ON THE ASYMPTOTIC BEHAVIOR OF SOME ALTERNATE SMOOTHING SERIES EXPANSION ITERATIVE METHODS

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On the Asymptotic Behavior of Some Alternate Smoothing(*) Series Expansion Iterative Methods

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§1. INTRODUCTION

Let us consider the problem of computing a function from its known line integrals, i.e., the mathematical model associated with X-ray transmission computed tomography. Given

$$\int_L f\, d\mu, \tag{1.1}$$

for every line L in the plane, or, to be more realistic, for a finite but very large number of lines, we want to approximate f, a real valued function with compact support in the plane. Two main approaches are know in the literature for

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solving this problem. The first, currently being used in commercial tomography, consists in computing an approximation to the inverse of the Radon transform, the operator defined by (1.1). Methods of this kind are known as transform methods (see [11]).

The second approach, in which we are interested now, consists in discretizing the problem at an early stage giving rise to a linear system of equations of the form

$$Ax = b, \tag{1.2}$$

where $A = \{a_{ij}\}$ is an $m \times n$ real matrix, $b = \{b_i\}$ the m-dimensional projection data (integral values) and $x = \{x_j\}$ the n dimensional image vector; x_j represents the average value of the function f inside the pixel (picture element) j; the left hand side of each equation in (1.2) is a discrete approximation of (1.1), so a_{ij} will be in general (but not always) the length of the intersection of pixel j and the i-th line. Because of several drawbacks like lack of continuity of the Radon transform, noise, and others, the model represented by (1.2) is sometimes replaced by inequalities in order to avoid inconsistency. So we obtain the new problem: compute $x \in \mathbb{R}^n$ such that

$$b - \varepsilon \leq Ax \leq b + \varepsilon, \tag{1.3}$$

where ε is a perturbation m-vector containing bounds on measurement errors. Taking into account that now (1.3) will not generally have a unique solution, an optimization approach may be used, reducing the problem to

$$\begin{aligned} &\text{optimize} \quad q(x), \\ &\text{s.t.} \qquad\; x \in C \end{aligned} \tag{1.4}$$

where q is either a quadratic or an entropy function. C is defined by (1.3) and some other inequalities containing information about the solution, like nonnegativity.

Methods for solving (1.2) – (1.4) are known as series expansion methods and they are iterative algorithms built to deal with very large and sparse problems. Typical sizes are $m = 10^4$, $n = 10^5$ and no more than 1% non zero entries of A. Up-dated reviews of this type of methods are [2], [3] and [4]. In [4] a complete discussion of the advantages and disadvantages of series expansion methods is presented.

As pointed out in [8], it has been found in practice that the efficiency of series expansion methods for image reconstruction can often by improved by applying between iterative steps certain processes to the image vectors. These processes have been referred to as "tricks" in the literature.

More precisely, suppose that the k-th iterative step of some series expansion method is defined by the function Q_k and the "trick" by φ_k, both mapping

n-dimensional vectors into n-dimensional vectors; then, the method combined with the sequence of tricks produces a new sequence defined by

$$\begin{aligned} \hat{x}^{k+1} &= Q_k(x^k) \\ x^{k+1} &= \varphi_k(\hat{x}^{k+1}). \end{aligned} \tag{1.5}$$

The main purpose of the transform φ_k is to incorporate a priori knowledge about desirable properties of the image vectors. One of these is reasonable smoothness and the corresponding "trick" in (1.5) is commonly known as selective smoothing, defined as follows (see [8] and [9] for a complete discussion about "tricks").

Let u_1 denote the density in a pixel and $u_2, \ldots, u_9$ the densities in its neighbors as shown in the diagram (1.6).

u_6	u_2	u_7
u_3	u_1	u_4
u_8	u_5	u_9

(1.6)

Let t, w_1, w_2, and w_3 be nonnegative real numbers called the threshold level and smoothing weights, respectively. Then the new value for u_1 is

$$\bar{u}_1 = \frac{w_1 u_1 + w_2 \sum_{j=2}^{5} h_j u_j + w_3 \sum_{j=6}^{9} h_j u_j}{w_1 + w_2 \sum_{j=2}^{5} h_j + w_3 \sum_{j=6}^{9} h_j} \tag{1.7}$$

where

$$h_j = \begin{cases} 1 & \text{if} \quad |u_j - u_1| \le t \\ 0 & \text{otherwise} \end{cases} \tag{1.8}$$

If the pixel is in the boundary and so u_j is undefined for some j's, we can set $h_j = 0$ for the corresponding j's or other alternatives that will be discussed later.

It is also possible to consider larger neighborhoods, but of course, this would increase the computational effort. In any case, it is easy to see that selective smoothing on a given image vector x is equivalent to compute Sx, where S is a stochastic matrix (nonnegative and with row sums equal to one) with row elements determined by the normalized weights. In the neighborhood pattern defined by (1.6) for instance, if i is the pixel being considered, take:

$$\tilde{w}_\ell = w_\ell / w, \qquad \text{for} \quad \ell = 1, 2, 3, \tag{1.9}$$

with

$$w = w_1 + w_2 \sum_{j=2}^{5} h_j + w_3 \sum_{j=6}^{9} h_j. \tag{1.10}$$

and then define $s_{ii} = \tilde{w}_1$, $s_{ij} = \tilde{w}_2$ if j is the up, down, right or left neighbor of pixel i (like pixels 2-5 with respect to pixel 1 in (1.6)), $s_{ij} = \tilde{w}_3$ if j is a diagonal neighbor of pixel i (like pixels 6–9 with respect to pixel 1 in (1.6)) and $s_{ij} = 0$ if j is not a neighbor of pixel i.

In any case, the sequence in (1.5) becomes

$$\begin{cases} \hat{x}^{k+1} = Q_k(x^k) \\ x^{k+1} = S\hat{x}^{k+1} \end{cases} \tag{1.11}$$

In this paper, we analyze the asymptotic behavior of (1.11) by considering reasonable hypothesis on S for Q_k defined so that it contains as particular cases a major iteration of several methods commonly used in image reconstruction, such as ART, ART for inequalities and others (see [8]).

Two main remarks have to be done regarding S. The first one is that for a finite value of t, S will depend on x^k; in this case we present examples for which the sequence (1.11) diverges; moreover, the limit points may be not fixed points of the algorithm. The second remark is that if $t = \infty$, S will be constant for every iteration and (1.11) convergent if S is symmetric with a positive diagonal. If S is nonsymmetric or with some zero diagonal elements, we give examples where the method diverges. This suggests that preserving symmetry of S by means of an appropriate choice of the boundary values (other than zero will give more stable results.

Finally, taking into account that if we choose S being the identity matrix, we obtain the method defined by Q_k, our presentation gives a unifying convergence approach for several series expansion iterative methods.

From now on $\langle \cdot, \cdot \rangle$ will denote the standard scalar product, $\|\cdot\|$ the induced square norm, I the identity matrix and e the vector with ones.

§2. PRELIMINARY RESULTS

Definition 1: Let $\mathcal{F}_1$ be the set of continuous functions $Q: \mathbb{R}^n \to \mathbb{R}^n$ satisfying:

1. $\|Q(x) - Q(y)\| \leq \|x - y\| \quad \forall x, y \in \mathbb{R}^n$
2. If $\|Q(x) - Q(y)\| = \|x - y\|$, then:
   a) $Q(x) - Q(y) = x - y$
   b) $\langle x - y, Q(y) - y \rangle = 0$

The set $\mathcal{F}_1$ contains all the projection operators used in the iterative algorithms developed below. We establish two properties of $\mathcal{F}_1$, for future reference.

Proposition 1. *$\mathcal{F}_1$ is closed under composition*

Proof: Let $Q_1, Q_2 \in \mathcal{F}_1$, $Q = Q_2 \circ Q_1$. Using condition 1. for Q_1 and Q_2:

$$\|Q(x) - Q(y)\| = \|Q_2(Q_1(x)) - Q_2(Q_1(y))\| \leq \|Q_1(x) - Q_1(y)\| \leq \|x - y\| \tag{2.1}$$

So condition 1. holds for Q.

Assume now $\|Q(x) - Q(y)\| = \|x - y\|$. Then, equality holds throughout (2.1), so:

$$Q_2(Q_1(x)) - Q_2(Q_1(y)) = Q_1(x) - Q_1(y) = x - y \tag{2.2}$$

using condition 2.a for Q_1 and Q_2. It follows that $Q(x) - Q(y) = x - y$. For 2.b:

$$\langle x - y, Q(y) - y\rangle = \langle x - y, Q_2(Q_1(y)) - Q_1(y)\rangle + \langle x - y, Q_1(y) - y\rangle$$

The second term vanishes because of condition 2.b for Q_1, and the first one, using (2.2) can be rewritten as $\langle Q_1(x) - Q_1(y), Q_2(Q_1(y)) - Q_1(y)\rangle$ which also vanishes, because of condition 2.b for Q_2. So, $\langle x - y, Q(y) - y\rangle = 0$. ∎

Proposition 2. *$\mathcal{F}_1$ is closed under strict convex combinations.*

Proof: Let $Q_1, \ldots, Q_m \in \mathcal{F}_1$, $\lambda_i \in \mathbf{R}_{>0}$ $(1 \leq i \leq m)$, $\sum_{i=1}^m \lambda_i = 1$, $Q = \sum_{i=1}^m \lambda_i Q_i$.

$$\begin{aligned}\|Q(x) - Q(y)\| &= \left\|\sum_{i=1}^m \lambda_i(Q_i(x) - Q_i(y))\right\| \leq \sum_{i=1}^m \lambda_i \|Q_i(x) - Q_i(y)\| \\ &\leq (\sum_{i=1}^m \lambda_i)\|x - y\| = \|x - y\|\end{aligned} \tag{2.3}$$

using condition 1. for the Q_i's. So condition 1. holds for Q. For condition 2., assume $\|Q(x) - Q(y)\| = \|x - y\|$. Then equality holds throughout (2.3) and $\sum_{i=1}^m \lambda_i \|Q_i(x) - Q_i(y)\| = \|x - y\|$. Since $\|Q_i(x) - Q_i(y)\| \leq \|x - y\|$ for each i, and $\lambda_i > 0$, it follows that $\|Q_i(x) - Q_i(y)\| = \|x - y\|$ for all i. Using 2.a and 2.b for the Q_i's:

$$Q_i(x) - Q_i(y) = x - y \tag{2.4}$$

$$\langle x - y, Q_i(y) - y\rangle = 0 \tag{2.5}$$

Multiplying (2.4) and (2.5) by λ_i and summing on i:

$$Q(x) - Q(y) = x - y$$

$$\langle x - y, Q(y) - y\rangle = 0$$

proving that Q satisfies condition 2. ∎

Definition 2: Let $\mathcal{F}_2$ be the set of functions $Q \in \mathcal{F}_1$ which, in addition to 1. and 2., satisfy:

3. $\forall S \in \mathbb{R}^{n\times n}$ the function $g: \mathbb{R}^n \to \mathbb{R}$ defined by $g(x) = \|x - SQ(x)\|^2$ attains its global minimum.

Condition 3. will be used to establish the existence of fixed points of the operators which define the 'smoothed' algorithms considered below.

Let S be a symmetric, stochastic $n \times n$ matrix with elements s_{ij} such that $s_{ii} > 0$ $(1 \leq i \leq n)$. By stochastic, we mean that $s_{ij} \geq 0$ $(1 \leq i, j \leq n)$ and $\sum_{j=1}^{n} s_{ij} = 1$ $(1 \leq i \leq n)$.

Proposition 3. *Let* $\gamma = \frac{1}{2} \min_{1 \leq i \leq n} s_{ii} > 0$. *Then* $\forall x \in \mathbb{R}^n$ $\|x\|^2 - \|Sx\|^2 \geq \gamma \|x - Sx\|^2$

Proof: For any i, using the stochasticity of S:

$$\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} s_{ij}s_{ik}(x_j - x_k)^2 = \frac{1}{2}\sum_{j=1}^{n} s_{ij}x_j^2(\sum_{k=1}^{n} s_{ik}) + \frac{1}{2}\sum_{k=1}^{n} s_{ik}x_k^2(\sum_{j=1}^{n} s_{ij})$$

$$-\sum_{j=1}^{n}\sum_{k=1}^{n} s_{ij}s_{ik}x_jx_k = \sum_{j=1}^{n} s_{ij}x_j^2 - (\sum_{j=1}^{n} s_{ij}x_j)^2$$

Summing on i the first and last member of the previous chain of equalities and rearranging:

$$\sum_{i=1}^{n}(\sum_{j=1}^{n} s_{ij}x_j)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij}x_j^2 - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}(x_j - x_k)^2(\sum_{i=1}^{n} s_{ij}s_{ik}) \Rightarrow$$

$$\|Sx\|^2 \leq \sum_{j=1}^{n} x_j^2(\sum_{i=1}^{n} s_{ij}) - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} s_{jj}s_{jk}(x_j - x_k)^2$$

$$\leq \|x\|^2 - \gamma\sum_{j=1}^{n}\sum_{k=1}^{n} s_{jk}(x_j - x_k)^2$$

$$\leq \|x\|^2 - \gamma\sum_{j=1}^{n}(\sum_{k=1}^{n} s_{jk}(x_j - x_k))^2$$

$$= \|x\|^2 - \gamma\sum_{j=1}^{n}(x_j - \sum_{k=1}^{n} s_{jk}x_k)^2 = \|x\|^2 - \gamma\|x - Sx\|^2$$

The first inequality results from the positivity of the s_{ij}'s, the second one from symmetry and stochasticity of S, the third one from convexity of $\|\cdot\|^2$ and the next equality from stochasticity of S again. ∎

Corollary 1.

i) $\|Sx\| \leq \|x\| \ \forall x \in \mathbb{R}^n$

ii) $\|Sx\| = \|x\| \Rightarrow Sx = x$

Proof: Immediate from Prop. 3. ∎

The matrix S represents the smoothing procedure explained in the introduction.

§3 DEFINITION OF THE GENERAL 'SMOOTHED' ALGORITHM AND CONVERGENCE RESULTS

We start by defining the basic 'smoothed' operator and establishing some of its properties.

Let $Q \in \mathcal{F}_2$, S be a symmetric, stochastic $n \times n$ matrix without zeroes in the diagonal. Define $T: \mathbb{R}^n \to \mathbb{R}^n$ by:

$$T(x) = SQ(x) \tag{3.1}$$

Proposition 4.

i) $\forall x, y \in \mathbb{R}^n \quad \|T(x) - T(y)\| \leq \|x - y\|$

ii) $\|T(x) - T(y)\| = \|x - y\| \Rightarrow T(x) - T(y) = Q(x) - Q(y) = Sx - Sy = x - y$

Proof:

i) Using Corollary 1.i and condition 1. on Q:

$$\|T(x) - T(y)\| = \|SQ(x) - SQ(y)\| \leq \|Q(x) - Q(y)\| \leq \|x - y\| \tag{3.2}$$

ii) If $\|T(x) - T(y)\| = \|x - y\|$, then equality holds throughout (3.2). From Corollary 1.ii and condition 2.a:

$$SQ(x) - SQ(y) = Q(x) - Q(y) = x - y = Sx - Sy \quad ∎$$

Let $F = \{x \in \mathbb{R}^n : T(x) = x\}$ and $g: \mathbb{R}^n \to \mathbb{R}$ defined by $g(x) = \|x - T(x)\|^2$.

Lemma 1. $F \neq \phi$.

Proof: By condition 3. g attains its minimum. Take z such that $g(z) \leq g(x)$ for all $x \in \mathbb{R}^n$ and let $y = T(z)$. Then:

$$\|T(z) - z\| \leq \|T(y) - y\| = \|T(T(z)) - T(z)\| \leq \|T(z) - z\| \tag{3.3}$$

using the definition of z in the first inequality and Prop. 4.i in the last one. From (3.3), using Prop. 4.ii:

$$\|T(z)-z\| = \|T(y)-y\| \Rightarrow T(z)-T(y) = Q(z)-Q(y) = z-y = Sz-Sy \tag{3.4}$$

So, using condition 2.b, (3.4), symmetry of S and the definition of y:

$$\begin{aligned} 0 = \langle y-z, Q(z)-z\rangle &= \langle S(y-z), Q(z)\rangle - \langle y-z, z\rangle \\ &= \langle y-z, SQ(z)\rangle - \langle y-z, z\rangle = \langle y-z, T(z)-z\rangle \\ &= \|T(z)-z\|^2 \Rightarrow T(z) = z \Rightarrow z \in F \Rightarrow F \neq \phi \end{aligned}$$ ∎

Proposition 5. *If $z \in F$ and $\|T(x)-z\| = \|x-z\|$, then $x \in F$.*

Proof: Since $z \in F$:

$$\|T(x)-z\| = \|x-z\| \Rightarrow \|T(x)-T(z)\| = \|x-z\|$$

By Prop. 4.ii:

$$x-z = T(x)-T(z) = T(x)-z \Rightarrow x = T(x) \Rightarrow x \in F$$ ∎

Now we define our general 'smoothed" algorithm:

$$\begin{aligned} &x^0 \in \mathrm{R}^n \\ &x^{k+1} = T(x^k) \end{aligned} \tag{3.5}$$

where T is as defined in (3.1).

Proposition 6. *If $z \in F$, then the sequence $\{\|x^k - z\|\}$ is decreasing.*

Proof: $\|x^{k+1}-z\| = \|T(x^k)-T(z)\| \leq \|x^k - z\|$ by Prop. 4.ii ∎

Corollary 2.. *The sequence $\{x^k\}$ is bounded.*

Proof: By Lemma 1, $F \neq \phi$. Take $z \in F$. Using recurrently Prop. 6:

$$\forall k \geq 0 \quad \|x^k - z\| \leq \|x^0 - z\|$$ ∎

Proposition 7. *If $\overline{x}$ is the limit of a subsequence of $\{x^k\}$, then $\overline{x} \in F$.*

Proof: By Lemma 1, $F \neq \phi$. Take $z \in F$. Assume $x^{j_k} \underset{k\to\infty}{\longrightarrow} \overline{x}$. Using Prop. 6:

$$\begin{aligned} &\|x^{j_{k+1}} - z\| \leq \|x^{j_k+1} - z\| \leq \|x^{j_k} - z\| \Rightarrow \\ &\|x^{j_{k+1}} - z\| \leq \|T(x^{j_k}) - z\| \leq \|x^{j_k} - z\| \end{aligned} \tag{3.6}$$

Since Q is continuous by Definition 1, T is also continuous. Taking limits in (3.6) as $k \to \infty$:

$$\|\overline{x}-z\| \leq \|T(\overline{x})-z\| \leq \|\overline{x}-z\| \Rightarrow \|T(\overline{x})-z\| = \|\overline{x}-z\|.$$

From Prop. 5, $\overline{x} \in F$ ∎

Theorem 1. *The sequence defined by (3.1), (3.5) is convergent.*

Proof: By Corollary 2, there exists a convergent subsequence with limit $\bar{x}$. By Prop. 7, $\bar{x} \in F$. Given $\varepsilon > 0$ take M such that $\|x^{j_M} - \bar{x}\| < \varepsilon$, and let $N = j_M$. For $k > N$, using Prop. 6:

$$\|x^k - \bar{x}\| \leq \|x^{j_M} - \bar{x}\| < \varepsilon.$$

It follows that $x^k \xrightarrow[k\to\infty]{} \bar{x}$. ∎

§4 SPECIFIC REALIZATIONS OF THE GENERAL ALGORITHM

Let C be a closed convex set in $\mathbb{R}^n$ and let π be the orthogonal projection onto C, i.e. $\pi\colon \mathbb{R}^n \to C$, $\pi(x) = \operatorname{argmin}_{y \in C} \|x - y\|$. π is well defined and continuous. Next, we introduce relaxation. Take $\alpha \in (0,2)$. Define $\pi_\alpha\colon \mathbb{R}^n \to \mathbb{R}^n$ as:

$$\pi_\alpha(x) = (1-\alpha)x + \alpha\pi(x) \tag{4.1}$$

Proposition 8. $\pi_\alpha \in \mathcal{F}_1$

Proof: By the Convex Separation Theorem [14, Th. 11.2], $\forall x, y \in \mathbb{R}^n$:

$$\langle x - \pi(x), \pi(y) - \pi(x)\rangle \geq 0 \tag{4.2}$$

$$\langle y - \pi(y), \pi(x) - \pi(y)\rangle \leq 0 \tag{4.3}$$

Adding (4.2) and (4.3):

$$\langle x - y, \pi(x) - \pi(y)\rangle \geq \|\pi(x) - \pi(y)\|^2 \tag{4.4}$$

Adding $\|x - y\|^2 - 2\langle x - y, \pi(x) - \pi(y)\rangle$ and multiplying by 2 on both sides of (4.4):

$$\begin{aligned} 2(\|x-y\|^2 - \langle x - y, \pi(x) - \pi(y)\rangle) &\geq 2\|\pi(x) - \pi(y) - (x-y)\|^2 \\ &\geq \alpha\|\pi(x) - \pi(y) - (x-y)\|^2 \end{aligned} \tag{4.5}$$

Expanding the last member of (4.5) and rearranging:

$$(\alpha - 2)\|x-y\|^2 + 2(1-\alpha)\langle x - y, \pi(x) - \pi(y)\rangle + \alpha\|\pi(x) - \pi(y)\|^2 \leq 0 \tag{4.6}$$

Multiplying by $\alpha > 0$ and adding $\|x-y\|^2$ on both sides of (4.6):

$$\begin{aligned}\|x-y\|^2 &\geq (1-\alpha)^2\|x-y\|^2 + 2\alpha(1-\alpha)\langle x-y, \pi(x)-\pi(y)\rangle \\ &\quad + \alpha^2\|\pi(x)-\pi(y)\|^2 = \|(1-\alpha)(x-y)+\alpha(\pi(x)-\pi(y))\|^2 \\ &= \|\pi_\alpha(x)-\pi_\alpha(y)\|^2 \end{aligned} \tag{4.7}$$

So, π_α satisfies condition 1. of Definition 1.

For condition 2.a, assume $\|\pi_\alpha(x)-\pi_\alpha(y)\| = \|x-y\|$. Then, equality holds throughout (4.7) and therefore throughout (4.6) and (4.5). Since $\alpha < 2$, equality in (4.5) implies $\|\pi(x)-\pi(y)-(x-y)\| = 0 \Rightarrow$

$$x - y = \pi(x) - \pi(y) \tag{4.8}$$

It follows from the definition of π_α that $x - y = \pi_\alpha(x) - \pi_\alpha(y)$, i.e. π_α satisfies condition 2.a.

Observe that equality in (4.5) implies equality in (4.4) and hence in (4.3). Using (4.8):

$$\langle x-y, \pi(y)-y\rangle = 0 \tag{4.9}$$

Multiplying (4.9) by α:

$$0 = \langle x-y, \alpha\pi(y)-\alpha y\rangle = \langle x-y, \pi_\alpha(y)-y\rangle$$

and π_α satisfies condition 2.b. ∎

Consider now closed convex sets $C_1, \ldots, C_m \subset \mathbb{R}^n$ and let P^i be the orthogonal projection onto C_i. For $\alpha \in \mathbb{R}$ define, as before:

$$P^i_\alpha(x) = (1-\alpha)x + \alpha P^i(x) \tag{4.10}$$

Take $\alpha_1, \ldots, \alpha_m \in (0,2)$ and define $P: \mathbb{R}^n \to \mathbb{R}^n$ by:

$$P = P^m_{\alpha_m} \circ \cdots \circ P^2_{\alpha_2} \circ P^1_{\alpha_1} \tag{4.11}$$

Take $\alpha \in (0,2)$ and $\lambda_i \in \mathbb{R}$ $(1 \leq i \leq m)$ satisfying $\lambda_i > 0$ $(1 \leq i \leq m)$, $\sum_{i=1}^m \lambda_i = 1$. Define $\overline{P}: \mathbb{R}^n \to \mathbb{R}^n$ by:

$$\overline{P} = \sum_{i=1}^m \lambda_i P^i_\alpha \tag{4.12}$$

Proposition 9. $P, \overline{P} \in \mathcal{F}_1$

Proof: Use Prop. 1 for P and Prop. 2 for $\overline{P}$. $P^i_\alpha \in \mathcal{F}_1$ because of Prop. 8. ∎

In order to establish convergence for the algorithms defined by (3.5) when P or $\overline{P}$ substitute for Q in (3.1), we need to show that P and $\overline{P}$ belong to $\mathcal{F}_2$, i.e. that they satisfy condition 3. in Definition 2. This is not true for general convex sets. Even when $S = I$, the operators $T(x) = SQ(x)$ with $Q = P$ or $Q = \overline{P}$ may fail to have fixed points if $C = \bigcap_{i=1}^{m} C_i$ is empty. Easy examples can be constructed, with $m = n = 2$, for which the algorithms defined by (3.1), (3.5) for such Q's diverge from any starting point [7]. When $S \neq I$ the algorithms may diverge even when $C \neq \phi$, as shown next.

Counterexample 1: Take $m = 1$, $n = 2$, $C_1 = \{(x, y): y^2 - x^2 \geq 1, y \geq 0\}$ $S = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Multiplying by S is the same as orthogonally projecting onto the diagonal of the first quadrant. Hence T consists on sequential orthogonal projections onto the epigraph of the hyperbola $y^2 - x^2 = 1$ and onto such diagonal. It is easy to see that the sequence defined by (3.5) diverges.

P and $\overline{P}$, however, satisfy condition 3, as shown below, in the linear case, i.e. when the sets C_i are hyperplanes or half spaces. This is the case of interest in image reconstruction.

Take $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Let a^i denote the rows of A. Consider the system:

$$Ax = b \tag{4.13}$$

Let C_i be the hyperplane defined by the i-th equation in (24), i.e.:

$$C_i = \{x \in \mathbb{R}^n : \langle a^i, x \rangle = b_i\}$$

In this case:

$$P^i(x) = x - \left(\frac{\langle a^i, x \rangle - b_i}{\|a^i\|^2} \right) a^i \tag{4.14}$$

Similarly, consider the system:

$$Ax \leq b \tag{4.15}$$

and take $C_i = \{x \in \mathbb{R}^n : \langle a^i, x \rangle \leq b_i\}$ so that:

$$P^i(x) = x - \max \left\{ 0, \frac{\langle a^i, x \rangle - b_i}{\|a^i\|^2} \right\} a^i \tag{4.16}$$

Observe that in the case of equalities, i.e. with P^i as in (4.14), both P and $\overline{P}$ are affine functions. In fact, it is easy to check that in the unrelaxed case $(\alpha_1 = \alpha_2 = \cdots = \alpha_m = 1)$ $P(x) = Bx + d$, with $B = I - A^T \tilde{A}^{-1} A$, $d = A^T \tilde{A}^{-1} b$, where $\tilde{A}$ is the lower triangular part of AA^T. Also, when $\alpha = 1$, $\overline{P}(x) = \overline{B}x + \overline{d}$ with $\overline{B} = I - A^T E A$, $\overline{d} = A^T E b$, where $E = \operatorname{diag}\left(\frac{\lambda_1}{\|a^1\|^2}, \ldots, \frac{\lambda_m}{\|a^m\|^2} \right)$.

Proposition 10. *P and $\overline{P}$, as defined by (4.11), (4.14) and (4.12), (4.14) respectively, belong to $\mathcal{F}_2$.*

Proof: In view of Prop. 9, only condition 3. in Definition 2 needs to be checked. Since P and $\overline{P}$ are affine, the functions $g(x) = \|x - SP(x)\|^2$, $\overline{g}(x) = \|x - S\overline{P}(x)\|^2$ are quadratic and bounded below (by 0). It follows that both of them attain their global minima. ∎

In order to prove the analogous result for the inequality case, it is convenient to introduce another class of functions. We call the intersection of a finite number of closed halfspaces a polytope. We say that a function $Q: \mathbb{R}^n \to \mathbb{R}^n$ is piece-wise affine over polytopes if there exist affine functions $U_1, \ldots, U_r: \mathbb{R}^n \to \mathbb{R}^n$ and polytopes $L_1, \ldots, L_r$ such that $\mathbb{R}^n = \bigcup_{j=1}^{r} L_j$ and $Q(x) = U_j(x)\ \forall x \in L_j$. In a similar way define a function $g: \mathbb{R}^n \to \mathbb{R}$ to be piece-wise quadratic over polytopes. Let $\mathcal{W}$ be the set of piece-wise affine functions over polytopes.

Proposition 11. *If $Q_1, Q_2 \in \mathcal{W}$, then:*

i) $Q_1 + Q_2 \in \mathcal{W}$

ii) $Q_2 \circ Q_1 \in \mathcal{W}$

Proof: Let U_{ij}, L_{ij} be the affine functions and polytopes associated with Q_i ($i = 1, 2,\ 1 \leq j \leq r(i)$). $Q_1 + Q_2$ is equal to the affine function $U_{1j} + U_{2k}$ on the polytope $L_{1j} \cap L_{2k}$. $Q_2 \circ Q_1$ is equal to the affine function $U_{2k} \circ U_{1j}$ on the polytope $U_{1j}^{-1}(L_{2k}) \cap L_{1j}$. ∎

Proposition 12. *P and $\overline{P}$, as defined by (4.11), (4.16) and (4.12), (4.16) respectively, belong to $\mathcal{W}$.*

Proof: P^i, as defined by (4.16), belongs to $\mathcal{W}$. In fact there are only two polytopes, which happen to be halfspaces. P^i is the identity on $\{x: \langle a^i, x\rangle \leq b_i\}$ and equal to the right hand side of (4.14) on $\{x: \langle a^i, x\rangle \geq b_i\}$. So $P^i_\alpha \in \mathcal{W}$, $\lambda_i P^i_\alpha \in \mathcal{W}$. By recurrent application of Prop. 11.i, $\overline{P} \in \mathcal{W}$. Using in a similar way Prop. 11.ii, $P \in \mathcal{W}$. ∎

Proposition 13. *P and $\overline{P}$, as defined by (4.11), (4.16) and (4.12), (4.16) respectively, belong to $\mathcal{F}_2$.*

Proof: In view of Prop. 9, only condition 3. in Definition 2 needs to be checked. By Prop. 12 the functions $g(x) = \|x - SP(x)\|^2$, $\overline{g}(x) = \|x - S\overline{P}(x)\|^2$ are piece-wise quadratic over polytopes and bounded below (by 0). Applying Frank-Wolfe's Theorem [14, Cor. 27.3.1] both g and $\overline{g}$ attain their minima on each polytope. Since there is a finite number of polytopes, they both attain their global minima. ∎

It follows from Theorem 1 and Props. 10 and 13 that the algorithms defined by:

$$x^0 \in \mathbb{R}^n \tag{4.17}$$

$$x^{k+1} = SQ(x)$$

with $Q = P$ or $Q = \overline{P}$, where P is defined by (4.11), and (4.14) or (4.16) and $\overline{P}$ is defined by (4.12), and (4.14) or (4.16) are all convergent. We proceed to identify their 'unsmoothed' versions, i.e., with $S = I$. (4.11), (4.14), (4.17) is ART, which can be seen as a relaxed version of Kaczmarz's algorithm [10]. (4.12), (4.14), (4.17) is simultaneous ART, or a relaxed version of Cimmino's algorithm [5]. (4.11), (4.16), (4.17) is ART for inequalities, equivalent to the SOR algorithm of Agmon, Motzkin and Schoenberg [1], [12]. (4.12), (4.16), (4.17) is the algorithm analyzed in [6]. So all these algorithms are convergent when combined with the smoothing procedure induced by the matrix S.

§5 GENERAL REMARKS

1. In the 'unsmoothed' case (i.e. with $S = I$) the algorithms discussed above converge to a point in $C = \bigcap_{i=1}^{m} C_i$ if $C \neq \phi$, i.e., in the linear case, to a solution of system (4.13) or (4.15) if the corresponding system is feasible. This is not true in general for $S \neq I$. In fact, that will hardly happen in the case of image reconstruction. Let x^* be the limit of the sequence defined by (4.17). For x^* to be feasible for (4.13) or (4.15) it is necessary that $x^* = Sx^*$. But if S is irreducible in the sense of Markov matrices, the only positive solution of $Sx = x$ is $x = \lambda e$ $(\lambda \in \mathbb{R}_{>0})$. In image reconstruction such an x represents a uniform image which is unlikely to be feasible. In this application irreducibility of S means that any pair of pixels can be joined by a path of consecutive neighbors, which is quite plausible. The matrix S defined by the neighborhood structure of (1.6), for instance, is irreducible.
2. The convergence proof given above does not require that systems (4.13), (4.15) be feasible. We mention that the simultaneous algorithms (i.e. with $\overline{P}$ as in (4.12)) without 'smoothing' converge, in the infeasible case, to a point which minimizes the weighted average (with weights λ_i) of the squares of the distances to the convex sets [6], [7].
3. Observe that the approach used in section 3 provides, when $S = I$, a unified framework for the convergence of successive (i.e. with P as in (22)) and simultaneous (i.e. with $\overline{P}$ as in (4.12)) projection algorithms for solving systems of linear equations or inequalities, and in general for the Convex Feasibility Problem. Another relation between these two types of algorithms is given by Pierra [13], who reduces the simultaneous algorithm to a succesive one in a different space. His results, however, cannot be directly applied in the case of unfeasible systems.

4. In connection with relaxation, we may consider also the case of variable relaxation, i.e., with α_i^k (depending on both the convex set C_i and the index of the iteration) instead of α_i (depending only on C_i). This is difficult to handle with our approach of operator generated sequences: $x^{k+1} = T(x^k)$. In fact, with variable relaxation we have $x^{k+1} = T_k(x^k)$. When $S = I$, the set F of fixed points is independent of α, so all the T_k's have the same set of fixed points and it is possible to accomodate variable relaxation in this framework, as done in [6]. For a general S, the set F changes with α and our approach seems unable to cope with this case. We mention also that the use of a different α_i for each convex set C_i in (4.12) amounts just to change the weights λ_i. That's why we use a common α for $\overline{P}$.
5. The hypotheses on S (symmetry and no zeroes in the diagonal) cannot be done without in the convergence proof, as seen in counterexamples 2 and 3 below. They are, however, reasonable in image reconstruction. When smoothing pixel i, its unsmoothed value must be taken into account, meaning $s_{ii} > 0$. Regarding symmetry, the value of s_{ij} depends on the proximity relation of pixels i and j which is generally symmetric. The exception are the pixels on the boundary, which have fewer neighbors. Symmetry can be preserved for such pixels if we give to its neighbors the same weights as that of neighbors of inner pixels, adding the weights of inexistent neighbors to the weight of the pixel itself, instead of redistributing them among the weights of the remaining neighbors.

In the following counterexamples we use the algorithm defined by (4.11), (4.14), (4.17) with $\alpha_i = 1$ $(1 \le i \le m)$.

Counterexample 2: (S nonsymmetric). $m = n = 2$. Take $\varepsilon > 0$ such that $2\varepsilon(1+\varepsilon^2) < 1$. Let

$$A = \begin{bmatrix} 1/\varepsilon & -1 \\ 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad S = \begin{bmatrix} 1-2\varepsilon(1+\varepsilon^2) & 2\varepsilon(1+\varepsilon^2) \\ \varepsilon^4 & 1-\varepsilon^4 \end{bmatrix}$$

$$x^0 = \begin{bmatrix} 2\varepsilon \\ 1+\varepsilon^2 \end{bmatrix}$$

It is easy to verify that $x^k = (1+\varepsilon^2)^k x^0$. So the sequence $\{x^k\}$ diverges.

Counterexample 3: (zeroes in the diagonal). $m = 1$, $n = 2$. $A = [1,1]$, $b = 4$ $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $x^0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. The sequence oscilates between $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

§6 SMOOTHING WITH A THRESHOLD RULE

We consider now the implementation of a threshold rule in the smoothing procedure. Under such a rule, when smoothing pixel i, a neighbor pixel j is taken into account only if $|x_i - x_j| \leq t$, where t is a given threshold level. Formally, we introduce the threshold rule in the general algorithm as follows.

Given $t \in \mathbb{R}_{>0}$ and a stochastic, symmetric $n \times n$ matrix S (with elements s_{ij}) without zeroes in the diagonal, consider the function $x \to S(x)$ (from $\mathbb{R}^n$ to $\mathbb{R}^{n \times n}$). Let $s(x)_{ij}$ be the elements of $S(x)$. $S(x)$ is defined by:

$$s(x)_{ij} = \begin{cases} s_{ij} & \text{if} \quad |x_i - x_j| \leq t \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

Next define $\tilde{S}(x) \in \mathbb{R}^{n \times n}$ with elements $\tilde{s}(x)_{ij}$ by:

$$\tilde{s}(x)_{ij} = \frac{s(x)_{ij}}{\sum_{k=1}^{n} s(x)_{ik}} \tag{5.2}$$

Given $Q \in \mathcal{F}_2$, let $V, T: \mathbb{R}^n \to \mathbb{R}^n$ be defined by:

$$V(x) = \tilde{S}(x)x \tag{5.3}$$

$$T(x) = V(Q(x)) \tag{5.4}$$

We study next algorithm (3.5) with T defined by (5.1) – (5.4).

Observe that in general $S(x)$ is symmetric but not stochastic, while $\tilde{S}(x)$ is stochastic but not symmetric. The algorithm (3.1), (3.5) can be seen as a particular case of this algorithm with $t = \infty$. Note that $S = S(e) = \tilde{S}(e)$.

Consider the set $K = \{(i,j): 1 \leq i < j \leq n\}$ and let $\mathcal{K}$ be the family of subsets of K. Given $J \in \mathcal{K}$ consider the region $R_J \subset \mathbb{R}^n$ defined as:

$$\begin{aligned} R_J = \{x \in \mathbb{R}^n : &|Q(x)_i - Q(x)_j| \leq t \quad \text{if} \quad (i,j) \in J, \\ &|Q(x)_i - Q(x)_j| > t \quad \text{if} \quad (i,j) \notin J\} \end{aligned}$$

Clearly $\mathbb{R}^n = \bigcup_{J \in \mathcal{K}} R_J$. For a fixed J, note that $x, y \in R_J \Rightarrow \tilde{S}(x) = \tilde{S}(y)$, i.e. $\tilde{S}(x)$ is constant on each non empty R_J. Let $\tilde{S}_J$ be such constant matrix. $T(x)$ can be rewritten as:

$$T(x) = \tilde{S}_J Q(x) \quad \text{if} \quad x \in R_J \tag{5.5}$$

Though in this form the operator T looks locally similar to the operator defined by (3.1), the algorithm defined by (3.5) with T as in (5.5) exhibits a much less regular behavior.

To begin with, $S(x)$ is not continuous in x, so T is possibly discontinuous in the boundary of each R_J. So, it may happen that the algorithm defined by (3.5) converges to a point x^* which is not a fixed point of T, as shown in counterexample 4 below.

Additionally, since S_J may be nonsymmetric, the algorithm may diverge, even if the original matrix S is symmetric, as shown in counterexample 5 below.

This last difficulty could be circumvented by a modification of the threshold rule so as to preserve symmetry, in a way similar to the treatment suggested in Remark 5 for pixels in the boundary. We can define $\tilde{S}(x)$ by

$$\tilde{s}(x)_{ij} = \begin{cases} s(x)_{ij} & \text{if } i \neq j \\ 1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} s(x)_{ij} & \text{if } i = j \end{cases} \tag{5.6}$$

instead of (5.2). In this situation, a local convergence theorem could be expected, in view of (5.5), establishing convergence for x^0 close enough to a fixed point of T.

Such is not the case. In counterexample 6 below, $\tilde{S}(x)$ is symmetric for all x, so that the sequence generated by (3.5) remains bounded for all x^0, but T has no fixed point and the sequence, starting at any x^0, oscilates between two regions. Parenthetically, observe that even when P as in (4.11) substitutes for Q in (5.4), T is not piece-wise affine over polytopes, because the regions R_J are not closed. Their closures $\overline{R}_J$ are polytopes, but then (5.5) does not hold with $\overline{R}_J$ instead of R_J.

We conclude that smoothing with a threshold $t < \infty$, if done systematically in an iterative way, does not seem to be a sound mathematical procedure.

In the following 3 counterexamples, the sequence $\{x^k\}$ is defined by (3.5), (5.1)-(5.4) with P, as defined by (4.11), (4.14), substituting for Q in (5.4) and $\alpha_i = 1$ $(1 \leq i \leq m)$.

Counterexample 4: (convergence to $x^* \notin F$) $m = n = 2$, $\epsilon \in (0,1)$, $t = 1$

$$S = \begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x^0 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

There are two regions: $R_K = \{(x_1, x_2): |x_1 + x_2 - 1| \leq 2\}$, $R_\phi = \{(x_1, x_2): |x_1 + x_2 - 1| > 2\}$ and $\tilde{S}_K = S$, $\tilde{S}_\phi = I$. So $\tilde{S}(x)$ is symmetric for all x. $x^0 \in R_\phi$ and it can be easily seen that:

$$x^k = \begin{bmatrix} 2(1+2^{-k}) \\ 1 \end{bmatrix} \in R_\phi \ \forall k \quad \Rightarrow \quad x^k \xrightarrow[k \to \infty]{} x^* = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

But $x^* \in R_K$ and

$$T(x^*) = \begin{bmatrix} 2-\epsilon \\ 1+\epsilon \end{bmatrix} \neq x^*.$$

Counterexample 5: (divergence). $m = 1$, $n = 3$, $A = [1, -1, 0]$, $b = 20$, $t = 25$

$$S = \frac{1}{4} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix} \qquad x^0 = \begin{bmatrix} 44 \\ 33 \\ 1 \end{bmatrix}$$

In this case $x^0 \in R_J$ with $J = \{(1,2)\}$ and

$$\tilde{S}_J = \frac{1}{20}\begin{bmatrix} 16 & 4 & 0 \\ 5 & 15 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

which is nonsymmetric. It can be easily checked that:

$$x^k = \begin{bmatrix} 44 + k/2 \\ 33 + k/2 \\ 1 \end{bmatrix} \in R_J \ \forall k.$$

The sequence $\{x^k\}$ diverges.

Counterexample 6: $(F = \phi)$. $m = n = 2$, $t = \frac{101}{100}$

$$S = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

There are two regions: $R_K = \{(x_1, x_2) : 2|x_2 - 3x_1 - 49| \leq 101\}$, $R_\phi = \{(x_1, x_2) : 2|x_2 - 3x_1 - 49| > 101\}$. $\tilde{S}_K = S$, $\tilde{S}_\phi = I$. So $\tilde{S}(x)$ is symmetric for all x, and:

$$T(x) = \begin{cases} SP(x) & \text{if} \quad x \in R_K \\ P(x) & \text{if} \quad x \in R_\phi \end{cases}$$

It can be easily checked that the only fixed point of P is:

$$x^* = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the only fixed point of SP is:

$$\bar{x} = \frac{1}{3}\left(\frac{7}{8}\right)^2 \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

But $x^* \in R_K$ and $\bar{x} \in R_\phi$. It follows that T has no fixed point.

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