ON PSEUDO–CONVEX POLYCIRCULAR DOMAINS IN BANACH SPACES

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Abstract. We introduce the notion of polycircular domain in a product of Banach spaces and we prove that pseudo–convexity of this kind of domain can be deduced from the pseudo–convexity of one of its projections plus another very natural condition. Afterwards we study the same type of question for more general domains when one of the Banach spaces has an unconditional Schauder basis.

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It is a natural question to ask for conditions under which properties of domains in a cartesian product of Banach spaces may be deduced from properties of their projections. This problem is more interesting when the domains under consideration are not the product of domains in the component spaces. In this note we introduce the notion of polycircular domain in a product of Banach spaces and we prove that pseudo–convexity of this kind of domain can be deduced from the pseudo–convexity of one of its projections plus another very natural condition. Afterwards we study the same type of question for more general domains when one of the Banach spaces has an unconditional Schauder basis.

Unless expressed explicitly otherwise we consider the complex Banach spaces E, F, G such that $E = F \times G \cong F \oplus G$, where $\oplus$ denotes the topological direct sum. If x is in E we may write $x = x_F + x_G$ to indicate the unique representation of x as the sum of an element of F with another of G. We also use π_F and π_G to denote the continuous projections from E onto F and G respectively.

1. Definition. An open connected subset U of E is called a *polycircular domain with space of symmetry F* if every u in U implies that $v + u_G$ belongs to U whenever v is in F and $||v|| = ||u_F||$. This domain is *complete* if $v + u_G$ is in U for every u in

U, v in F with $||v|| \leq ||u_F||$.

We remark that the above definition is inspired by the usual concept of polycircular domain in $\mathbb{C}^n$ (see [4]): an open connected subset U of $\mathbb{C}^n$ is a polycircular domain with plane of symmetry $z_1 = 0$ if $(z_1 e^{i\mu}, z_2, \ldots, z_n)$ belongs to U whenever $(z_1, z_2, \ldots, z_n) \in U$ and μ is in $\mathbb{R}$; and it is complete if $(\lambda z_1, z_2, \ldots, z_n)$ is in U for $(z_1, z_2, \ldots z_n) \in U$ and $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$. It is clear that this definition gives many others domains (even in $\mathbb{C}^n$) not covered by the classical concept.

The following theorem gives a characterization of complete polycircular domains.

2. Theorem. U is a complete polycircular domain in E with space of symmetry F if, and only if, there are a domain B in G and a lower semicontinuous function ρ from B into $\mathbb{R}_+$ such that

$$U = \{x + y;\ x \in F,\ y \in B \quad \text{and} \quad ||x|| < \rho(y)\} \qquad (*)$$

Proof. It is quite obvious that if U has the form (*) then it is a complete polycircular domain with space of symmetry F.

Now we suppose that U is a complete polycircular domain with space of symmetry F. We have $\pi_G(U)$ equals to $U \cap G = B$ and we define ρ from $\pi_G(U)$ into $\mathbb{R}_+$ by

$$\rho(u_G) = \sup\{r;\ r > 0 \quad \text{and} \quad B_r^F(0) + u_G \subset U\}$$

where $B_r^F(0)$ denotes the open ball in F with center 0 and radius r. In order to show that ρ is lower-semicontinuous we consider g in $\pi_G(U)$ and $\alpha > 0$ such that $\rho(g) > \alpha$. It is clear that there is $r > \alpha$ such that $B_r^F(0) + g \subset U$. We consider $\varepsilon = 2^{-1}(r - \alpha) > 0$ and w in F with $||w|| = r - \varepsilon > 0$. There is $\delta > 0$ such that U contains the set $w + B_\delta^G(g)$. Thus, if v is in $B_\delta^G(g)$, we have $w + v$ in U and, for every t in F, with $||t|| \leq ||w|| = r - \varepsilon$ we get $t + v$ in U. Therefore $B_{r-\varepsilon}^F(0) + v$ is contained in U and $\rho(v) \geq r - \varepsilon > \alpha$ for every v in $B_\delta^G(g)$. Now we show that

(i) $U \subset \{x + y;\ x \in F,\ y \in \pi_G(U) \text{ and } ||x|| < \rho(y)\}$.

If $u \in U$ then $u_G \in \pi_G(U)$ and there is $\delta > 0$ such that $B_\delta^F(u_F) + u_G$ is contained in U. If $u_F = 0$, then it is clear that $\rho(u_G) > \delta > 0 = ||u_F||$. If $u_F \neq 0$ we consider $w = (2||u_F||)^{-1}\delta u_F$. Thus w is in $B_\delta^F(0)$ and $u_F + w + u_G = (1 + (2||u_F||)^{-1}\delta)u_F + u_G$ is in U with the norm of the first term of the sum equals to $||u_F|| + 2^{-1}\delta$. Hence $\rho(u_G) \geq ||u_F|| + 2^{-1}\delta > ||u_F||$ and (i) is true. We also have:

(ii) $\{x + y;\ x \in F,\ y \in \pi_G(U) \text{ and } ||x|| < \rho(y)\} \subset U$.

If $x+y$ is in the left-hand side set in (ii) and r is such that $\|x\| < r < \rho(y)$ we get $B_r^F(0) + y \subset U$ and $x+y \in U$. Hence (ii) is true and (*) follows.

3. Theorem. Let $U = \{x+y;\ x \in F,\ y \in B,\ \|x\| < \rho(y)\}$ be a complete polycircular domain with space of symmetry F. Then U is pseudo-convex if, and only if, B is pseudo-convex in G and $-\log\rho$ is plurisubharmonic in B.

Proof. (1) If U is pseudo-convex, since $B = U \cap G$, it follows that B is pseudo-convex in G. We note that if $b \in B$ and $x \in F$,

$$d_U(b;x) = \sup\{s > 0;\ b + \lambda x \in U,\ |\lambda| < s\}.$$

Hence $\rho(b) \leq d_U(b;x)$ for every x in F of norm 1. On the other hand, if $r < \inf\{d_U(b;x);\ x \in F,\ \|x\| = 1\}$ we have $b + \lambda x$ in U for every x in F of norm 1 and λ in $\mathbb{C}$, $|\lambda| \leq r$, i.e., U contains the set $B_r^F(0) + b$. Hence $r \leq \rho(b)$. As a consequence of these facts we have $\rho(b) = \inf\{d_U(b;x);\ x \in F,\ \|x\| = 1\}$, and consequently

$$-\log\rho(b) = \sup\{-\log d_U(b;x);\ x \in F,\ \|x\| = 1\}.$$

Since U is pseudo-convex, $-\log d_U(\,\cdot\,;x)$ is plurisubharmonic in B for every x in F of norm 1. It follows that $-\log\rho$ is plurisubharmornic in B.

(2) If B is pseudo-convex and $-\log\rho$ is plurisubharmonic in B, it follows that $F \oplus B$ is pseudo-convex and $\log\|x_F\| - \log\rho(x_G) = \varphi(x)$ defines a plurisubharmonic function in $F \oplus B$. Consequently, since $U = \{x \in F \oplus B;\ \log\|x_F\| - \log\rho(x_G) < 0\}$, U is pseudo-convex.

For the basic results on pseudo-convexity and plurisubharmonicity in Banach spaces see Mujica [3].

Now we have the following consequences of Theorem 2.

3. Corollary. Let $U = \{x+y;\ x \in F,\ y \in B,\ \rho(y) > \|x\|\}$ be a complete polycircular domain with space of symmetry F.

(1) If U is a domain of existence in E and G is separable, then B is a domain of existence in G and $-\log\rho$ is plurisubharmonic in B.

(2) If U is a domain of holomorphy in E and G is separable with the bounded approximation property, then B is a domain of holomorphy in G and $-\log\rho$ is plurisubharmonic in B.

Proof. (1) Is a consequence of Theorem 2 and Proposition 11.7.(b) of [3]. (2) is a consequence of Theorem 2 and the Theorem of Gruman–Kiselman (see [1]).

4. Corollary. Let $U = \{x + y;\ x \in F,\ y \in B,\ \rho(y) > \|x\|\}$ be a complete polycircular domain in E with space of symmetry F. If E is separable with the bounded approximation property, then the following conditions are equivalent:

(1) U is pseudo-convex.
(2) U is a domain of existence.
(3) U is a domain of holomorphy.
(4) B is pseudo-convex and $-\log \rho$ is plurisbharmonic in B.
(5) B is a domain of existence in G and $-\log \rho$ is plurisubharmonic in B.
(6) B is domain of holomorphy and $-\log \rho$ is plurisubharmonic in B.

The proof is a direct consequence of Theorem 2 and the Theorem of Gruman-Kiselman.

From now we consider F with a normalized unconditional Schauder basis $(e_j)_{j=1}^{\infty}$. In this case there is another natural way of generalizing the classical definition of polycircular domain. In order to study this new concept we fix our notation. If $x \in F$ we write $x = \sum_{j=1}^{\infty} x_j e_j$ and for $\lambda = (\lambda_j)_{j=1}^{\infty} \in \mathbb{C}^{\mathbb{N}}$ we denote by x_λ the formal sum $\sum_{j=1}^{\infty} \lambda_j x_j e_j$. We denote by $\Delta^{\mathbb{N}}$ the set of all $\lambda \in \mathbb{C}^{\mathbb{N}}$ such that $|\lambda_j| \leq 1$ for every $j \in \mathbb{N}$ and by $S^{\mathbb{N}}$ the set of all $\lambda \in \mathbb{C}^{\mathbb{N}}$ such that $|\lambda_j| = 1\ \ \forall j \in \mathbb{N}$. In F we always consider its equivalent norm having the following property:

$$\|x\| = \sup\{\|\sum_{j=1}^{k} \lambda_j x_j e_j\|;\ \ k \in \mathbb{N},\ \ \lambda \in \Delta^{\mathbb{N}}\}.$$

Then it can be proved that $D_x = \{x_\lambda \in F;\ \ \lambda \in \Delta^{\mathbb{N}}\}$ is a compact subset of F. We recall that a connected open subset V of F is a *Reinhardt domain* if $x_\lambda \in V$ whenever $x \in V$ and $\lambda \in S^{\mathbb{N}}$. See Matos-Nachbin [3] for properties on pseudo-convexity of these domains. Now we introduce the notion of semi-Reinhardt domain in E.

5. Definition. An open connected subset U of E is called a *semi-Reinhardt domain with space of symmetry* F if $(u_F)_\lambda + u_G \in U$ for $u \in U$ and $\lambda \in S^{\mathbb{N}}$. It is *complete* if $u \in U$ and $\lambda \in \Delta^{\mathbb{N}}$ imply that $(u_F)_\lambda + u_G \in U$.

It is clear that if U is a (complete) semi-Reinhardt domain in E with space of symmetry F and $x \in \pi_G(U)$, then $\pi_F(U(x))$ is a (complete) Reinhardt domain in F, where $U(x)$ denotes the section of U at x. It is also clear that a (complete) polycircular domain in E with space of symmetry F is a (complete) semi-Reinhardt domain.

6. Proposition. *If U is a complete semi-Reinhardt domain in E with space of symmetry F, then V coincides with the set*

$$V = \{x + y;\ x \in F,\ y \in U \cap G \ \text{and} \ \inf_{\lambda \in \Delta^{\mathbb{N}}} d_U(y; x_\lambda) > 1\}.$$

Proof. If $x + y \in V$ with $x \in F$, $y \in U \cap G$ and $d_U(y; x_\lambda) \geq \alpha > 1$ for all $\lambda \in \Delta^{\mathbb{N}}$, it follows that $x_\lambda + y \in U$ for all $\lambda \in \Delta^{\mathbb{N}}$. Hence $x + y \in U$.

On the other hand if $x + y \in U$ with $x \in F$ and $y \in G$, then $x_\lambda + y \in U$ for all $\lambda \in \Delta^{\mathbb{N}}$. In particular, for $\lambda_j = 0,\ j \in \mathbb{N}$, we get $y \in U \cap G$. Since D_x is compact in F, $D_x + y \subset U$ and d_U is lower semi-continuous in $U \times E$, we have $d_U(y; x_\lambda) > 1$ for all $\lambda \in \Delta^{\mathbb{N}}$ and

$$\inf_{t \in D_x} d_U(y, t) = d_U(y, t_0)$$

for some $t_0 \in D_x$, hence > 1. Therefore $x + y \in V$. Q.E.D.

If G has also a normalized unconditional Schauder basis $(b_j)_{j=1}^{\infty}$, then the union of the basis of F and G is an unconditional normalized Schauder basis in E. It follows that a complete semi-Reinhardt domain in E with space of symmetry F is a complete Reinhardt domain if, and only if, B is a complete Reinhardt domain in G.

7. Theorem. *If U is a complete semi-Reinhardt domain in E with space of symmetry F then the following conditions are equivalent:*

(a) *U is pseudo-convex*

(b) *$B = U \cap G$ is pseudo-convex in G and $-\log d_U$ is plurisubharmonic in $B \times F \cong B \oplus F$.*

Proof. If U is pseudo-convex, then $B = U \cap G = \pi_G(U)$ is pseudo-convex and $-\log d_U$ is plurisubharmornic in $U \times E$, in particular in $B \times F$.

Now we suppose B pseudo-convex and $-\log d_U$ plurisubharmonic in $B \times F$. We define

$$\rho(b, x) = \sup\{-\log d_U(b; x_\lambda);\ \ \lambda \in \Delta^{\mathbb{N}}\}$$

for $b \in B$, $x \in F$. Since $\rho_\lambda(b, x) = -\log d_U(b, x_\lambda)$ defines a plurisubharmonic function in $B \times F$ for every $\lambda \in \Delta^{\mathbb{N}}$, we have ρ plurisubharmonic in $B \times F$ if ρ is upper semi-continuous in $B \times F$. This will be true if we prove that

$$\delta(b, x) = \inf\{d_U(b, x_\lambda)\ \ \lambda \in \Delta^{\mathbb{N}}\}$$

is lower semicontinuous in $B \times F$. In order to prove this we consider $\delta(b, x) > r > s$. Henxe $b + \mu x_\lambda \in U$ for all $|\mu| \leq r$ and $\lambda \in \Delta^{\mathbb{N}}$. Now if we use the continuity of the sum operation in $G \times F$ and a compacity argument (relative to the compact sets

D_x and $\Delta_r = \{\mu \in \mathbb{C};\ |\mu| \leq r\})$ we find open balls centered at 0 V and W in G and F such that $(b+v) + \mu(x+w)_\lambda \in U$ for all $v \in V$, $w \in W$, $|\mu| \leq r$ and $\lambda \in \Delta^N$. This implies that $\delta(b+v, x+w) \leq r$ for all $v \in V$ and $w \in W$. Hence we have proved that ρ is plurisubharmonic in $B \times F$. Since B is pseudo-convex it follows that $B \times F$ is pseudo-convex and $\{(b,x) \in B \times F;\ \rho(b,x) < D\} \cong U$ is a pseudo-convex open subset of $B \times F \cong F \oplus B \subset E$. Q.E.D.

8. Corollary. If G is separable with the bounded approximation property and U is a complete semi-Reinhardt domain in E with space of symmetry F, then the following conditions are equivalent:

(1) U is pseudo-convex.

(2) U is a domain of existence in E.

(3) U is a domain of holomorphy.

(4) B is pseudo-convex in G and $-\log d_U(\ \cdot\ ,\ \cdot\ )$ is plurisubharmonic in $F \times B \cong F \oplus B$.

(5) B is a domain of existence in G and $-\log d_U(\cdot, \cdot)$ is plurisubharmonic in $F \times B$.

(6) B is a domain of holomorphy and $-\log d_U(\ \cdot\ ,\ \cdot\ )$ is plurisubharmonic in $F \times B$.

This a consequence of Theorem 7 and the Theorem of Gruman–Kiselman. In this situation E is separable and has the bounded approximation property.

9. Theorem. If G has unconditional normalized Schauder basis and U is a complete Reinhardt domain in E, then the six conditions of Corollary 8 are also equivalent to:

(a) U is logarithmically convex and modularly decreasing.

(b) U is the domain of existence of a holomorphic function representable by multiple power series.

(c) U is the domain of convergence of a multiple power series around 0.

This is a consequence of Corollary 8 and Theorem 3.4 of [2], plus the remark made after the proof of Proposition 6.

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