PERIODIC ORBITS NEAR THE BOUNDARY OF A 3-DIMENSIONAL MANIFOLD

J. Sotomayor and *M. A. Teixeira*

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Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
IMECC – UNICAMP
Caixa Postal 6065
13.081 – Campinas – SP
BRASIL

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1- INTRODUCTION

In this paper we study one parameter families of vector fields X_λ depending on a real parameter λ, on a 3-dimensional smooth manifold M, with boundary $S = \partial M$. Our main interest is to describe the orbit structure of the family, for parameter values near 0, on a small neighborhood of a periodic orbit γ_o of X_o, tangent to S.

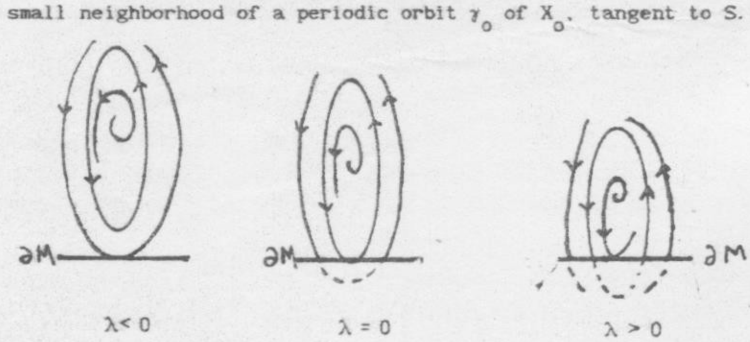

Figure 1 - The family X_λ

This paper continues the study performed in [ST], where the orbit structure near a point at which S and the family of singular points of the family meet in general position, was described.

We assume that the intersection between the manifold S and the surface of periodic orbits of the family X_λ is transversal and occurs at a point of quadratic contact (i.e. fold point) between S and the hyperbolic periodic orbit γ_o of the vector field X_o.

There are three essentially different generic cases to consider, according to the relative position of the spectrum of the Poincaré map of the periodic orbit γ_o and the unitary circle, assuming that the eigenspaces and the tangent space to the fold curve in S meet transversally.

a) The eigenvalues are real, distinct from each other and different from 1 in absolute value, and the corresponding eigenspaces are transversal to S.

b) The eigenvalues are complex conjugate; we assume further that the arguments $\pm\theta$ of the eigenvalues satisfy: $n\theta \neq k\pi$, for $n, k \in \mathbb{N}$.

c) The eigenvalues are real, equal, negative and different from 1, but the linear part is not diagonalizable. Furthermore, the eigenspace is transversal to the fold line.

Concerning the above case a), there are tree subcases to consider:

a_1) The eigenvalues in absolute value are both larger or smaller than 1.

a_2) Both eigenvalues are negative and in absolute value one of them is larger while the other is smaller than one.

a_3) The diffeomorphism is of saddle type but it is disjoint of case a_2.

Denote by χ^r the space of all C^r vector fields tangent to M, endowed with the C^r topology, where r is big enough and finite. In this paper we aim to describe the qualitative changes (bifurcations) that occur near a periodic orbit γ_o of a vector field X_o under the conditions described above. For this purpose we will assume that M is compact and regard the family X_λ as a curve in the space χ^r.

1.1 Definition: Denote by Q(T) the space of vector fields in χ^r such that:

i) all of its critical points are hyperbolic and contained in the interior of M;

ii) all the points of tangency of the vector field with S are folds or cusps (to be defined below);

iii) all of its periodic orbits of period less than or equal to T are hyperbolic and contained in the interior of M.

iv) the stable and unstable manifolds of the vector field meet transversally;

v) all stable and unstable manifolds of the vector field meet the boundary, transversally.

1.2 Proposition: The set Q(T) is residual in χ^r.

Proof.- Similar to that of Kupka -Smale Theorem [MP_1].

Let p∊S and $f: M,S \to \mathbb{R},0$ be a C^∞ local implicit representation of S at p with $df(p) \neq 0$.

1.3 Definition. We say that $p \in S$ is an S-singular (resp. S-regular) point of $X \in \chi^r$ if $Xf(p) = 0$ (resp. $Xf(p) \neq 0$). We denote by S_X the set of all S-singular points of X.

1.4 Definition. We say that $p \in S$ is a fold (resp. cusp) point of X if If $Xf(p) = 0$ and $X^2f(p) \neq 0$ (resp. $Xf(p) = X^2f(p) = 0$ and $\{df(p), d(Xf(p)), d(X^2f(p))\}$ are linearly independent).

1.5 Definition: Let $X \in \chi^r$, γ be a periodic orbit of X tangent to a 2-dimensional subamnifold S of M and Y be in a small neighborhood of X in χ^r. The vector fields X and Y are said to be S-equivalent at γ if there are a neighborhood V of γ in M and a homeomorphism h mapping W to itself, preserving S, and sending orbits of X onto orbits of Y.

2- Statement of results

Denote by $Q_1(T)$ the set of vector fields in χ^r which verify conditions i, ii, iv and v, but violate iii, in the sense that exatly one of the hyperbolic periodic orbits assumed to be less than T, is tangent to M at a single point of quadratic contact.

We consider the following subsets of $Q_1(T)$: $Q_1(T, a_i)$, $Q_1(T,b)$, $Q_2(T,c)$ which verify the conditions a_i, b and c, for i=1, 2, 3, defined in Section 1 respectively.

Theorem 1: The set $Q_1(T)$ is a submanifold of class C^{r-1} and codimension one in χ^r. Each subclass $Q_1(T,a_i)$, $i = 1,2,3$ as well as the subclass $Q_1(T,b)$, is an open submanifold of $Q_1(T)$. The subclass $Q_2(T,c)$ is a submanifold of codimension two of χ^r. Also, the union $Q_{11}(T) = Q_1(T,a_1) \cup Q_1(T,b) \cup Q_2(T,c)$ is still a codimension one submanifold of χ^r.

Theorem 2: Along each one of the manifolds $Q_1(T,a_1)$, $Q_1(T,a_2)$, $Q_1(T,b)$ and $Q_{11}(T)$ the topological type of the vector fields is locally constant (according to Definition 1.5).

Theorem 3: The submanifold $Q_1(T,a_3)$ is further foliated by codimension one submanifolds (defined by the levels of the quotient between the logarithms of the absolute values of the eigenvalues), along which the topological type is locally constant.

Denote by Z^r the space of all C^1 mappings ζ: $[-\varepsilon,\varepsilon] \to \chi^r$. endowed with the C^1 topology. We say that ζ_1 and ζ_2 are C^0 equivalent if there is a homeomorphism $h : [-\varepsilon,\varepsilon] \to$ Homeo.(M) such that $h(\lambda)$ is an S-equivalence between $\zeta_1(\lambda)$ and $\zeta_2(\lambda)$ (in the sense of definition 1.5).

Theorem 3: Consider X_o in $Q_1(T)$ and γ_o its periodic orbit of period less than T, tangent to the boundary $S = \partial M$. The one parameter family in Z^r, $\zeta(\lambda) = X_\lambda$, $\lambda \in (-\varepsilon, \varepsilon)$ is locally structurally stable around γ_o provided that :

i) $\zeta(-c\ ,\ c) \subseteq Q(T) \cup Q_1(T,a_1) \cup Q_1(T,a_2) \cup Q_1(T,b) \cup Q_2(T,c)$.

ii) ζ is transversal to $Q_1(T,a_1) \cup Q_1(T,a_2) \cup Q_1(T,b) \cup Q_2(T,c)$.

iii) $\zeta(-c)$ and $\zeta(c)$ are in $Q(T)$.

2.1- Remark: If instead of the boundary of M we consider S to be a distinguished 2-dimensional submanifold of M, we obtain a different results for the submanifolds $Q_1(T,a_3)$, $Q_1(T,b)$, and $Q_2(T,c)$ defined analogously to those considered in the statement of Theorem 1 (we use the same notations for both situations). In fact, we have the following proposition.

2.2 Proposition: Let M be a compact C^∞ 3-dimensional manifold and let S be a compact 2-dimensional submanifold of M.

1- $Q_1(T,b)$ is a codimension one and open submanifold of χ^r foliated by codimension one submanifolds corresponding to the value of the argument of the eigenvalue . If the argument of the eigenvalue is rational then the topological type is locally constant along the leaves.

2- $Q_1(T,a_3)$ is a codimension one and open submanifold of χ^r foliated by codimension one submanifolds corresponding to the quotient between the logarithms of the absolute values of the eigenvalues.

3- On $Q_1(T,a_1) \cup Q_2(T,c)$ the topological type is not locally constant; however on $Q_1(T,a_1)$ it is so.

2.3 Remark: A similar situation has been found in [S_3] concerning regularity properties of the Bifurcation Set.

3- Poincare Mapping

In this section we deal with the simultaneous behavior of a couple diffeomorphism-curve in the plane. All results of this section will be used in the sequel.

Let $X \in Q_1(T)$, γ be the periodic orbit of X tangent to ∂M , $\rho: R^2,0 \Rightarrow R^2,0$ be the 1^{th} return mapping associated to X and γ and L be the curve in $R^2,0$ representing the singular set of X ; this means that the orbit of X passing through any point of L is tangent to ∂M (of course, $R^2,0$ represents a 2-dimensional submanifold transverse to X around a point of γ). The study is reduced to the analysis of a pair (ρ,α) where ρ is a planar diffeomorphism and $\alpha: R,0 \Rightarrow R^2,0$ is a C^∞ imbedding with $Im.\alpha = L$ (see Figure 2).

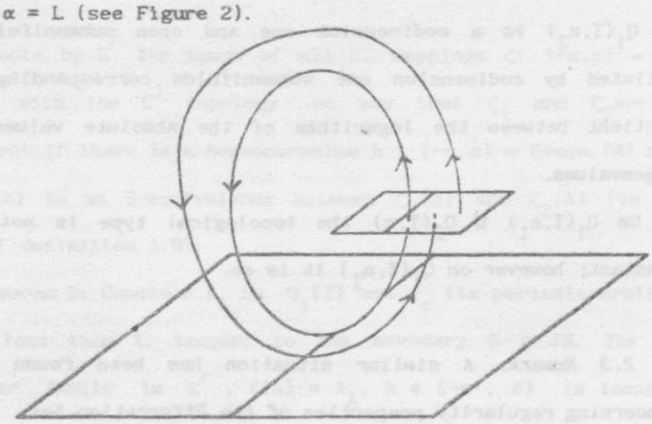

Figure 2 - The Poincare mapping

We have to introduce the following concepts.

Denote by I_0 the space of C^∞ imbedding $\alpha: R,0 \to R^2$ and D the space of C^r-diffeomorphisms $\phi : R^2,0 \to R^2,0$ both endowed with the C^r-topology.

3.1 Definition: We say that two pairs (ϕ_o,α_o) and (ϕ,α) in $G_0 = D \times I_0$ are conjugate if there is a C^0-conjugacy $h: R^2,0 \to R^2,0$ between ϕ_o and ϕ and such that $h(Im(\phi_o)) \subset Im(\phi)$.

In what follows we consider I_1 as being the space of C^∞ imbedding $\alpha \in I_0$ such that $\alpha(0)=0$. Observe that $G_1 = D \times I_1$ is a codimension one submanifold of G_0.

3.2 Proposition : The pair (ϕ_o,α_o) is structurally stable in G_1 provided that : i)the eigenvalue λ_1,λ_2 of $\phi_o'(0)$ are real and distinct ; ii)the eigenspaces of $\phi_o'(0)$ are transverse to $Im(\alpha_o)$ at 0.

Proof: Denote by $\Sigma_1(a_1)$ the set of all $(\phi,\alpha) \in G_1$ satisfying the above conditions i) and ii). Of course $\Sigma_1(a_1)$ is open in G_1.

Let $(\phi_o,\alpha_o) \in \Sigma_1(a_1)$ and $L_o = Im.\alpha_o$. Assume for instance that $0<\lambda_1<\lambda_2<1$.

We show the stability of (ϕ_o,α_o) in G_1.

Let D_o be a very small fundamental domain of ϕ_o, bounded by the curves C_o and $\phi_o(C_o)$ (C_o is a closed curve as it is shown in the Figure below).

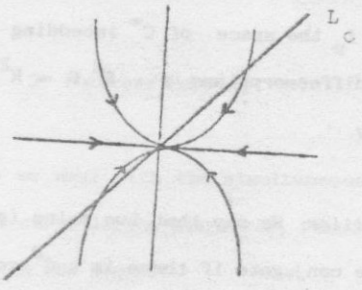

Figure 3 - Nodal singularity

We consider D_0, $D_1 = \phi_o(D_0)$, $D_2 = \phi_o(D_1)$, ..., $D_n = \phi_o(D_{n-1})$, ..., $L_0 = L \cap D_0$,

$L_1 = \phi_o^{-1}(L \cap D_1)$, ..., $L_n = \phi_o^{-n}(L \cap D_n)$...

We have $L_i \subset D_0$ for every i ; moreover L_i converge to T_2, where T_2 is the eigenspace of $\phi_o'(0)$ associated to λ_2.

For any small perturbation (ϕ,α) of (ϕ_o,α_o) in G_1 we get similar objects

$$\lambda_1', \lambda_2', T_1', T_2', D_n', L_n'.$$

We shall construct a conjugacy between the pairs above by proceeding in the following way:

We define the required homomorphism by firstly sending D_0 in D_0' in such a way that L_n goes to L_n'. The extension of the conjugacy to a full neighborhood of 0 is immediate.∎

The following result is related with the condition a_1.

given in the Introduction.

3.3 Lemma: Let $(\phi_o, \alpha_o) \in G_1$. Assume that the eigenvalues of $\phi_o'(0)$ are real and distinct and that there is one eigenspace associated to ϕ_o which is tangent to the curve $L_o = \mathrm{Im}.\alpha_o$ at 0. Then (ϕ_o, α_o) is not structurally stable in G_1.

Proof- It is enough to observe that arbitrarily close to (ϕ_o, α_o) there is in G_1 an element (ϕ, α) such that either

i) the curve $L = \mathrm{Im}.\alpha$ meets the strong invariant manifold of ϕ twice (see Figure 4), or

ii) the saturates of L_o and L, by ϕ_o and ϕ respectively have distinct topological portraits (as shown in Figure 5).■

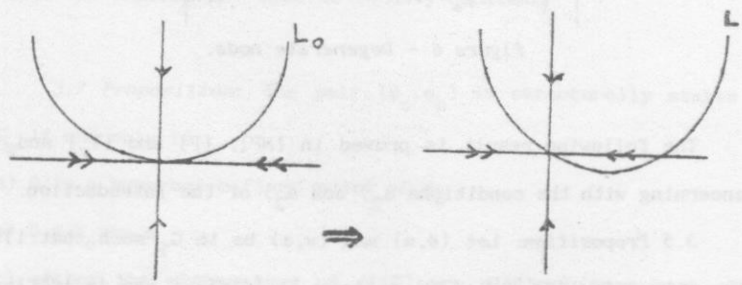

Figure 4 - Degenerate node: case i).

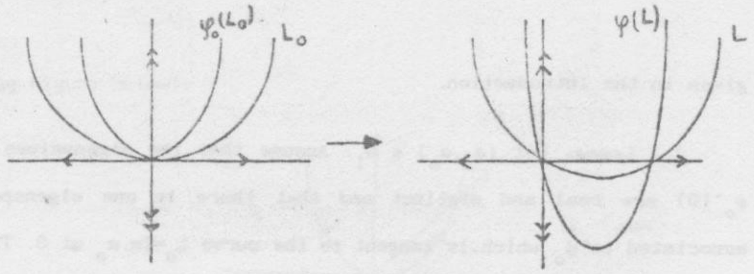

Figure 5 - Degenerate node: case ii).

3.4- *Remark:* Let $(\phi_o,\alpha_o) \in G_1$. Assume that the eigenvalues of ϕ_o are real and equal. Then (ϕ_o,α_o) is never structurally stable in G_1. We have just to observe that ϕ_o can be approximated by diffeomorphisms which are of focus type and to look at the saturate of L_o by ϕ.

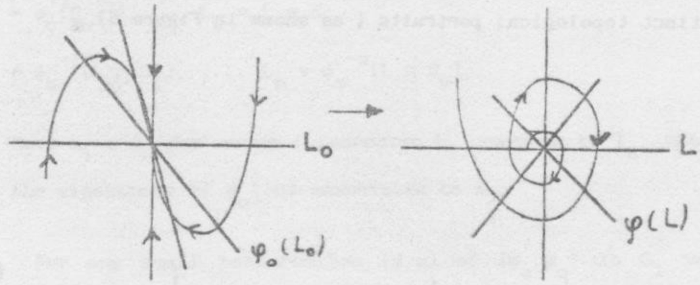

Figure 6 - Degenerate node.

The following result is proved in [MP], [P] and $[T_2]$ and is concerning with the conditions a_2) and a_3) of the Introduction.

3.5 *Proposition*: Let (ϕ,α) and (υ,α) be in G_1 such that: i) the eigenvalue of $\phi'(0)$ and $\upsilon'(0)$ satisfy $|\lambda_1|<1<|\lambda_2|$ and $|\mu_1|<1<|\mu_2|$ respectivelly; ii) the eigenspaces of $\phi'(0)$ and $\upsilon'(0)$ are transverse to $L=\mathrm{Im}.\alpha$ at 0. Then (ϕ,α) and (υ,α) are C^o-conjugated if and only if $\log|\lambda_1|/\log|\lambda_2| = \log|\mu_1|/\log|\mu_2|$.

3.6.a *Corollary*: Let $(\phi_o, \alpha_o) \in G_1$. Assume that the eigenvalue of $\phi_o'(0)$ satisfy $|\lambda_1| < 1 < |\lambda_2|$. Then (ϕ_o, α_o) is never structurally stable in G_1.

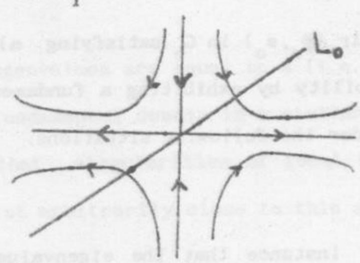

Figure 7 - Saddle singularity.

Denote by $\Sigma_1(a_2, a_3)$ the set of all (ϕ, α) in G_1 satisfying the conditions i) and ii) of Proposition 3.5..

3.6.b-*Corollary*: The set $\Sigma_1(a_2, a_3)$ is foliated by codimension one submanifolds (defined by the levels of the quotient between the logarithms of the absolute values of the eigenvalues), along which the topological type is locally constant.

3.7 *Proposition*: The pair (ϕ_o, α_o) is structurally stable in G_o if and only if :

a) 0 is a hyperbolic fixed point of ϕ_o ;

b) $0 \notin L_o = \mathrm{Im}.\alpha_o$;

c) either the eigenvalues of $\phi'(0)$ are distinct from each other or they are equal but $\phi'(0)$ is diagonalizable. In the last case we impose that L_o is transverse to the associated eigenspace.

d) if ϕ is of node type then L_o is transverse to the strong invariant manifold of ϕ;

e) if ϕ is of saddle type then L_o is transverse to the invariant manifold of ϕ.

Proof: The proof of the necessary condition is trivial. We concentrate on the sufficient condition.

Given a pair (ϕ_o, α_o) in G_o satisfying a), b), c, d and e) we prove its stability by exhibiting a fundamental domain in each case. We consider the following situations:

i) Node case:

Assume for instance that the eigenvalues of $\phi_o'(0)$ satisfy $0<\lambda_1< \lambda_2<1$; the other cases are treated in a similar way.

We choose a fundamental domain D bounded by a closed curve C (surrounding 0) and $\phi_o(C)$, such that L is completely contained in D, as shown in the figure below.

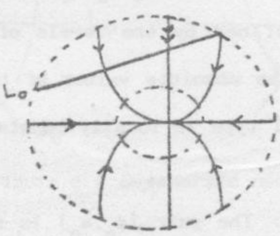

Figure 8 - Fundamental domain of a node - regular case.

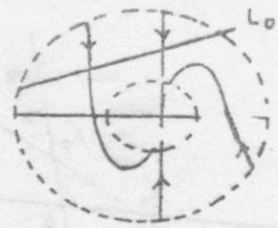

Figure 9 - Fundamental domain of a node - degenerated case.

If the eingenvalues are equal to λ (i.e. a degenerate node) we may choose a fundamental domain in a similar way as above. We must just observe that singularities of focal type or non degenerate nodal type exist arbitrarily close to this singularity.

h) Saddle case:

Assume first that the eigenvalues of $\phi_o'(0)$ satisfy $0<\lambda_1<1<\lambda_2$ and let T_1, T_2 be the respective eigenspaces. We may, and do, restrict the analysis to a region (quadrant) determined by the invariant manifolds of ϕ_o. We consider without lost of generality that ϕ_o is a linear diffeomorphism.

Consider the following objects:

$L_1^+ = \phi_o(L),\ L_2^+ = \phi_o^2(L), \ldots\ldots L_n^+ = \phi_o^n(L),$

$L_1^- = \phi_o^{-1}(L),\ L_2^- = \phi_o^{-2}(L), \ldots\ldots\ldots\ldots L_n^- = \phi_o^{-n}(L),$

$A_o^+ = L \cap L_1^+,\ A_o^- = L \cap L_1^-,\ y_o = T_1 \cap L,\ x_o = T_2 \cap L$

$y_1 = L_1^+ \cap T_1,\ x_1 = L_1^- \cap T_2, \ldots\ldots, y_n = L_n^+ \cap T_1,\ x_n = L_n^- \cap T_2$

$B_1 = L_1^+ \cap L_1^-.$

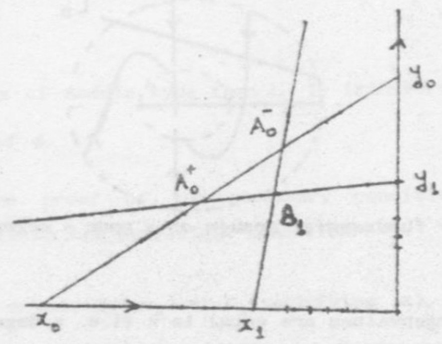

Figure 10 - Saddle singularity: a fundamental domain

For a pair (ϕ,α) arbitrarily close to (ϕ_o,α_o) in G_o we have similar objects:

L', L'^+_1, L'^-_1, ..., L'^+_n, L'^-_n....
A'^+_o, A'^-_o, x'_o, y'_o, ..., x'_n, y'_n, ..., B'_1.

We construct a C^o equivalence between (ϕ_o,α_o) and (ϕ,α) as follows:

1- Define some how a homomorphism from the arc of L : $[A_o^+ \; A_o^-]$ to the arc of L':$[A'^+_o A^-_o]$ and extend it to: from $[A_o^- \; B_1]$ onto $[A'^-_o \; B_1']$ in $\phi(L)$ and $\phi(L')$ respectively.

2- Draw "lines" (more precisely, smooth family of curves C_υ, $\upsilon \in [0,1]$) transverse to the stable invariant manifoldof ϕ_o, with C_o = $[A_o^+, y_1])$ joining A_o^+ to $[y_o, \; y_1]$ and A_o^- to $[x_o, x_1]$. We do the same for $(\phi \; ,\alpha)$ and the required conjugacy must preserve these objects. In this way the equivalence is also extended to the arcs $[x_o \; x_1]$ and $[y_o \; y_1]$ of the invariant manifolds.

3-Now we extend the equivalence to a full neighborhood of 0 in a straighforward and natural way.

iii)Focal case:

In this case the construction of the fundamental domain is similar to that obtained in the Nodal case.

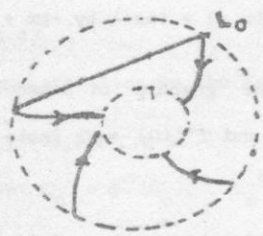

Figure 11 - Focal singularity- a fundamental domain.

The above arguments allow us to finish the present proof without any difficulty. ∎

3.7.a *Remark*: Denote by Σ_o the collection of all elements in G_o which are structurally stable in G_o. Of course this set is characterized by the last proposition and is open and dense in G_o.

3.8 *Proposition*: Let (ϕ_o, α_o) be in G_1 such that the eigenvalues of $\phi_o'(0)$ are $= a \pm ib$ with $b \neq 0$. Then : i) the rotation angle associated to ϕ_o is a topological invariant for the structural stability in G_1;

ii)assume that the rotation angle ϕ_o is rational and the contact between $L = \mathrm{Im}.\alpha_o$ and the invariant foliation of ϕ_o is parabolic. Then any small perturbation (ϕ, α) of (ϕ_o, α_o) in G_1 is conjugate to (ϕ_o, α_o)provided that the rotation angle of the both diffeomorphisms coincide.

Proof: Part i) is obvious.

Let us prove part ii).

Consider, via Stenberg Theorem that, the diffeomorphism in question in its normal form

$$\phi_o(x,y) = (ax + by, -bx + ay)$$

and $L = \text{Im}.\alpha_o$ as being the graphic of a smooth mapping $y = f(x)$ with $f(0) = f'(0) = 0$ and $f''(0) \neq 0$ (note that $f'(0) = 0$ is not a restriction).

Let (ϕ,α) be a small perturbation of (ϕ_o,α_o) in G_1.

In polar coordinates, we have $\phi_o(z) = \rho e^{i\theta} z$; assume for instance that $0<\rho<1$.

Let $\theta \in Q$ with $\theta = p/q$.

For $n = kq$, we get

$$\phi_o^n(z) = \rho^n e^{in\theta} z$$

In this way we have $\phi_o^n(\rho^n z) = 0$ if and only if $(\rho^n y - f(\rho^n x)) = 0$.

So $y = f(\rho^n x)/\rho^n$

Since $f(x) = ax^2$+h.o.t with $a \neq 0$, we get $y = \rho^n ax^2 + \rho^{2n}\,o(x^3)$.

Now we are able to choose in a natural way a fundamental domain for a conjugacy between (ϕ_o,α_o) and (ϕ,α), as follows.

Let D be a fundamental domain for the mapping ϕ_o^q, bounded by C_o and $C_1 = \phi^q(C_o)$ (see Figure 12).

Call $C_n = \phi_o^{nq}(C_o)$.

Denote by D_n the annulus bounded by C_n and C_{n+1}, $L_n = L \cap D_n$ and $K_n = \phi_o^{-nq}(L_n)$, for $n = 0,1,2,\ldots$.

Observe that $L \cap C_o = \{a_o\} \cup \{b_o\}$, $K_j \cap K_l = \emptyset$ for $j \neq l$, $K_n \cap C_o = \{a_n\} \cup \{b_n\}$, with $(a_n) \to p_o$ and $(b_n) \to p_o$, as is shown in Figure 13.

We restrict our attention to the following objects D_o , L_o and $\phi^{-j}(L_o)$, for $j = 0,1,\ldots,q-1$.

As before , let $C_o^1 = \phi^{-1}(C_o)$, , $L_o^k = L \cap D_o^k$ and $K_o^k = \phi_o^{-k}(L_o^k)$ where D_o^k is the region bounded by C_o^k and C_o^{k+1} with $k=0,1,\ldots,q-2$.

Observe that $K_o^k \cap L_o^o$ contains at most k points.

So D_o^o is a fundamental domain for (ϕ_o,α_o) , recalling that L_o^o and all points in $K_o^k \cap L_o^o$ must be distinguished in the process of construction of the present conjugacy.■

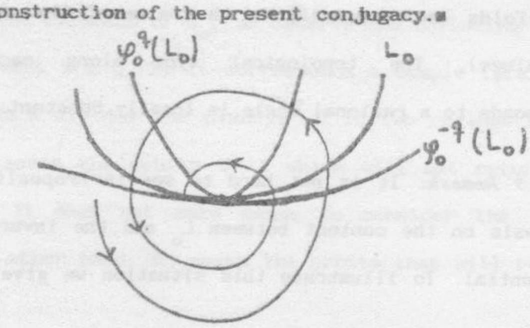

Figure 12 - Focus - rational case.

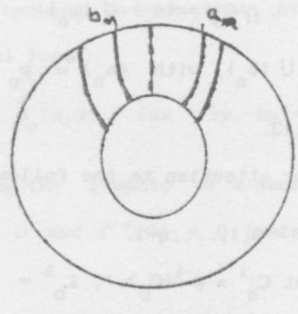

Figure 13 - Focal fundamental domain.

Call $\Sigma_1(b)$ the set in G_1 constituted by the elements (ϕ,α) which satisfy the hypothesis of Proposition 3.8.

3.8.a *Corollary:* The set $\Sigma_1(b)$ is foliated by codimension one submanifolds (corresponding to the value of the argument of the eigenvalues). The topological type along each leaf that corresponds to a rational angle is locally constant.

3.9 *Remark:* It is not hard to see in Proposition 3.8 that, the hypothesis on the contact between L_o and the invariant foliation is essential. To illustrate this situation we give the following result:

3.10 *Proposition:* Let ϕ_o be in *D* and α_o and α be in I_1. Assume further that

i) ϕ_o is a linear diffeomorphism with eigenvalue λ satisfying $|\lambda|>1$, $\lambda = |\lambda| e^{i\tau 2\pi}$, τ being irrational;

ii) $L_o = \mathrm{Im.}\ \alpha_o$ is a straight line trought the origin.

iii) $L = Im.\ \alpha$ is a curve in $R^2,0$ with non zero curvature in the origin.

Then (ϕ_o, α_o) and (ϕ_o, α) are not C^o conjugate.

Proof: First of all observe that $\phi_o{}^n(L_o) \cap L_o = \{0\}$ for $n\neq 0$. Since $\phi_o{}^n(L)$, is getting more and more straight, there are intersections of $\phi_o{}^n(L)$ and L for suitable n (namely those n with $n.\tau$ near zero mod(1), arbitrarily near (but different from) the origin. This implies that (ϕ_o, α_o) and (ϕ_o, α) are not C^o conjugated. Moreover, if β is a diffeomorfism of $R^2,0$ such that $\beta(L) \subseteq L_o$ we deduce immediately that (ϕ_o, α_o) and $(\beta \circ \phi_o \circ \beta^{-1}, \alpha_o)$ are not C^o conjugate.∎

3.11 Remark: Concerning the investigation of the dynamics of vector fields defined on a manifold with boundary by means the analysis of the elements of G_o, we observe the following :

a) For an element $X \in Q_1(T)$ it corresponds a couple (ϕ, α) in G_1. So that $L = Im.\alpha$ divides the plane $R^2,0$ in two regions R_1 and R_2, where R_1 meets the orbits of X which will not return to the plane. On R_1 it does not make sense to consider the image $\phi(R_1)$. On the other hand, R_2 meets the orbits that will return to the section.

b) In the case that $X \in Q(T, a_2)$, any orbit of the vector field will exit M after at most two returns.

c) In the case that $X \in Q(T, b)$ and $X \in Q_2(T, c)$, any orbit will exit M after finitely many returns.

In conclusion, the above considerations together with the results of this section lead us to characterize immediately the qualitative behavior of a vector field in $Q_1(T)$ near the boundary of a manifold . We have the following result.

3.12 Corollary: Let $(\phi_o,\alpha) \in G_1$ such that ϕ_o satisfies one of the conditions a_2, b, c, given in Section 1. Then there is a positive and finite number k such that $\phi^k(p)$ and $\phi^{-k}(q)$ are in R_1 for every p, q in R^2,0.

4- Codimension one singularities

Denote by Φ^Γ the space of C^1 mappings $\zeta : J \to G_o$, with J = $[-\varepsilon,\varepsilon]$ endowed with the C^1 topology. We say that $\lambda_o \in J$ is an ordinary point of $\zeta \in \Phi^\Gamma$ if there is a neighborhood N of λ_o such that $\zeta(\lambda_o)$ is C^o conjugate to $\zeta(\lambda)$ for every $\lambda \in N$; if λ_o is not an ordinary value of ζ, it is called a bifurcation value of ζ.

4.1 Definition: We say that ζ_1 and ζ_2 in Φ^Γ are conjugate if there is a homeomorphism h:J $\to$ Homeo.(M) such that $h(\lambda)$ is a conjugacy between $\zeta_1(\lambda)$ and $\zeta_2(h(\lambda))$. With this concept, the structural stability in Φ^Γ is defined in an obvious way.

Let us denote by A^Γ , the collection of the elements $\zeta \in \Phi^\Gamma$ such that :

1) $\zeta(J) \subset \Sigma_o \cup \Sigma_1(a_1)$.
2) ζ is transversal to $\Sigma_1(a_1)$.
3) $\zeta(a)$ and $\zeta(b)$ are in Σ_o.

We have the following result:

4.2 ***Proposition***: **Any $\zeta \in A^{\Gamma}$ is Structurally Stable in Φ^{Γ}.**

Proof: The proof follows immediately from Propositions 3.2 and 3.7.■

Denote by F_2 the foliation of $\Sigma_1(a_2,a_3)$ described in Corollary 3.6.b.

4.3 ***Proposition***: **Let $\zeta \in \Phi^{\Gamma}$ be transversal to $\Sigma_1(a_2,a_3)$ with $\zeta_o(0) \in \Sigma_1(a_2,a_3)$ and ζ_1 be a small perturbation of ζ_o in Φ^{Γ}. Then ζ_o and ζ_1 are conjugate provided that $\zeta_o(0)$ and $\zeta_1(0)$ are in the same leaf of F_2.**

Proof: The proof follows from Propositions 3.6 and 3.7.■

Denote by F_3 the foliation of $\Sigma_1(b)$ described in Corollary 3.8.a.

4.4 ***Proposition***: **Let $\zeta_o \in \Phi^{\Gamma}$ with $\zeta(0) \in \Sigma_1(b)$. Assume that ζ is transverse to $\Sigma_1(b)$ and let ζ_1 be a small perturbation of ζ_o in Φ^{Γ}. Then ζ_o and ζ_1 are conjugate provided that $\zeta_o(0)$ and $\zeta_1(1)$ are in the same leaf of F_3 and this leaf must correspond to a rational argument of the eigenvalues.**

Proof: The proof follows from Propositions 3.7 and 3.8.■

5- The manifold $Q_1(T,b)$ (the irrational case)

Let (ϕ,α) be in $\Sigma_1(b)$ such that the argument of the

eigenvalues of $\phi'(0)$ is irrational. Assume that ϕ is linear and in polar coordinates (r,θ), the curve $L = \mathrm{Im}.\alpha$ is given by the graphic of a C^{∞} mapping:

$$r = a\sin(\theta) + h_1(\sin(\theta))$$

with $a\neq 0$ and h_1 being the higher order terms of the mapping.
Let $\phi(r,\theta) = (\rho r, \mu + \theta)$ with $0<\rho<1$ and μ being irrational.

5.1 Proposition : Assume that $h_1 \equiv 0$ and let β be a small pertubation of α in I_1 in a way that $L' = \mathrm{Im}.\beta$ is given by $\theta = b\sin(r)$. We have that (ϕ,α) and (ϕ,β) are conjugate in G_1.

Proof: First of all, observe that

$$\phi^n(r,\theta) = (\rho^n r, n\mu + \theta).$$

The points in $\phi^m(L) \cap \phi^n(L)$, $n \in Z$, are defined by the values of θ, which satisfy:

$$\rho^{n-m}\sin(\theta - n\mu) = \sin(\theta - m\mu).$$

In this way, we determine a sequence θ_{nm} in $r = 0$ (and so a sequence x_{nm} in L), which does not depend on the particular value of a.

Consider now the curve $C = \{(r,\theta) : r = r_o\}$, with r_o small.

The points in $C \cap \phi^n(L)$ are defined by the expression

$$y_n = (\mathrm{arc.\ sin.}(r_o / a\rho^n)) + n\mu.$$

Consider the curve $C' = \{(r,\theta) : r = r_1\}$ where r_1 satisfies $r_1 = (b / a)r_o$. Observe that the points in $C' \cap \phi^n(L')$ are defined by the same expression given above, where $L' = \mathrm{Im}.\ \beta$.

Now we may choose a fundamental domain for (ϕ,α) on which we are able to construct the required conjugacy. ∎

5.2 *Remark:* We observe that the assumption "ϕ is linear" is not a restriction due to Stenberg Theorem and the condition "$a \neq 0$" means that the contact between L and the invariant foliation of ϕ, in cartesian coordinates, is parabolic. The last result remains true in the case $|\rho| \neq 1$.

6- Proof of results

6.1- Proof of Theorem 1.

Let $X_o \in Q_1(T)$, and γ_o be the periodic trajectory of X_o tangent to $S = \partial M$ at a point p.

We choose coordinates in M around p such that the manifold S ∂M is locally represented by the mapping $f(x,y,z) = z$, and γ_o is C^r parametrized by an imbedding $\alpha_o : (-\varepsilon, \varepsilon) \to M$ given by $\alpha_o(t) = (0,t,t^2) + h.o.t.$.

The correspondence $Y \to \alpha_Y$ is C^r where Y is a vector field in a neighborhood of X_o in χ^r, α_Y is a C^r parametrization of the periodic orbit γ_Y of Y close to γ_o (we observe that such γ_Y can be off M).

Let V be a small neighborhood of X in χ^r.

Define $F : V \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ given by $F(Y,t) = (\pi^3\alpha_Y)(t)$ where $(\pi^3\alpha_Y)$ is the third component of the mapping α_Y.

We have $F_t(X_o,0) = 0$, $F_{tt}(X_o,0) \neq 0$ (say >0). In this way there exists a map $t = t(Y)$ defined in a neighborhood of X_o

satisfying $F_t(Y, t(Y)) = 0$.

Define $G(Y) = F(Y, t(Y))$. It satisfies :

i) If $G(Y) > 0$ then there exist t_1, t_2 in a neighborhood U of 0 with $t_1 < t(Y) < t_2$ satisfying $F(Y, t_1) = F(Y, t_2) = 0$.

ii) $F(Y, t) = 0$ only if $t = t(Y)$.

iii) If $G(Y) < 0$ then $F(Y, t) < 0$ for every t in U.

Observe that, $DG(X_o)$ is surjective.

It is obvious that $Q_1(T, a_1)$, as well as $Q_i(T, b)$, $i = 1,2,3$. are open sets of Q_1. Moreover, the proof of the regularity of $Q_{11}(T)$ is completely similar to the proof of the main result in $[S_3]$. Basically, the proof that $Q_2(T, c)$ is a codimension two submanifold of χ^r is contained in $[T_3]$.

Now the conclusion of the theorem is immediate.∎

6.2- Proof of Theorem 2

This proof follows from 3.2, 3.3, 3.11 and 3.12.∎

6.3- Proof of Theorem 3

This proof follows from 3.5.∎

6.4- Proof of Theorem 4

This proof follows from 3.8, 3.8.a, 3.10, 3.11, 3.12 and 4.2.∎

6.5- Proof of Proposition 2.2

This proof follows from 3.2, 3.3, 3.5 , 3.8 , and 3.8.a.∎

7 - References

[F]- M. Feigin: On the structure of C-bifurcation boundaries of Piecewise Continuous System, PMM. (42), 5, (1978), 820-829.

[M]- W. de Melo: Moduli of stability of 2-dimensional diffeomorfisms. Topology 19 (1980), 9-12.

[MP_1]- W. de Melo & J. Palis: Geometric Theory of Dynamical Systems, an Introduction, Springer Velag, (1982).

[MP_2]- W. de Melo & J. Palis: Moduli of stability for diffeomorfisms. Lecture Notes in Math. , 819, Springer Verlag (1890), 315-339.

[P]- J. Palis: A differentiable invariant of topological conjugacies and moduli of stability; Asterisque, (51), (1978), 335-346.

[S_1]- J. Sotomayor: Generic one parameter families on two dimensional manifolds; Publ. Math. IHES., (43), (1974).

[S_2]- J. Sotomayor: Structural Stability in manifolds with boundary , in "Global Analysis and its applications ", (3), IEAA, Vienna , (1974), 167-176.

[S_3]- J. Sotomayor. Stability and regularity in bifurcations of planar vector fields. Proceedings of the Royal Society of Edinburgh, to appear.

[ST]- J. Sotomayor & M.A. Teixeira: Vector fields near the boundary of a 3-manifold: Lecture Notes in Mathematics, Springer-Verlag, 1331, (1988), 169-195.

[T_1]- M.A. Teixeira: Generic bifurcations in manifolds with boundary, J. of Diff. Eq., (25), (1978), 65-89.

[T_2]- M.A. Teixeira: Stability conditions for discontinuous vector fields, J. of Diff. Eq., to appear.

[T_3]- M.A. Teixeira: Generic bifurcation of certain singularities , Boll. dell'Unione Mat. Italiana, V. 9 (2), (1979), 238-254.

J. Sotomayor
IMPA
Estrada Dona Castorina 110
22.460 Rio de Janeiro, RJ
BRAZIL

M.A. Teixeira
IMECC-UNICAMP
Caixa Postal 6065
13.081 Campinas,
BRAZIL

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