector imaginary part of $\mathcal{B}_2\mathcal{B}_1$ with the vector imaginary part of $\mathcal{BR}$ in eq.(46), we get that the direction of the rotation axis is given by $\vec{\nu}_2\times\vec{\nu}_1$. Then dividing the imaginary vector part by the scalar real part in eq.(91) and eq.(46) and equating these terms, it results:

$$\begin{aligned} tg(\theta/2) &= \frac{\sin\alpha\; tgh\,(\nu_1/2)\; tgh\,(\nu_2/2)}{1+\cos\alpha\; tgh\,(\nu_1/2)\; tgh\,(\nu_2/2)} \\ &= \frac{v_1v_2\sin\alpha}{(1+1/\gamma_1)(1+1/\gamma_2)+v_1v_2\cos\alpha} \end{aligned} \tag{92}$$

where α is the angle between $\vec{v}_1$ and $\vec{v}_2$.

Thomas Precession: The Thomas precession is a phenomenon that appears when we consider the motion of a particle such that its acceleration has a component orthogonal to the velocity. The particle's world line in $\mathcal{M}$ can be considered as an integral line Γ of a reference frame $\mathbf{z}$ comoving with the particle. To each $p\in\Gamma$ we can associate an "instantaneous" rest inertial frame (irif). We can show that the irif's do not have the same orientation as the spatial coordinate axes. In order to relate the irif's associated to the different points of the particle's world line, let us consider an inertial frame $\mathbf{z}_0$ (the laboratory). The irif associated to the particle at the instant t in $\mathbf{z}_0$ will be denoted $\mathbf{z}_1$, it has velocity $\vec{v}$ relative to $\mathbf{z}_0$. Then the coordinates naturally adapted to $\mathbf{z}_1$ [14] are transformed into the coordinates of $\mathbf{z}_0$ by a Lorentz boost $\mathcal{B}(\vec{v})$. The irif associated to the particle at the time $t+\delta t$ will be denoted $\mathbf{z}_2$ and has velocity $(\vec{v}+\delta\vec{v})$ relative to $\mathbf{z}_0$. The coordinates naturally adapted to $\mathbf{z}_2$ can be obtained from the coordinates of $\mathbf{z}_0$ by a Lorentz boost $\mathcal{B}(-\vec{v}-\delta\vec{v})$. Then, $\mathbf{z}_1$ and $\mathbf{z}_2$ are related by the composition of the boosts

$\mathcal{B}(-\vec{v}-\delta\vec{v})\mathcal{B}(\vec{v})$ and it follows that the space axes of z_1 are rotated by an angle θ (given by eq.(92)) relative to the axes of z_2. For a circular motion, $\delta\vec{v}\cdot\vec{v}=0$ and we get, by using the above equation with the approximations $\cos\alpha \simeq -1$, $\sin\alpha \simeq |\delta\vec{v}|/v$, that

$$tg(\theta/2)=\frac{\gamma^2 v|\delta\vec{v}|}{2(1+\gamma)} \tag{93}$$

For small velocities, i.e., $|\vec{v}|<<1$, $tg\theta/2\simeq\theta/2$ and

$$\theta=\frac{\gamma^2 v|\delta\vec{v}|}{1+\gamma} \tag{94}$$

The direction of the rotation axis is given by $-(\vec{v}+\delta\vec{v})\times\vec{v}$ and the rotation frequency is

$$\vec{w}=-\frac{\gamma^2}{1+\gamma}\frac{\delta\vec{v}}{\delta t}\times\vec{v} \tag{95}$$

The Thomas frequency is, of course, $\vec{w}_T=-\vec{w}$ since it is defined as how much the axes of z_2 are rotated relative to the axes of z_1.

8. Conclusions.

We have obtained in this paper with the Clifford algebra methodology a finite closed formula for a generic Lorentz transformation $M\in SL(2,\mathbb{C})$ [the master equation, eq.(32)] together with the physical interpretation of the parameters. Our master equation is to be regard as a generalization of the well known equations describing transformations of boosts and rotations. Our presentation is sufficiently clear and can be understood even with a minimum knowledge of Clifford algebras. The master equation (eq.(32)) and its consequences can be exposed by the Pauli algebra only, by representing the objects of the space–time algebra $\mathbb{R}_{1,3}$ in the former algebra.

Using our master equation we studied several important topics as e.g., the composition of two boosts and the Thomas precession. Our treatment is to be compared with other approaches using Clifford algebras [3,4,5,6] and matrix algebra methods, [11,12,13] and also with the traditional method presented, e.g., in [16].

Besides the results presented above let us mention that the Lorentz transformations have a surprising dynamical interpretation [4,7], i.e. the solution of the motion's equations for a charged particle (Lorentz force) can be expressed as a Lorentz transformation whose generators are intimately associated with the electromagnetic field. This is possible because the electromagnetic field has the same nature as the generators of the Lorentz transformations [7].

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