A MULTIPLIER THEOREM
ON WEIGHTED ORLICZ SPACES

B. Bordin
and
J. B. Garcia

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Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
Caixa Postal 6065
13.081 – Campinas – SP
BRASIL

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A Multiplier Theorem on Weighted Orlicz Spaces

B. Bordin[1]
IMECC — UNICAMP
Caixa Postal 6065
13.081 - Campinas - SP - Brasil
and
J. B. Garcia
Universidade Federal de Mato Grosso do Sul
Departamento de Matemática
79.100 - Campo Grande - MS - Brasil

Abstract

In this note we establish a theorem for multipliers of Mihlin–Hörmander type in weighted Orlicz spaces.

1. Introduction

Let $m(x)$ be a bounded function on $\mathbb{R}^n$ and consider, for f in the Schwartz class $S(\mathbb{R}^n)$, the multiplier operator T_m defined by $(T_m f)(x) = (\widetilde{\widehat{m}} * f)(x)$, where $\widetilde{\widehat{m}}(t) = \widehat{m}(-t)$ and $\widehat{m}$ is the Fourier transform of m. We say that the multiplier m satisfies the Mihlin–Hörmander condition if, for a real number s greater than or equal to 1, a positive integer k and a multi-index $\gamma = (\gamma_1, \cdots, \gamma_n)$ of non negative integers γ_j with $|\gamma| = \sum \gamma_j$, we have

$$\sup_{R>0}\Big(R^{s|\gamma|-n} \int_{R<|x|<2R} |D^\gamma m(x)|^s dx\Big)^{1/s} < +\infty$$

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for all $|\gamma| \leq k$.

In 1960 L. Hörmander [6], proved that T_m is (after extension) bounded from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for all $1 < p < +\infty$ with $s = 2$ and $k > n/2$.

Afterwards, Hörmander's theorem has been generalized by several authors. For instance in [5], H. Triebel gives a vectorial version for these operators by considering, e.g., the space $L^p(\ell^q)$.

In [2], D.S. Kurtz–R.L. Wheeden proved the following theorem, for the weighted L^p-space $L^p_{(w)} = \{f \; : \; (\int_{\mathbb{R}^n} |f(x)|^p w(x) dx)^{1/p} < +\infty\}$:

1.1 Theorem. Let s and k real numbers such that $1 < s \leq 2$ and $n/s < k \leq n$.

(i) If $n/k < p < +\infty$ and $w \in A_{p.k/n}$

or

(ii) if $1 < p < (n/k)'$ and $w^{-1/(p-1)} \in A_{p'.k/n}$,

then T_m is bounded from $L^p_{(w)}$ into $L^p_{(w)}$.

Moreover,

(iii) if $w^{n/k} \in A_1$, then T_m is of w-weak type $(1,1)$, i.e.,

$$w(\{\, x \in \mathbb{R}^n \; : \; |Tf(x)| > \lambda\}) \leq c \cdot \lambda^{-1} \cdot \|f\|_{L^1_{(w)}}, \quad \lambda > 0.$$

A vectorial version of this theorem was given by D. L. Fernandez in [1], considering the weighted space $L^p_{(w)}(\ell^q)$.

In this note we consider the problem of obtaining a theorem of Kurtz–Wheeden type for weighted Orlicz spaces $L^\Phi_w(\mathbb{R}^n) = \{f \quad : \int \Phi(|f(x)|w(x))dx < +\infty\}$ (the precise definition is given bellow).

Our result depends essentially on an interpolation result for weighted Orlicz spaces and generalizes parts (i) and (ii) of theorem 1.1.

2. Preliminary definitions

A real continuous convex function Φ defined on the interval $[0, +\infty)$ is called a N–function if $\lim_{t\to 0} \Phi(t)/t = 0$ and $\lim_{t\to +\infty} \Phi(t)/t = +\infty$.

Given a N–function Φ, we can associate its conjugate function denoted by Φ^* and defined by

$$\Phi^*(t) = \max_{s>0}(st - \Phi(s)),$$

for all $t \geq 0$.

Hereafter, we shall always consider a N–function Φ satisfying a Δ_2–condition

($\Phi \in \Delta_2$), i.e., we shall suppose that there exists a number $a > 1$ such that

$$\Phi(2t) \leq a\Phi(t)$$

for all $t > 0$.

Given a N-function Φ, the Orlicz space $L^{\Phi}(\mathbb{R}^n, \mathbb{R}) = L^{\Phi}$ consists of all Lebesgue measurable functions f on $\mathbb{R}^n$ such that

$$\rho(f, \Phi) = \int_{\mathbb{R}^n} \Phi(|f(x)|)dx < +\infty.$$

The space L^{Φ} is complete when endowed with the norm (called Orlicz's norm)

$$||f||_{\Phi} = \sup\{ \int_{\mathbb{R}^n} |f(x).g(x)|dx \ : \ \rho(g, \Phi^*) \leq 1\}.$$

Let w be a positive function on $\mathbb{R}^n$. The weighted Orlicz space $L^{\Phi}_w(\mathbb{R}^n, \mathbb{R}) = L^{\Phi}_w$ is the space of all functions f such that $f.w$ belongs to L^{Φ}. The L^{Φ}_w-norm of f is given by

$$||f||_{\Phi,w} = \sup\{ \int_{\mathbb{R}^n} |f(x).g(x)|dx \ : \ \rho(g.w^{-1}, \Phi^*) \leq 1\}.$$

In this note we shall need to consider the numbers

$$\alpha = \lim_{s \to 0^+} [-\log(\sup_{t>0} \Phi^{-1}(t)/\Phi^{-1}(s.t))/\log s]$$

and

$$\beta = \lim_{s \to +\infty} [-\log(\sup_{t>0} \Phi^{-1}(t)/\Phi^{-1}(s.t))/\log s],$$

called Boyd's indices (see[4]) associated to the N-function Φ, where Φ^{-1} is the inverse of Φ.

For more details on Orlicz spaces we refer to Krasnosel'skii-Rutickii's book [7].

We shall denote by $L^p_w(\mathbb{R}^n)$ the space of all measurable functions f such that

$$||f||_{p,w} = (\int_{\mathbb{R}^n} |f(x).w(x)|^p dx)^{1/p} < +\infty.$$

We remember that a positive and locally integrable function w is a weight in the Muckenhoupt class A_p ($w \in A_p$), $1 < p < +\infty$, if for all cubes Q on $\mathbb{R}^n$, the inequality

$$(|Q|^{-1} \int_Q w(x)dx)(|Q|^{-1} \int_Q w(x)^{-1/(p-1)}dx)^{p-1} < +\infty$$

holds, where $|Q|$ stands for the Lebesgue measure of Q.

3. The multiplier theorem

We shall need the following interpolation theorem:

3.1. Theorem. Let Φ be a N-function which satisfies, as well as its conjugate, the Δ_2-condition. Let α and β be the Boyd indices and let $p_\Phi = \beta^{-1}$ and $q_\Phi = \alpha^{-1}$. Let r_1 and r_2 be real numbers such that, $1 \leq r_1 < q_\Phi$ and $p_\Phi < r_2 < +\infty$. Let T be a quasi-linear operator on $L^{r_1}_w(\mathbb{R}^n) + L^{r_2}_w(\mathbb{R}^n)$ such that

$$(1) \qquad |\{ x \in \mathbb{R}^n : |Tg(x)w(x)| > \lambda \}| \leq (c/\lambda^{r_j}) \int_{\mathbb{R}^n} |g(x)w(x)|^{r_j} dx$$

$j = 1, 2$. Then T is well defined on L^Φ_w and there exists a finite constant C such that

$$(2) \qquad \|Tf\|_{\Phi,w} \leq C\|f\|_{\Phi,w}$$

for all $f \in L^\Phi_w$.

Proof. Let $f \in L^\Phi_w$ and let $\lambda > 0$. We decompose f as $f = f_1 + f_2$ where $f_1 = f\chi_{A_\lambda(f)}$, $f_2 = f - f_1$ and $A_\lambda(f) = \{ x \in \mathbb{R}^n : |f(x)w(x)| > \lambda \}$. Since by hypothesis $r_1 < q_\Phi$ and $\Phi \in \Delta_2$ we obtain $t^{r_1} \leq c_1\lambda^{r_1}\, \Phi(t)/\Phi(\lambda)$, for all $t > \lambda$ and for some constant c_1. Therefore,

$$\int_{\mathbb{R}^n} |f_1(x)w(x)|^{r_1}\, dx \leq c_1(\lambda^{r_1}/\Phi(\lambda)) \int_{\mathbb{R}^n} \Phi(|f(x)w(x)|)dx$$

and whence $f_1 \in L^{r_1}_w$.

On the other hand, since $p_\Phi < r_2 < +\infty$ and $\Phi \in \Delta_2$, there exists constant c_2 such that $t^{r_2} \leq c_2\lambda^{r_2}\Phi(t)/\Phi(\lambda)$ for all $t \leq \lambda$. Therefore,

$$\int_{\mathbb{R}^n} |f_2(x)w(x)|^{r_2}\, dx \leq c_2(\lambda^{r_2}/\Phi(\lambda)) \int_{\mathbb{R}^n} \Phi(|f(x)w(x)|)dx$$

and whence $f_2 \in L^{r_2}_w$.

Then we have that L^Φ_w is contained in $L^{r_1}_w + L^{r_2}_w$, i.e., T is well defined on L^Φ_w.

Let us prove 3.1(2). From the quasi-linearity of the operator T, with constant C, and condition 3.1(1) we get

$$\begin{aligned}
(3) \qquad \int_{\mathbb{R}^n} \Phi(|(Tf(x).w(x)|)dx &= \int_0^\infty \varphi(\lambda)|A_\lambda(Tf)|d\lambda \\
&\leq \int_0^\infty \varphi(\lambda)|A_{\lambda/2C}(Tf_1)|d\lambda + \int_0^\infty \varphi(\lambda)|A_{\lambda/2C}(Tf_2)|d\lambda \\
&\leq c(r_1) \int_B |f(x).w(x)|^{r_1}\Big(\int_0^{|f(x).w(x)|} \lambda^{-r_1}\varphi(\lambda)d\lambda\Big)dx \\
&\quad + c(r_2) \int_B |f(x).w(x)|^{r_2}\Big(\int_{|f(x).w(x)|}^\infty \lambda^{-r_2}\varphi(\lambda)d\lambda\Big)dx \\
&= I_1 + I_2
\end{aligned}$$

where φ is the density function of Φ and $B = \{ x \in \mathbb{R}^n : |f(x)| > 0\}$.

Since $\Phi \in \Delta_2$ and $r_1 < q_\Phi$, there exists a constant $c' > 1$ such that

$$s^{r_1-1}\,\varphi(t) \le c'\,\varphi(st) \tag{4}$$

for all $s > 1$ and $t > 0$. Also, since $p_\Phi < r_2 < +\infty$, there exists a constant $c'' > 1$ such that

$$\varphi(st) \le c''\, s^{r_2-1}\,\varphi(t) \tag{5}$$

for all $0 < s < 1$ and $t > 0$.

From 3.1(4) we obtain

$$\int_0^{|f(x).w(x)|} \lambda^{-r_1}\varphi(\lambda)d\lambda \le C'\,\Phi(|f(x).w(x)|)|f(x).w(x)|^{-r_1}$$

and whence,

$$I_1 \le C(r_1)\int_{\mathbb{R}^n} \Phi(|f(x).w(x)|)dx.$$

Analogously, from 3.1(5) we get

$$I_2 \le C(r_2)\int_{\mathbb{R}^n} \Phi(|f(x).w(x)|)dx.$$

Inserting the estimates obtained for I_1 and I_2 in 3.1(3) we obtain

$$\int_{\mathbb{R}^n} \Phi(|Tf(x).w(x)|)dx \le C \int_{\mathbb{R}^n} \Phi(|f(x).w(x)|)dx$$

The norm inequality 3.1(2) follows from the fact that T is positively homogeneous, i.e., $|T(\lambda\cdot u)| = |\lambda||T(u)|$ for all scalar λ, and the equivalence of Orlicz's norm and Luxemburg's norm, namely

$$||f||_{(\Phi,w)} = \inf\{k : \int_{\mathbb{R}^n} \Phi(|f(x)|.w(x)/k)dx \le 1\}.$$

The proof is complete.

Now we are in condition to prove the following multiplier theorem:

3.2. Theorem. Let T_m be the multiplier operator as given in section 1 with m satisfying the Mihlin-Hörmander condition for $1 < s \le 2$ and $n/s < k \le n$. Let Φ be a N–function satisfying the Δ_2-condition jointly with its conjugate function Φ^*. Let p_Φ, q_Φ and p_{Φ^*}, q_{Φ^*} the Boyd's exponents with respect to Φ and Φ^* respectively.

(i) If $n/k < q_\Phi < +\infty$ and $w^{p_\Phi} \in A_{q_\Phi.k/n}$,

or

(ii) if $1 < p_\Phi < (n/k)'$ and $w^{-p_{\Phi^*}} \in A_{q_{\Phi^*}.k/n}$

then T_m is bounded from L^Φ_w into L^Φ_w.

Proof (i).

Step 1: There exists p_0 satisfying $n/k < p_0 < q_\Phi$ such that $w^{p_\Phi} \in A_{p_0.k/n}$. Since $q_\Phi \leq p_\Phi$ it follows that $p_0 < p_\Phi$ and consequently $w^{p_0} \in A_{p_0.k/n}$. From the Kurtz-Wheeden theorem 1.1, we have that T_m is bounded from $L^{p_0}_w$ into $L^{p_0}_w$.

Step 2: From the weight theory we have that if $w^{p_\Phi} \in A_{q_\Phi.k/n}$, there exists $\varepsilon > 0$ such that $w^{p_\Phi+\varepsilon} \in A_{q_\Phi.k/n}$. Let $p_1 = p_\Phi + \varepsilon$. Then $w^{p_1} \in A_{q_\Phi.k/n}$ and consequently $w^{p_1} \in A_{p_1.k/n}$ since $q_\Phi < p_1$. The Kurtz-Wheeden theorem 1.1 implies that T_m is bounded from $L^{p_1}_w$ into $L^{p_1}_w$.

Step 3: The boundedness of T_m on $L^{p_j}_w$, j=0.1. implies condition 3.1(1). Moreover, we have that $1 < p_0 < q_\Phi \leq p_\Phi < p_1 < +\infty$. Then by the interpolation theorem we obtain that T_m is bounded from L^Φ_w into L^Φ_w.

Proof (ii). We have that

$$
\begin{aligned}
(1) \qquad ||T_m f||_{\Phi,w} &= \sup\{ \int_{\mathbb{R}^n} |T_m f(x).g(x)|dx \ : \rho(g.w^{-1}, \Phi^*) \leq 1\} \\
&= \sup\{ \int_{\mathbb{R}^n} |f(x).T^*_m g(x)|dx \ : \rho(g.w^{-1}, \Phi^*) \leq 1\} \\
&\leq \sup\{ ||f||_{\Phi,w}.||T^*_m g||_{\Phi^*,w^{-1}} \ : \rho(g.w^{-1}, \Phi^*) \leq 1\}
\end{aligned}
$$

where T^*_m is the adjoint operator of T_m.

Taking into account that $1/p_\Phi + 1/q_{\Phi^*} = 1$ and $p_\Phi < (n/k)'$, it follows that $q_{\Phi^*} > n/k$. Since $w^{-p_{\Phi^*}} \in A_{q_{\Phi^*}.k/n}$ by hypotesis (i) we obtain

$$||T^*_m g||_{\Phi^*,w^{-1}} \leq c.||g||_{\Phi^*,w^{-1}}.$$

Inserting in 3.2(1) we get the desired result.

Remark 1. Theorem 3.2 generalizes Kurtz-Wheeden theorem. In fact, if we take $\Phi(t) = t^p$, in the above theorem, we have that $p_\Phi = q_\Phi = p$ and $p_{\Phi^*} = q_{\Phi^*} = p'$ with $1/p + 1/p' = 1$.

Remark 2. As consequence of theorem 3.2. the multiplier operator T_m is bounded, e.g., on the weighted Orlicz space $(L^p(1+\log^+ L))_w$ for p and w in the conditions of our theorem. where $f \in (L^p(1+\log^+ L))_w$ if

$$\int_{\mathbb{R}^n} (|f(x)|w(x))^p(1+\log(|f(x)|w(x)))dx < +\infty.$$

Remark 3. A vectorial version of theorem 3.2 can be obtained, considering the weighted Orlicz spaces $L^{\Phi}_w(\mathbb{R}^n, \ell^q)$.

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