FINITE FORM OF PROPER ORTHOCRONOUS LORENTZ TRANSFORMATIONS AND ITS DYNAMICAL INTERPRETATION

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RELATÓRIO TÉCNICO Nº 32/89

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Setembro – 1989

Finite Form of Proper Orthocronous Lorentz Transformations and its Dynamical Interpretation

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Abstract: In this paper we present the finite form of a generic proper orthocronous Lorentz transformations (denoted $L_+^\uparrow$), or in other words, we give to the exponential of a generic generator of $SL(2,\mathbb{C})$ (the universal covering group of $\mathcal{L}_+^\uparrow$) a closed form, which represents a generalization of the well known exponential forms for pure boosts and pure rotations. Besides this algebraic result concerning $SL(2,\mathbb{C})$, we show that there exists a surprising dynamical interpretation of the transformations of $\mathcal{L}_+^\uparrow$ when applied to the relativistic four vector velocity namely, that these transformations yield the integral solution of the equation of motion of a charged particle under the action of electric and magnetic fields in an arbitrary reference frame.

1. The Exponential Form of the Transformations of $\mathcal{L}_+^\uparrow$

The developments that follow are mainly based on the well known fact that the proper orthocronous Lorentz transformations $L_+^\uparrow$ can be described by the elements of $SL(2,\mathbb{C})$[1,2] (which is the universal covering group of $\mathcal{L}_+^\uparrow$).

The elements of $SL(2,\mathbb{C})$ by definition are the 2×2 complex unimodular matrices. Each 2×2 complex matrix (an element of $\mathbb{C}(2)$) can be represented by a linear combination of the base $\beta = \{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ where σ_i are the Pauli matrices. There exists an isomorphism among the four-vectors $u \in \mathbb{R}^{1,3}$ (where $\mathbb{R}^{1,3}$ is the Minkowski space) and the 2×2 matrices U that are linear combinations of the base β with real coefficients. We have:

$$\mathbb{R}^{1,3} \ni u \mapsto U = u^0 + \vec{u} \equiv u^0\mathbb{1} + u^i\sigma_i \in \mathbb{C}(2). \tag{1}$$

If g is the Lorentz metric then $g(u,u) = (u^0)^2 - (u^i)^2$ is represented by $g(u,u) \mapsto \det U = (u^0)^2 - (\vec{u})^2$.

The matrices $\pm M \in SL(2,\mathbb{C})$ will be called for short in what follows Lorentz transformations. The action of $\mathcal{L}_+^\uparrow$ on a four vector U is given by[3]

$$U' = MUM^+ \tag{2}$$

where M^+ denotes the hermitian conjugated of M.

We can write the operator M as a linear combination of the basis β with complex coefficients, i.e.:

$$M = w + \vec{H} \tag{3}$$

The unimodularity condition, i.e., $\det M = 1$ is equivalent to the condition

$$M\widetilde{M} = \widetilde{M}M = 1 \tag{4}$$

where the operation $\sim$ called space time reversion is defined by

$$M = w + \vec{H} \Longrightarrow \widetilde{M} = w - \vec{H} \tag{5}$$

A direct calculation shows that

$$M\widetilde{M} = w^2 - \vec{H}^2 \tag{6}$$

Now, the square of a complex vector can be calculated using the Pauli product of two arbitrary vectors. We have the well known formula[3,4]

$$\vec{H}\vec{F} = \vec{H}\cdot\vec{F} + i\vec{H}\times\vec{F} \tag{7}$$

where

$$\vec{H}\cdot\vec{F} = \frac{1}{2}(\vec{H}\vec{F} + \vec{F}\vec{H}) = \Sigma_{i=1}^3 H_i F_i \tag{8}$$

and

$$i\vec{H} \times \vec{F} = \frac{1}{2}(\vec{H}\vec{F} - \vec{F}\vec{H}) = \Sigma_{i=1}^{3} i\varepsilon_{ijk} H_i F_j \sigma_k \tag{9}$$

These equations are trivial consequences of the well known formulas, $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$ and $\frac{1}{2}(\sigma_i\sigma_j - \sigma_j\sigma_i) = i\varepsilon_{ijk}\sigma_k$.

From these equations it follows that the square of a complex vector is a complex number. This means that we can parametrize w^2 and $\vec{H}^2$ in eq.(6) by hyperbolic functions. We put

$$w^2 = \cosh z \quad ; \quad \vec{H}^2 = \sinh z \tag{10}$$

where $z \in \mathbb{C}$.

The scalar part of the operator M, the complex number w is determined by using the square-root function which exists in the complex field. We have $w = \cosh z$. We parametrize $\vec{H}$ as follows

$$\vec{H} = \hat{F} \sinh z \tag{11}$$

where the complex "normalized" vector $\hat{F}$ ($\hat{F}^2 = 1$) is given by

$$\hat{F} = \frac{\sinh z^+}{|\sinh z|^2} \vec{H} \tag{12}$$

In this way the operator M defined by eq.(3) assumes the following form in terms of the parameters z and $\hat{F}$

$$M = \cosh z + \hat{F} \sinh z \tag{13}$$

This is a finite form for the operator $M \in SL(2,\mathbb{C})$ representing a generic Lorentz transformation of $\mathcal{L}_+^\uparrow$ [3]. This is a really convenient form since it is the finite form of the exponential of a complex vector denoted $\vec{F}$, related to the parameters z and $\hat{F}$ by

$$\vec{F} = z\hat{F} \quad , \quad \vec{F}^2 = z^2 \quad ; \quad \hat{F} = \frac{z^+ \vec{F}}{|z|^2} \, , \tag{14}$$

as it can be seen at once through the series expansions[3]

$$M = \exp(\vec{F}) = 1 + \vec{F} + \frac{\vec{F}^2}{2} + \frac{\vec{F}^3}{3!} + \cdots \tag{15}$$

The argument of the exponential, the complex vector $\vec{F}$, define the operator M and is said to be the generator of the transformation. $\vec{F}$ can be written as the sum of two euclidian vectors $\vec{E}, \vec{B} \in \mathbb{R}^3$, i.e.

$$\vec{F} = \vec{E} + i\vec{B} \tag{16}$$

When the generator $\vec{F}$ of the transformation M has only a real part, i.e., $\vec{F} = \vec{E}$, the transformation is said to be a boost. In this case the operator M in eq.(13) ($z = x = |\vec{E}|$, $\hat{F} = \vec{E}/E$), is given by [3]

$$M = \cosh x + \frac{\vec{E}}{|\vec{E}|} \sinh x \tag{17}$$

Observe that the boost operators are hermitian.

When the generator $\vec{F}$ of the transformation M has only an imaginary part, i.e., $\vec{F} = i\vec{B}$, the transformation is said to be a spatial rotation. In this case the operator M in eq.(13) $(z = iy = i|\vec{B}|\, ,\ \hat{F} = \vec{B}/B)$ is given by

$$M = \cos y + i\frac{\vec{B}}{B}\sin y \tag{18}$$

Observe that the rotation operators are unitary, $M^{+} = \widetilde{M} = M^{-1}$.

The solution of eq.(2) for the transformation of a relativistic four-vector U under the action of a Lorentz transformation $L_{+}^{\uparrow} \in \mathcal{L}_{+}^{\uparrow}$ is given by [3]:

$$\textbf{Scalar part :}\quad u^{0'} = \frac{u^0}{|z|^2}\{|z|^2|\cosh z|^2 + (E^2 + B^2)|\sinh z|^2\}$$

$$-2\vec{u}\cdot\frac{(\vec{E}\times\vec{B})}{|z|^2}|\sinh z|^2 + \frac{\vec{u}\cdot\vec{E}}{|z|^2}f(x,y) + \frac{\vec{u}\cdot\vec{B}}{|z|^2}g(x,y) \tag{19a}$$

$$\textbf{Vector part :}\quad \vec{u}' = \frac{u^0}{|z|^2}\{\vec{E}f(x,y) + \vec{B}g(x,y) + 2\vec{E}\times\vec{B}|\sinh z|^2\}$$

$$+\frac{\vec{u}}{|z|^2}\{|z|^2|\cosh z|^2 - \frac{\vec{E}^2 + \vec{B}^2}{|z|^2}|\sinh z|^2\} - \frac{\vec{u}\times\vec{E}}{|z|^2}g(x,y) \tag{19b}$$

$$+\frac{\vec{u}\times\vec{B}}{|z|^2}f(x,y) + \frac{2|\sinh z|^2}{|z|^2}\{(\vec{u}\cdot\vec{E})\vec{E} + (\vec{u}\cdot\vec{B})\vec{B}\}$$

where the functions $f(x,y)$ and $g(x,y)$ are defined by

$$f(x,y) = x\sinh 2x + y\sin 2y \tag{20a}$$

$$g(x,y) = y\sinh 2x - x\sin 2y \tag{20b}$$

The variables x and y are such that $z = x + iy$ can be obtained from the equation

$z^2 = \vec{F}^2$, by observing that $\vec{F}^2 = \vec{F} \cdot \vec{F} = \vec{E}^2 - \vec{B}^2 + 2i\vec{E} \cdot \vec{B}$. We get:

$$x^2 = \frac{1}{2}\{|z|^2 + (\vec{E}^2 - \vec{B}^2)\} \tag{21a}$$

$$y^2 = \frac{1}{2}\{|z|^2 - (\vec{E}^2 - \vec{B}^2)\} \tag{21b}$$

and also

$$|z|^2 = \sqrt{(\vec{E}^2 - \vec{B}^2)^2 + 4(\vec{E} \cdot \vec{B})^2} \tag{21c}$$

Eqs(19a) and (19b) reduce to well known formulas in the cases $\vec{E} = 0$ (rotation) and $\vec{B} = 0$ (pure boost)[3,5].

2. Dynamical Interpretation of the Transformations $L_+^\uparrow$.

The result we want to show in this section is that a transformation of $\mathcal{L}_+^\uparrow$ conveniently parametrized gives the integral solution for the motion of a charged particle under the action of an electric and magnetic field. This result, which has not been exploited in the literature, [3,5] is due to the fact that the electromagnetic field (which as well known, is represented by a two–form) is represented by objects of the same mathematical nature as the generators of $SL(2,\mathbb{C})$. * The electric field can be written as a linear combination of boost generators, whereas the magnetic field can be written as a linear combination of the rotation generators, which makes this representation of the fields significative[9], since we know that a charged particle in the presence of a constant electric field suffers an acceleration in the direction of the field, and in the presence of a constant magnetic field the charged particle suffers a rotation[3,5]. In resume, in the Pauli algebra which is isomorphic to $\mathbb{C}(2)$ [3,9,10], the electromagnetic field is represented by a complex vector analogous to the one given by eq.(16). [3]

We now show that a transformation $M \in SL(2,\mathbb{C})$, parametrized in a convenient way when applied to the relativistic four–vector velocity of a charged particle is such that the transformed four–vector satisfies the Lorentz force equation with constant electric and magnetic fields. We start by writing the generator of the Lorentz transformation as

$$\vec{f} = \frac{\alpha}{2}\vec{F}_T \tag{22}$$

(*) The proof of this result involves knowledge of the structure of the Clifford algebra $\mathbb{R}_{1,3}$ and the fact that $Spin_+(1,3) \simeq SL(2,\mathbb{C}) \subset \mathbb{R}_{1,3}^+ \simeq \mathcal{P} \simeq \mathbb{C}(2)$ ($\mathcal{P}$ is the Pauli–algebra and $\mathbb{R}_{1,3}^+$ is the even subalgebra of $\mathbb{R}_{1,3}$). For more details see references[6,7,8,9,10,11,12].

where $\vec{F} = \vec{E} + i\vec{B}$ is the electromagnetic field, $\alpha \in I\!R$ is a constant and τ is an appropriate real parameter. We denote by $M(\tau)$ the transformation generated by $\vec{f}$ and rewrite eq.(2) as

$$U(\tau) = M(\tau)UM^{+}(\tau) \tag{23}$$

where $U(\tau) = (u^0(\tau), \vec{u}(\tau))$ and $U = (u^0, \vec{u})$.

Observe that in eq.(23) the initial four-vector U does not depend on the parameter τ.

By differentiating both members of eq.(23) in relation to τ we have

$$\frac{dU}{d\tau} = \frac{\alpha}{2}(\vec{F}U(\tau) + U(\tau)\vec{F}^{+}) \tag{24}$$

where we used that from $M = e^{\frac{\alpha}{2}\vec{F}\tau}$ it follows $dM/d\tau = \frac{\alpha}{2}\vec{F}M$; the fact that $\vec{F}$ commutes with M, and that $dM^{+}/d\tau = \alpha/2\ M^{+}\vec{F}^{+}$. This result, eq.(24) can be obtained rigorously from the finite form for the operator M (eq.(13)).

Now, from the Pauli product of two vectors (eq.(7)), taking into account that $\vec{F}^{+} = \vec{E} - i\vec{B}$, it follows that the right hand side of eq.(24) is the Lorentz force, i.e.

$$\frac{1}{2}(\vec{F}U + U\vec{F}^{+}) = u^0\vec{E} + \vec{u}\cdot\vec{E} + \vec{u}\times\vec{B} \tag{25}$$

We can then write the scalar and vector parts of eq.(24) as:

$$\frac{du^0}{d\tau} = \alpha\vec{u}\cdot\vec{E} \tag{26a}$$

$$\frac{d}{d\tau}\vec{u} = \alpha(u^0\vec{E} + \vec{u}\times\vec{B}) \tag{26b}$$

From eq.(26a) and (26b) we see that $\alpha = q/mc$, where q is the charge of the particle, m its mass, c is the velocity of light and τ must be identified with the proper time parameter along the world line of the charged particle. With these identifications, the velocity four-vector U given by eq.(23) is indeed the integral solution of eqs(26a) and (26b) for constant electric and magnetic fields.

In eq.(23) U is interpreted, of course, as the initial four-vector velocity since for $\tau = 0$ it is $M(0) = \mathbb{1}$ and $U(0) = U$.

We mention that there are another simple problems that can be solved with the above technique. These are the cases where there exists only the electric field or only the magnetic field and in both cases the fields have a constant direction varying only their intensity [and in a convenient way such that from Maxwell's equations no other fields arise]. In these cases the solution of the motions equations (eq.(23)) are given by transformations $M \in SL(2,\mathbb{C})$ in which the generator has the form

$$\vec{f} = \int d\tau z(\tau)\hat{F} \tag{27}$$

and the electric or magnetic field is given by

$$\vec{F} = z(\tau)\hat{F} \tag{27a}$$

We emphasize also that the above technique can be extended to deal with the general problem of motion of a charged particle in presence of an arbitrary electromagnetic field, i.e., the solution of the motion's equations still can be written as a Lorentz transformation as in eq.(23). We must only change the generator of the Lorentz transformation (eq.(22)) conveniently, as we did at eq.(27)[13]. With this new technique we will study some approximations[14,15] that can be useful in some practice applications.

3. Conclusions.

The fact that we obtained the transformations $M \in SL(2,\mathbb{C})$, which represent a proper orthocronous Lorentz transformations $L_+^\uparrow \in \mathcal{L}_+^\uparrow$ in a finite form (eq.(13)), easily related to the exponential of the generator of the transformation (eqs(14) and (15)) is a step that "completes" the study of the group $SL(2,\mathbb{C})$ by permitting, among the others the treatment in closed form of the composition of two transformations of $\mathcal{L}_+^\uparrow$ (as e.g. the important problem of the composition of two boosts[3]) instead of the traditional treatment via the Campbel–Baker–Hausdorff formula[5].

The finite form of a generic proper orthocronous Lorentz transformation is by itself a gratifying result, but this is increased by the physical interpretation presented in this paper: namely, that the integral solution for the motion of a charged particle in a electromagnetic field is equivalent to a transformation $M \in SL(2,\mathbb{C})$ (eq.(23)) whose generator is related to the electromagnetic field (eq.(22) and (27)). This result provides a new method to solve the problem of the motion of a charged particle, which is usually represented by a system of differential equations[14].

Acknowledgments: The authors are grateful to CNPq (Brasil), CAPES (Brasil) and CNR (Italia) for partial support, and to Professor E. Recami for suggestion and a reading of the manuscript and to Professor Umberto Bartocci and Professor G. Arcidiacono for the hospitality at the Univ. of Perugia.

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