MINIMAL REALIZATIONS UNDER CONTROLLABILITY

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Abstract. We study minimal realizations of controllable systems. By an approach which parallels the one used by Bastos Gonçalves [1] but taking into account only positive times a co-distribution on the cotangente bundle is introduced. From this co-distribution weak minimal realizations are constructed. A finiteness condition for the existence of minimal realizations is given.

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INTRODUCTION

In this paper we consider the problem of constructing minimal realizations of control systems with output whose associated dynamics satisfies the controllability assumption.

By a control system we understand here a quadruplet $\sigma = (M, \Sigma, \mathbb{R}^s, h)$ where M is a smooth manifold, $h : M \to \mathbb{R}^s$ is smooth and Σ - the associated dynamics - is a family of vector fields on M. This family induces a "group"

$$G_\Sigma = \{X^1_{t_1} \circ \ldots \circ X^k_{t_k} : X^j \in \Sigma,\ t_j \in \mathbb{R}\}$$

and a semi-group

$$S_\Sigma = \{X^1_{t_1} \circ \ldots \circ X^k_{t_k} : X^j \in \Sigma,\ t_j \geq 0\}$$

of local diffeomorphisms on M (with compositions where allowed). Here X_t stands for the flow generated by the vector field X. The orbit of $x \in M$ by σ (or Σ) is the set $G_\Sigma(x) = \{\varphi(x) : \varphi \in G_\Sigma\}$. These orbits partitions M in a foliation with singularities ([7], [10]) and the system is said to be transitive if M itself is the orbit of every $x \in M$. The set $S_\Sigma(x) = \{\varphi(x) : \varphi \in S_\Sigma\}$ is the forward orbit of x under σ (or Σ) and the system is said to be controllable from $x \in M$ in case $S_\Sigma(x) = M$. It is said to be controllable in case $S_\Sigma(x) = M$ for every $x \in M$.

Following Gonçalves [1] we say that two elements $x, y \in M$ are quasi-indistinguishable by σ if $h \circ \varphi(x) = h \circ \varphi(y)$ $\forall \varphi \in G_\Sigma$. They are said to be indistinguishable if $h \circ \varphi(x) = h \circ \varphi(y)$ for all $\varphi \in S_\Sigma$.

σ is called observable if no two points of M are indistinguishable and σ is weakly observable at $x \in M$ if there exist a neighbourhood U of x such that every $y \in U$ is not indistinguishable from x. In a similar way we define the notions of quasi-observability and weak-quasi-observability.

In the foregoing we shall look for minimal realizations of σ, i.e., we search for a transitive and observable system $\tilde{\sigma} = (\tilde{M}, \tilde{\Sigma}, I\!R^r, \tilde{h})$ and a smooth and onto submersion π such that the vector fields in σ are projectable by π_*, $\pi_*(\Sigma) = \tilde{\Sigma}$ and $h = \tilde{h} \circ \pi$. We also consider the corresponding notions of weak-minimality, quasi-minimality and weak-quasi-minimality.

Many authors have dealt with these problems under various assumption on M, Σ and $\sim$. Apart from the basic paper by Sussmann [9] we mention here the works of Hermann and Krener [2] where a co-distribution in T^*M is defined in order to obtain weak minimal realizations of σ. In [1] Basto Gonçalves introduces another co-distribution from wich the construction of a weak quasi minimal realization is obtained in a quotient manifold. We mention also the results of Gauthier and Bonnard [3], [4] where the construction of minimal realizations from weak minimal systems is made.

In this paper we introduce another co-distribution in T^*M. In the presence of controllability it is easily shown that this co-distribution is integrable. Quotientizing M by the leaves of this integrable distribution we get at once a weak minimal realization of a controllable system. We also introduce the notion of Σ-finite equivalence relation. With this notion we give a sufficient condition for the existence of minimal realization for controllable systems. The main technique used is a result on isotropy semi-groups proved in § 3 wich appeared originally in [6].

§ 2. WEAK MINIMAL REALIZATIONS

Let $\sigma = (M, \Sigma, I\!R^r, h)$ be a transitive control system and assume that the vector fields in Σ are globally defined in M and complete.

In [1] a quasi minimal realization for this kind system is constructed on the quotient manifold of M by the leaves of the co-distribution

$$\Delta = \text{Span}\{\varphi^* dh_i : \varphi \in G_\Sigma \ , \ i = 1, \dots, s\} \subset T^*M \ .$$

Thus $ker\Delta$ defines an integrable distribution on M and if Δ stands also for the equivalence relation on M define by $x\Delta y$ iff x and y are in the same leaf of $Ker\Delta$, $\pi : M \to M/\Delta$ is a submersion and $\Sigma/\Delta = (M/\Delta, \pi_*(\Sigma), \mathbb{R}^s, \overline{h})$ with $\overline{h} \circ \pi = h$ defines a quasi-minimal realization of Σ.

In order to study weak minimal realization, we consider here the co-distribution

$$\Delta^+ = \text{Span}\{\varphi^* dh_i : \ \varphi \in S_\Sigma, \ i = 1, \dots, s\} \subset T^*M$$

Clearly, $\Delta^+ \subset \Delta$ so that $Ker\Delta \subset Ker\Delta^+$. Also, if $\varphi \in S_\Sigma$ then $\varphi^*(\Delta^+) \subset \Delta^+$ and hence $\varphi_* Ker\Delta^+ \subset Ker\Delta^+$. Note however, that these inequalities may be strict and possibly not true for $\varphi \in G_\Sigma$ (see example 2.4 below).

Proposition 2.1. We have,

a) $\Delta^+ = \Delta$ iff $\dim(\Delta^+)$ is constant.

b) If Σ is controllable then $\Delta^+ = \Delta$.

Proof. a) Since Σ is transitive and Δ is G_Σ-invariant, the dimension of Δ remains constant on M. Conversely, if $\dim(\Delta^+)$ is constant, then $\varphi^*(\Delta^+) = \Delta^+$ for $\varphi \in S_\Sigma$ so that $(\varphi^{-1})^*(\Delta^+) = \Delta^*$ hence Δ^+ is G_Σ-invariant. Since $dh_i \in \Delta^+$ for $i = 1, \dots, s$ and $\Delta^+ \subset \Delta$ it follows that $\Delta^+ = \Delta$.

b) If Σ is controllable and $x, y \in M$ then there are $\varphi, \psi \in S_\Sigma$ with $\varphi(x) = y$ and $\psi(y) = x$. Thus $\psi^*(\Delta_y^+) \subset \Delta_x^+$ and $\varphi^*(\Delta_x^+) \subset \Delta_y^+$ so that the dimension of Δ^+ is constant on M hence $\Delta^+ = \Delta$ □.

Bearing in mind this proposition and the result on quasi-minimality appearing in [1] mentioned above, the construction of minimal realizations for controllable systems can be done at once such a construction is obtained for systems satisfying $\Delta = \Delta^+ = T^*M$. For the sake of completeness we include the following statements.

Proposition 2.2. The unobservability equivalence relation $\sim$ is discrete in case $\Delta = \Delta^+ = T^*M$.

Proof. For $x \in M$ we can choose indices $i_j \in \{1, \ldots, s\}$ and $\varphi_j \in S_\Sigma$, $j = 1, \ldots, n$, $n = \dim M$ such that if $f_j = h_{i_j} \circ \varphi_j$ then $T^*M = \Delta^+ = \{df_1, \ldots, df_n\}$. If $y \sim x$ then $\varphi_j(y) \sim \varphi_j(x)$ so $f_j(y) = f_j(x)$. However $x \to (f_1(x), \ldots, f_n(x))$ is locally a diffeomorphism around x. Therefore there exist a neighbourhood V of x such that

$$\{x\} = C(x) \cap V$$

and $\sim$ is discrete □

Corollary 2.3. If Σ is controllable then Σ admit a weak minimal realization.

Proof. Is immediate after propositions 2.1 and 2.2 □

Example 2.4. On $\mathbb{R}^2$ take the family of vector fields $D = \left\{\frac{\partial}{\partial x}, \pm\frac{\partial}{\partial y}\right\}$ and the output map $h(x, y) = h(x)$ a smooth function of x alone wich is zero for $x \geq 0$ and strictly decreasing for $x < 0$. We have for $(x, y) \in \mathbb{R}^2$, *Ker* $\Delta(x, y)$ is spanned by $\frac{\partial}{\partial y}$, while

$Ker\ \Delta^+(x,y) = Ker\ \Delta(x,y)$ in case $x < 0$ and $Ker\ \Delta^+(x,y) = T_{(x,y)}\mathbb{R}^2$ if $x > 0$. For $t < 0$ the diffeomorphism $\varphi_t \in G_\Sigma$ define by $\varphi_t(x,y) = (x+t,y)$ does not preserve $Ker\ \Delta^+$.

§ 3. ISOTROPY SEMI-GROUPS

For a given system Σ, we have denoted by G_Σ and S_Σ the system group and semi-group respectively. With the assumption that the vector fields in Σ are globally defined and complete, G_Σ is properly a group and S_Σ a semi-group of diffeomorphism of M. In this case, S_Σ is a subsemigroup of G_Σ and generates G_Σ in the sense that every $\varphi \in G_\Sigma$ is a finite product of elements of S_Σ and $S_\Sigma^{-1} = \{\varphi^{-1} : \varphi \in S_\Sigma\}$. Next we shall see that in the presence of controllability, the isotropy semi-group at $x \in M$, also generates the isotropy group. For $x \in M$, set

$$G_x = \{\varphi \in G_\Sigma \ : \ \varphi(x) = x\}$$

and

$$S_x = \{\varphi \in S_\Sigma \ : \ \varphi(x) = x\}.$$

G_x is the isotropy group at x and S_x the isotropy semi-group. Clearly, $S_x = S_\Sigma \cap G_x$ and G_x is a subgroup of G_Σ. Similarly S_x is a subsemi-group of S_Σ. In general, it may happens that the size of S_x is too small compared with that of G_x. However, in case Σ is controllable we can state.

Proposition 3.1. Assume Σ is controllable. Then, for all $x \in M$ S_x generates G_x.

Proof. Throughout we fix $x \in M$. We shall regard formally G_Σ is a principal bundle over M with typical fibre G_x. For every $y \in M$, we put:

$$E_y = \{p \in G_\Sigma \; : \; p(x) = y\}$$

Since the elementsin G_Σ are globally defined diffeomorphism in M, there is for each $p \in Ey$ the identification (bijection) $a \in G_x \to p\,a = p \circ a \in E_y$.

Under this identification it is possible to parameterize the elements in S_y fixing y by elements of G_x. In fact if $q \in S_\Sigma$ is such that$q(y) = y$ then there exists a unique $a \in G_x$ with $q\,p = p\,a$. Taking into account this parameterization we put for each $p \in E_y$

$$S_p = \{a \in G_x \; : \; \exists q \in S_y \;\text{ with }\; q\,p = p\,a\}.$$

We shall show that for each $p \in E_y$, S_p generates G_x. This is clearly equivalent to the statement of the proposition. We prove first:

"for $\varphi \in S_\Sigma$ and $p_1 \in E_y$, let $p_2 = \varphi p_1$, then

(#)

$$\text{Span}(S_{p_1}) = \text{Span}(S_{p_2})"$$

Put $z = \varphi(y)$ and take $a \in S_{p_2}$ and $\psi \in S_\Sigma$ such that $\psi(z) = y$. Then,

$$p_2 a \in S_\Sigma \circ p_2 = S_\Sigma \circ \varphi \circ p_1 \subset S_\Sigma \circ p_1.$$

So if we define $b \in G_x$ by means of $\psi \circ p_2 a = p_1 b$ then $p_1 b \in S_\Sigma \circ p_1$ and $b \in Sp_1$. We have,

$$\varphi^{-1}\psi^{-1}p_1 b = \varphi^{-1}p_2 a = p_1 a$$

and

$$\psi\varphi p_1 = p_1\bar{a} \;\text{ with }\; \bar{a} \in Sp_1.$$

Therefore,

$$p_1 a = \varphi^{-1}\psi^{-1}p_1 b = p_1 \cdot \bar{a}^{-1}b$$

so that

$$a = \bar{a}^{-1}b \text{ and } a \in \mathrm{Span}(Sp_1).$$

Since $a \in Sp_2$ was taken arbitrary, this means that

$$\mathrm{Span}(S_{p_2}) \subset \mathrm{Span}(S_{p_1})$$

To see the other inclusion, reverse time by taking

$$-D = \{-x | x \in D\} \text{ instead of } D.$$

Since

$$S_{-\Sigma} = S_\Sigma^{-1}$$

S_p is changed into S_p^{-1}. So applying the same reasoning to $\varphi^{-1} \in S_\Sigma^{-1}$ instead of φ. We get

$$\mathrm{Span}(S_{p_1}^{-1}) \subset \mathrm{Span}(S_{p_2}^{-1})$$

so that

$$\mathrm{Span}(S_{p_1}) = \mathrm{Span}(S_{p_2})$$

as claimed. Now, take $p \in E_y$ and $\psi \in S_\Sigma$ and put $z = \psi^{-1}(y)$. By controllability there exist $\varphi \in S_\Sigma$ with $\varphi(y) = z$. Then

$$\psi\varphi p \in E_y \text{ so } \psi\varphi p = pa \text{ with } a \in S_p.$$

Hence $\psi^{-1}p = \varphi p a^{-1}$, i.e., $\psi^{-1}p \in S_\Sigma \circ pb$ with $b \in \mathrm{Span}(S_p)$. Applying this relation successively, we see that for every $\theta \in G_\Sigma$, $\theta p \in S_\Sigma \circ pb$, $b \in gen.(S_p)$. In particular, this happens if $\theta \in G_y$. The proof of the proposition follows now from the fact that every $a \in G_x$ is uniquely defined by means of

$$\psi p = pa \ , \ \text{ with } \psi \in G_y \quad \square$$

§ 4. Σ-FINITE EQUIVALENCE RELATIONS

As we saw before the construction of minimal realizations of controllable systems reduces to the case where the unobservability equivalence relation is discrete. We show below that minimal realizations for controllable systems exist in case this discrete equivalence relation satisfies a certain finiteness condition.

Starting with a system such that each element in Σ is a globally defined and complete vector field in M we construct the family Σ^2 of vector fields on $M \times M$ bu putting

$$\Sigma^2 = \{X \oplus X \ : \ X \in \Sigma\}$$

Associated to this family we have a group G_{Σ^2} and a semi-group S_{Σ^2} of diffeomorphisms on $M \times M$. Of course, if $\pi : M \times M \to M$ denotes the projection onto the first coordinate, then

$$\pi(G_{\Sigma^2}(x,y)) = G_\Sigma(x) \ , \ \ \forall y \in M.$$

Let G_x be the isotropy subgroup of G_Σ at $x \in M$. The orbits $G_x(y)$ of G_x partition M in a foliation with singularities ([7]). Also, from the construction of G_{Σ^2}, we have at once that for $x, y \in M$

$$(x, G_x(y)) = G_{\Sigma^2}(x,y) \cap \pi^{-1}(x)$$

Definition. Let $R \subset M \times M$ be a discrete equivalence relation. We say that R is Σ-finite in case

i) R is S_{Σ^2}-invariant (i.e. $\varphi(x,y) \in R$ if $(x,y) \in R$ and $\varphi \in S_{\Sigma^2}$)

ii) For each $x \in M$ the intersection of $R(x)$ with every orbit of G_x is at most finite.

Proposition 4.1. Suppose Σ is controllable and let R be a Σ-finite equivalence relation. Then

a) $G_x(R(x)) = R(x)$ for every $x \in M$, so that $G_x(y)$ is finite for $y \in R(x)$.

b) For every $x \in M$, $y \in R(x)$ dim $G_{\Sigma^2}(x,y) =$ dim M.

Proof. a) Take $\varphi \in S_x$ and $y \in R(x)$. By assumption $R(x) \cap G_x(y)$ is finite and $\varphi(R(x) \cap G_x(y)) \subset R(x) \cap G_x(y)$. It follows that $\varphi(R(x) \cap G_x(y)) = R(x) \cap G_x(y)$. Since y is arbitrary this implies that $\varphi(R(x)) = R(x)$ and hence that $\varphi^{-1}(R(x)) = R(x)$. And (a) follows from Proposition 3.1.

b) From (a) and the remark made previously it follows that $G_{\Sigma^2}(x,y) \cap \pi^{-1}(x)$ is finite. This implies that dim $G_{\Sigma^2}(x,y) =$ dim $G_\Sigma(x) =$ dim M because Σ is assumed to be controllable and hence transitive □

Under the conditions of this proposition we have thus that for $y \in R(x)$ $G_{\Sigma^2}(x,y) \cap \pi^{-1}(x)$ is finite.

We can now prove

Proposition 4.2. With the assumptions of proposition 4.1 if $y \in R(x)$ then Σ^2 is controllable on $G_{\Sigma^2}(x,y)$.

Proof. Let m be the order of $X = G_{\Sigma^2}(x,y) \cap \pi^{-1}(x)$. Make G_x act on X by $p(x,y) =$

$(x, p(y))$, $(x,y) \in X$, $p \in G_x$. This action defines a homomorphism ρ of G_x into the permutations group of m elements. This group being finite, the group $G_x/Ker\rho$ is a finite group. This being so $S_x/Ker\rho = G_x/Ker\rho$. In fact, $S_x/Ker\rho$ is a semi-group of $G_x/Ker\rho$, which by proposition 3.1 generates $G_x/Ker\rho$. From this it is easily seem that

$$S_x/Ker\rho = G_x/Ker\rho$$

Since the action of G_x on X depends only on the action of $G_x/Ker\rho$ in X, it follows that S_x is transitive on X. Controllability now follows as in the proof of the main result in [5] □

Corollary 4.3. Under the conditions of Proposition 4.1 G_Σ preserves R.

Proof. Take $x, y \in M$ such that $y \in R(x)$ and $\varphi \in G_\Sigma$. Put $x_1 = \varphi(x)$ and $y_1 = \varphi(y)$. Then $(x_1, y_1) \in G_{\Sigma^2}(x,y)$ and since Σ^2 is controllable on the orbit $G_{\Sigma^2}(x,y)$, there exist $\psi \in S_\Sigma$ such that $x_1 = \psi(x)$, $y_1 = \psi(y)$. So that $(x_1, y_1) \in R$ and R is preserved by G_Σ □

Theorem 4.4. Suppose that Σ is controllable and that the unobservability relation $\sim$ is Σ-finite. Then there exist a minimal realization for Σ.

Proof. From de above corollary we conclude that $\sim$ is G_Σ-invariant. The theorem thus follows from Theorem 11 in [8].

Remarks. 1) In the proofs above we used heavely the action of G_x on $G_{\Sigma^2}(x,y) \cap \pi^{-1}(x)$. By the very construction of G_x, it is not difficult to look at it as a subgroup of the fundamental group of M. This should establish a bridge between our method and those employed by Gauthier and Bonnard [4].

2) Theorem 4.4 show that in case Σ is controllable, the equivalence classes of a Σ-finite equivalence relation R are finite and have the same number. This follows from the existence of M/R.

3) From proposition 4.2 we have in particular that Σ^2 restricted to $G_{\Sigma^2}(x,y)$ has the accessibility property. It is an interesting question to know wether this is a consequence of the controllability of Σ alone.

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