New Sum Rules of Special Functions

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RELATÓRIO TÉCNICO Nº 25/89

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Julho – 1989

New Sum Rules of Special Functions

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1. Introduction

In a recent paper Dattoli and Dipace([1]) have devised a generating function method to derive bilinear sum rules involving Bessel and Laguerre polynomials which emerge from the following mathematical problems, free-electron-laser([2]) and plasma physics([3]). In another recent paper Montaldi-Zucchelli([4]) using a very simple method have obtained without solving the differential equation bi-bilinear sum rules for Laguerre and Jacobi polynomials and sum rules for Bessel functions.

We note here that both Davoli-Dipace and Montaldi-Zucchelli bilinear sum rules of special functions involve only functions of the first kind. We call first kind the functions which are regular solutions of a given differential equation near at

([1]) G. Dattoli and A. Dipace: Nuovo Cimento B, **87**, 50 (1985)
([2]) W. B. Becker: Opt. Commun. **33**, 69 (1980)
([3]) G. Dattoli, A. Dipace and A. Torre: Nuovo Cimento B, **90**, 85 (1985)
([4]) E. Montaldi and G. Zucchelli: Nuovo Cimento B, **102**, 229 (1988)

the origin and the second kind the functions which are regular solution of the given differential equation at infinity.

The aim of this paper is show that it is possible to obtain bilinear sum rules of special function where not only first kind are involved. Indeed, the sum rules to be presented below involve functions of the first and second kinds and different arguments.

The possibility of obtaining these new sum rules is due to global technique, namely the Green's function technique.

The systematics of the Green's function technique is based in writting the Green's function associated with a differential equation in two different ways. Firstly we write the Green's function associated with the differential equation by means of Sturm–Liouville method([5]) and secondly by means of Coulomb Green's function([6]).

This paper is organized as follow: in Section 2 we present the outline of the method; in Section 3 we calcule bilinear sum rules of special function and in Section 4 we present our conclusions.

2. Outline of the method

To discuss the Green's function technique we consider here only the Jacobi differential equation.

The Green's function associated to Jacobi differential equation satisfies the following non homogeneous differential equation

$$\mathcal{L}\, G(x,x') = \delta(x-x') \tag{2.1}$$

where $\mathcal{L}$ is the Jacobi differential operator defined by

$$\mathcal{L} = (1-x^2)\frac{d^2}{dx^2} + [2(\beta-\alpha) - 2(\alpha+\beta+1)x]\frac{d}{dx} + \mu(\mu+2\alpha+2\beta+1)$$

where $2\alpha > -1$ and $2\beta > -1$.

Two linearly independent solutions of the homogeneous differential equation are

$$P_\mu^{(2\alpha,2\beta)}(x) \quad \text{and} \quad Q_\mu^{(2\alpha,2\beta)}(x)$$

where the first (first kind) is regular at origin and the second (second kind) is regular at infinity.

([5]) E. Capelas de Oliveira: Master Thesis, IFGW - UNICAMP (1979)

([6]) L. C. Hostler: J. Math. Phys. 11, 2966 (1970)

Using the Sturm–Liouville method[5] to calculate the Green's function associated with Jacobi differential equation, we obtain

$$(2.2)\quad G(x,x') = 2^{-2(\alpha+\beta)}\frac{\Gamma(\mu+1)\Gamma(\mu+2\alpha+2\beta+1)}{\Gamma(\mu+2\alpha+1)\Gamma(\mu+2\beta+1)}P_\mu^{(2\alpha,2\beta)}(x_<)Q_\mu^{(2\alpha,2\beta)}(x_>)$$

where $x_<(x_>)$ is the lesser (greater) of $x(x')$.

We must now calculate the Green's function via Coulomb Green's function. The Green's function for the Schrōdinger's differential equation with a Coulomb potential was obtained by Hostler[6] making an expansion of the Green's function is partial waves and using the Sturm–Liouville method in the radial differential equation.

From this we take the Fourier transform of the Jacobi differential equation and get a differential equation which is identified with the radial differential equation of the Schrōdinger's differential equation with Coulomb potential. As an integral representation, in terms of Whittaker functions for the radial Coulomb Green's function is known[6], we can calculate the inverse Fourier transform and then obtain the Green's function to Jacobi differential equation.

The Jacobi functions have the following integral representation[7]

$$2^{-n}\frac{\Gamma(\nu+n+1)}{\Gamma(\nu+m+1)}\frac{\Gamma(2\nu+2)}{\Gamma(\nu-m+1)}Q_{\nu-n}^{n-m;n+m}(x) = \frac{1}{2}\int_0^\infty dte^{-xt}t^{n-1}M_{m;\nu+1/2}(2t)$$

$$2^{-n}\frac{\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)}P_{\nu-n}^{(n-m,n+m)}(x) = (-1)^{\nu-n+1}\frac{1}{2\pi i}\int_\infty^{0+} dte^{-xt}t^{n-1}W_{m;\nu+1/2}(2t)$$

where $M_{\mu;\nu}(x)$ and $W_{\mu;\nu}(x)$ are the Whittaker functions, being the first regular at origin and the second regular at infinity.

Introducing the above expressions in Eq. 2.2 we obtain for the Green's function the following expression:

$$(2.3)\quad \begin{aligned} G(x,x') &= \frac{1}{2}\frac{\Gamma(\nu-n+1)}{\Gamma(2\nu+2)}\frac{\Gamma(\nu-m+1)}{\Gamma(\nu+n+1)}(-1)^{\nu-n+1}\,.\\ &\cdot\frac{1}{2\pi i}\int_0^\infty\int_\infty^{0+} e^{-xt-x't'}(tt')^{n-1}M_{m;\nu+1/2}(2t)W_{m;\nu+1/2}(2t)dtdt' \end{aligned}$$

where the parameters are conveniently identified.

([7]) J. Bellandi F$^{\underline{o}}$, E. Capelas de Oliveira and H. G. Pavão, Rev. Bras. Fis. 12, 600, (1982) The product of two Whittaker functions which appear in this equation is the Green's function for the radial Schrödinger differential equation with the Coulomb potential and here it is $t' > t$.

Introducing the Coulomb Green's function in Eq. (2.3) and making an integration the Green's function for the Jacobi differential equation results,

$$G(x,x') = (-1)^{\nu-n+1} \frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(2\nu+2)} \cdot$$

$$\cdot \frac{1}{2\pi i} \int_{\infty}^{0+} dv (shv)^{2\nu+1} cth^{2m} v/2 \left\{ (x + chv)(x' + chv) \right\}^{-\nu-n-1} \cdot \tag{2.4}$$

$$\cdot \, {}_2F_1 \left[\nu+n+1;\ \nu+n+1;\ 2\nu+2;\ \frac{sh^2 v}{(x+chv)(x'+chv)} \right]$$

where ${}_2F_1(a,b;c;x)$ is a hypergeometric function and $x' > x$.

We note that, identifying the two Green's functions we obtain an integral representation for the product of two linearly independent solutions of Jacobi differential equation with different arguments and of different kinds. Using the expansion of the hypergeometric function(8) we obtain a new sum rule for Jacobi functions that have as particular cases, Gegenbauer, Legendre and Tchebishef functions and polynomials(9).

3. Bilinear Sum Rule of Jacobi Functions

Using the above integral representation we obtain a bilinear sum rule of Jacobi functions (and polynomials) and as a by product the bilinear sum rule of Legendre functions (and polynomials).

We introduce a change of variable of the type

$$chv = \frac{1+\xi^2}{1-\xi^2}$$

in Eq. (2.4) and we use an integral representation for the hypergeometric function in terms of Bessel functions(10) $J_\mu(x)$ and $Y_\mu(x)$. We get

(8) A. Erdélyi: Higher Transcendental Function – New York Mc Graw-Hill (1953)

(9) E. Capelas de Oliveira, Ph.D.'s thesis, IFGW–UNICMAP (1982)

(10) I. S. Gradshtein and I. M. Ryzhik, Table of Integrals, Series and Products. Acad. Press., Inc. (1980)

$$G(x,x') = \frac{2\pi}{\text{sen}\{\pi(\nu - n + 1/2)\}} \alpha^{-m} \beta^{-n} \oint_{\eta=0} d\eta\, \eta^{-2m+2n-1}$$

(3.1)
$$\cdot \int_0^\infty J_{2\nu+1}(2t) \left(\frac{z_1 - z_2/\eta^2}{z_1 - z_2\eta^2} \right)^n Y_{2n}\left\{ (z_1 - z_2/\eta^2)^{1/2} (z_1 - z_2\eta^2)^{1/2} t \right\} dt$$

where the parameters are defined by

$$\alpha^2 = \frac{(x+1)(x'+1)}{(x-1)(x'-1)} \quad \beta^2 = (x^2 - 1)(x'^2 - 1) \quad z_1^2 = \alpha\beta \quad z_2^2 = \frac{\beta}{\alpha}$$

Integrating in η – the unitary circle around the origin – with the parametrization $\eta^2 = \exp(i\phi)$ and using the Graf sum([11]) we obtain the following integral representation for the Green's function

$$G(x,x') = \pi \frac{\beta^{-n} \alpha^{-m}}{\text{sen}\{\pi(\nu - n + 1/2)\}} \cdot$$
$$\cdot \int_0^\infty J_{2\nu+1}(2t) Y_{m+n}(z_1 t) J_{m-n}(z_2 t) dt$$

Multiplying both members of the above equation by the weight function and canceling the term $\alpha^{-m}\beta^{-n}$ we get:

(3.2)
$$(x-1)^{\frac{n-m}{2}} (x+1)^{\frac{n+m}{2}} (x'-1)^{\frac{n-m}{2}} (x'+1)^{\frac{n+m}{2}} G(x,x') =$$
$$= \frac{\pi}{\text{sen}\{\pi(\nu - n + 1/2)\}} \int_0^\infty J_{2\nu+1}(2t) Y_{m+n}(z_1 t) J_{m-n}(z_2 t) dt.$$

Then, to obtain a bilinear sum rule for Jacobi function we multiply both members of the above equation by $\exp(im\rho)$ and sum in the m–parameter. We obtain

(3.3)
$$\sum_{m=-\infty}^{\infty} 2^{-n} \frac{\Gamma(\nu - n + 1)}{\Gamma(\nu - m + 1)} \frac{\Gamma(\nu + n + 1)}{\Gamma(\nu + m + 1)} W(x,x') P_{\nu-n}^{(n-m;\, n+m)}(x) Q_{\nu-n}^{(n-m;\, n+m)}(x') e^{im\rho} =$$
$$= \frac{\pi}{\text{sen}\{\pi(\nu - n + 1/2)\}} \int_0^\infty J_{2\nu+1}(2t) dt \sum_{m=-\infty}^{\infty} Y_{m+n}(z_1 t) J_{m-n}(z_2 t) e^{im\rho}$$

where $W(x,x')$ is the weight function.

([11]) W. Magnus, F. Oberhettinger and R. P. Soni, Formulas and Theorems for the Special function of Mathematical Physics – New York (1966)

To perform the sum in right side of Eq. (3.3) we use the Graf sum rule[11] and expand the hypergeometric function in terms of the Jacobi function. We then get the following bilinear sum rule,

$$\sum_{m=-\infty}^{\infty} 2^{-2n} \frac{\Gamma(\nu-n+1)}{\Gamma(\nu-m+1)} \frac{\Gamma(\nu+n+1)}{\Gamma(\nu+m+1)} (x-1)^{\frac{n-m}{2}} (x+1)^{\frac{n+m}{2}} (x'-1)^{\frac{n-m}{2}} (x'+1)^{\frac{n+m}{2}} \cdot$$

$$\cdot P_{\nu-n}^{(n-m;n+m)}(x) Q_{\nu-n}^{(n-m;n+m)}(x') e^{im\rho} =$$

$$= e^{i(\rho+2\psi)n} \left(\frac{\omega}{2}\right)^{2n} Q_{\nu-n}^{(0;2n)} \{xx' - (x^2-1)^{1/2}(x'^2-1)^{1/2} \cos\rho\}$$

where $x > 1$, $x' > 1$, $0 < \psi < \pi/2$, $\rho \in \mathbb{R}$ and the parameters ψ and ω are given by

$$e^{2i\psi} = \frac{z_1 - z_2 e^{-i\rho}}{z_1 - z_2 e^{i\rho}} \qquad \omega^2 = (z_1 - z_2 e^{i\rho})(z_1 - z_2 e^{-i\rho})$$

Now, in the case $|x| < 1$, for angular solutions, we obtain a sum rule to Jacobi polynomials,

$$\sum_{m=-\infty}^{\infty} 2^{2n} \frac{\Gamma(\nu-n+1)\Gamma(\nu+n+1)}{\Gamma(\nu-m+1)\Gamma(\nu+m+1)} (-1)^n (1-\cos\theta)^{\frac{n-m}{2}} (1+\cos\theta)^{\frac{n+m}{2}} \cdot$$

$$\cdot (1-\cos\theta')^{\frac{n-m}{2}} (1+\cos\theta')^{\frac{n+m}{2}} P_{\nu-n}^{(n-m;n+m)}(\cos\theta) Q_{\nu-n}^{(n-m;n+m)}(\cos\theta') e^{im\rho} \tag{3.4}$$

$$= e^{i(\rho+2\psi)n} \left(\frac{1+\cos\gamma}{2}\right)^n Q_{\nu-n}^{(0;2n)}(\cos\gamma)$$

where $\cos\gamma = \cos\theta \ \cos\theta' + \operatorname{sen}\theta \ \operatorname{sen}\theta' \ \cos\rho$, $0 < \theta < \pi/2$, $0 < \theta' < \pi/2$, $0 < \psi < \pi/2$ and $\rho \in \mathbb{R}$.

This sum rule is a generalization of the well known Legendre sum rule which we obtain making $n = 0$ in Eq. (3.4)

$$\sum_{m=-\infty}^{\infty} (-1)^m P_\nu^{-m}(\cos\theta) Q_\nu^m(\cos\theta') e^{im\rho} = Q_\nu(\cos\gamma)$$

where $\cos\gamma = \cos\theta \ \cos\theta' + \operatorname{sen}\theta \ \operatorname{sen}\theta' \ \cos\rho$, $0 < \theta < \pi/2$, $0 < \theta' < \pi/2$, $0 < \theta + \theta' < \pi$ and $\rho \in \mathbb{R}$.

4. Conclusions

We have obtained a bilinear sum rule for Jacobi functions (polynomials) by means of the Green's function technique. This bilinear sum rule is a generalization of the Legendre sum rule. We note that this technique is a global one since it permits us to write bilinear sum rules for both Jacobi functions and polynomials.

We must remember that this Green's function technique when used for the confluent hypergeometric functions gives directly the integral representation for the Green's function. This is important since then it is not necessary to calculate the inverse Fourier transform.

We believe that this technique is a great tool in the description of the motion of the spherical symmetric top because the differential equation parametrized, in terms of the Euler angles must be written there as a Jacobi differential equation which are irreductibles representations of the rotation group.

We also believe that this technique should be important in the study of the quasi unitary unimodular matrix representation of $SU(2)$ since the Jacobi functions are related with the generalized spherical harmonics.

The author express his gratitude to Prof. W. A. Rodrigues Jr. for bringing to his attention the paper by Montaldi and Zucchelli and for many discussions.

Riassunto:

In questo lavoro si utilizza il metodo della funzione di Green per dedurre nuove regole di somma per le funzione di Jacobi che è la generalisazione della regola di somma delle funzione di Legendre.

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