SLICE–PRODUCTS IN BIVARIATE APPROXIMATION

João B. Prolla

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ABSTRACT: Let $\{n_k\}$ and $\{m_j\}$ be two sequences of non-negative integers. Let G, H and W be the closed linear span of the sets $\{s^{n_k}\ ;\ k = 1,2,3,...\}$, $\{t^{m_j}\ ;\ j = 1,2,3,...\}$ and $\{s^{n_k} t^{m_j}\ ;\ k, j = 1,2,3,...\}$ in C(S), C(T) and C(S × T), respectively, where S and T are closed and bounded intervals in $\mathbb{R}$. We give conditions under which W is equal to the slice–product G # H. Recall that a function $f \in C(S \times T)$ belongs to G # H, by definition, if and only if, for every pair $(s,t) \in S \times T$, the sections f_t and f_s belong to G and H respectively, where f_t is the mapping $x \in S \longmapsto f(x,t)$ and f_s is the mapping $y \in T \longrightarrow f(s,y)$.

Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
IMECC – UNICAMP
Caixa Postal 6065
13.081 – Campinas – SP
BRASIL

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§1. Introduction

Suppose S and T are closed and bounded intervals in $\mathbb{R}$. Let $C(S)$, $C(T)$ and $C(S \times T)$ be the corresponding spaces of continuous real–valued functions. Let $\{n_k\}$ and $\{m_k\}$ be two sequences (finite or infinite) of non–negative interges, and consider the linear span W of $\{s^{n_k} t^{m_j};\ k, j = 1, 2, 3, \ldots\}$. We are interested in finding its uniform closure $\overline{W}$ in $C(S \times T)$. In general W is not an algebra and we cannot use the Stone–Weierstrass theorem to describe its uniform closure. However, if the set $N = \{n_k;\ k = 1, 2, 3, \ldots\}$ is such that $N + N \subset N$, then the linear span G of $\{s^{n_k};\ k = 1, 2, 3, \ldots\}$ is an algebra, even though W is still not an algebra. Using a vector–valued version of the Stone–Weierstrass Theorem (see Theorem 3 below) we describe the uniform closure of W in $C(S \times T)$ as the set of all functions $f \in C(S \times T)$ such that, for every pair of points $(s, t) \in S \times T$, the section $x \in S \mapsto f(x, t) = f_t(x)$ belongs to the uniform closure of G, and the other section $y \in T \mapsto f(s, y) = f_s(y)$ belongs to the uniform closure of the linear span of $\{t^{m_k};\ k = 1, 2, 3, \ldots\}$ in $C(T)$. (See Theorem 2 below.)

To state an abstract version of the above bivariate polynomial approximation problem let us first introduce some notation. Throughout this paper, whenever X is a compact Hausdorff space, and E is a normed space, $C(X; E)$ denotes the normed space of all continuous functions $f : X \to E$ equipped with the topology of uniform convergence, given by the sup–norm

$$\|f\| = \sup\{\|f(x)\|_E;\ x \in X\}.$$

When $E = \mathbb{R}$, we write $C(X) = C(X; \mathbb{R})$.

Suppose now that S and T are two compact Hausdorff spaces, and that two non–empty subsets $G \subset C(S)$ and $H \subset C(T)$ are given. We are interested in the following bivariate approximation problem:

Q. Which functions $f \in C(S \times T)$ can be uniformly approximated on $S \times T$ by functions of the form

$$w(s,t) = \sum_{i=1}^{n} g_i(s) h_i(t)$$

where $g_i \in G$, $h_i \in H$ $(i = 1, \ldots, n)$, and $n \in \mathbb{N}$ is arbitrary.

Let us denote by $G \otimes H$ the subset of $C(S \times T)$ of such functions $w(s,t)$, and by $G \bar{\otimes} H$ its uniform closure in $C(S \times T)$. Then our problem can be stated as follows.

Q. Characterize those $f \in C(S \times T)$ that belong to $G \bar{\otimes} H$.

In order to answer this question, let us recall the definition of the *slice-product* $G \# H$ introduced by Eifler [4]: a function $f \in C(S \times T)$ belongs to $G \# H$ if, and only if, for every pair (s,t) its sections f_s and f_t belong to H and G respectively, where f_s is the mapping $y \in T \mapsto f(s,y)$, and f_t is the mapping $x \in S \mapsto f(x,t)$. Clearly, the following inclusions hold

$$G \otimes H \subset G \# H \subset \overline{G} \# \overline{H}$$

where $\overline{G}$ (resp. $\overline{H}$) denotes the uniform closure of G (resp. H) in $C(S)$ (resp. $C(T)$). On the other hand, the slice-product of two closed sets is closed. Hence

$$G \bar{\otimes} H \subset \overline{G} \# \overline{H}.$$

Our objective is to find properties of G and H that will make true the equality

$$G \bar{\otimes} H = \overline{G} \# \overline{H}.$$

If *both* G and H are unital subalgebras, then $G \otimes H$ is a unital subalgebra of $C(S \times T)$ and the classical Stone–Weierstrass theorem for subalgebras describes $G \bar{\otimes} H$, and it is easy to prove that $G \bar{\otimes} H = \overline{G} \# \overline{H}$. But when G or H is not assumed to be an algebra, then that theorem cannot be used because now $G \otimes H$ is not a subalgebra of $C(S \times T)$. However a stronger version of that theorem, namely its *vector-valued version for modules* solves the problem if G or H is assumed to be a subalgebra. Notice that in this case $G \otimes H$ is a vector subspace of $C(S \times T)$. We state our solution when G is assumed to be a subalgebra, but of course a similar proof establishes the same result when H is assumed to be a subalgebra (and G is arbitrary).

Theorem 1. *If G is a subalgebra of $C(S)$ and H is a non-empty subset of $C(T)$, then*

$$G \bar{\otimes} H = \overline{G} \# \overline{H}.$$

We will postpone the proof of Theorem 1 and will first establish some corollaries in bivariate polynomial approximation.

Theorem 2. *Let $\{n_k\}$ and $\{m_k\}$ be two sequences of non-negative integers, and assume that $n_1 < n_2 < \cdots < n_k < \cdots$ and $N = \{n_k;\ k = 1,2,3,\ldots\}$ is such that $N + N \subset N$. Let S and T be two closed and bounded intervals in $\mathbb{R}$. The closed linear span of the set $\{s^{n_k}t^{m_j};\ k, j = 1,2,3,\ldots\}$ in $C(S \times T)$ is $V\#W$, where V is the closed linear span of $\{s^{n_k};\ k = 1,2,3,\ldots\}$ in $C(S)$, and W is the closed linear span of $\{t^{m_j};\ j = 1,2,3,\ldots\}$ in $C(T)$.*

Proof. Let G be the linear span of the set $\{s^{n_k};\ k = 1,2,3,\ldots\}$ in $C(S)$, and let H be the linear span of $\{t^{m_k};\ k = 1,2,3,\ldots\}$ in $C(T)$. Then $G \otimes H$ is the linear span of $\{s^{n_k}t^{m_j};\ k, j = 1,2,3,\ldots\}$ in $C(S \times T)$. The semi-group property $N + N \subset N$ implies that G is a subalgebra of $C(S)$. By Theorem 1, $\overline{G \otimes H} = \overline{G}\#\overline{H} = V\#W$. □

The strong version of the Stone–Weierstrass theorem that we will use in the proof of Theorem 1 is the following. (See Theorem 1.26, Prolla [8]).

Theorem 3. *Let X be a compact Hausdorff space and let E be a normed space. Let W be a vector subspace of $C(X;E)$ which is an A-module for some subalgebra $A \subset C(X)$, i.e., $AW \subseteq W$. Then, for each $f \in C(X;E)$, there exists some equivalence class $[x]$ (mod. A) such that*

$$\mathrm{dist}(f;W) = \mathrm{dist}(f\,[x];\ W[x]).$$

Let us explain the notation used above. For each $x \in X$, we denote by $[x]$ the equivalence class of x modulo the following equivalence relation:

$$x \equiv y \ (\mathrm{mod.}\ A) \Leftrightarrow \varphi(x) = \varphi(y) \quad \text{for all} \quad \varphi \in A.$$

Then $[x] = \{y \in X;\ \ \varphi(x) = \varphi(y) \text{ for all } \varphi \in A\}$. Notice that $[x]$ is closed in X, and therefore a compact Hausdorff space. For every function $f \in C(X;E)$, we denote by $f[x]$ the restriction of f to $[x]$. Then $f[x]$ belongs to $C([x];E)$. Finally, if $W \subset C(X;E)$ then

$$W[x] := \{g[x];\ g \in W\} \subset C([x];E)$$

and $\mathrm{dist}(f[x];W[x])$ is measured in the space $C([x];E)$. Hence

$$\mathrm{dist}(f[x];W[x]) = \inf_{g\in W} \|f[x] - g[x]\| =$$

$$= \inf_{g\in W} \sup_{y\in[x]} \|f(y) - g(y)\|_E.$$

When $[x]$ is the singleton $\{x\}$ one identifies $f[x]$ with $f(x)$ and $W[x]$ with the subspace $W(x) = \{g(x);\ g \in W\} \subset E$.

Not only Theorem 3 will be used in the proof of Theorem 1, but it can be used to find dense subspaces of $C[a, b]$ which are not subalgebras.

Theorem 4. *Let $M = \{m_k\}$ be a sequence of positive integers in arithmetic progression, with $m_{k+1} - m_k = r \quad (k = 1, 2, 3, \ldots)$.*

(a) *If r is an odd positive integer, let $T = [c, d] \subset \mathbb{R}$.*

(b) *If r is an even positive integer, let $T = [c, d] \subset \mathbb{R}_+$ or $T = [c, d] \subset \mathbb{R}_-$.*

Let W be the linear span of $\{t^{m_1}, t^{m_2}, \ldots, t^{m_k}, \ldots\}$ in $C(T)$. Then

(1) *if $0 \notin T$, W is dense in $C(T)$;*

(2) *if $0 \in T$, W is dense in the set $\{f \in C(T);\ f(0) = 0\}$, and the linear span of $\{1, t^{m_1}, t^{m_2}, \ldots, t^{m_k}, \ldots\}$ is dense in $C(T)$.*

Proof. Let A be the algebra of all multipliers of W:

$$A = \{\varphi \in C(T);\ \varphi g \in W \quad \text{for all} \quad g \in W\}.$$

Clearly, the function t^r belongs to A. In both cases (a) and (b), the function t^r separates the points of T. Hence, for any $t \in T$, $[t]$ (mod. A) is the singleton $\{t\}$.

Case (1). Then $W(t) = \mathbb{R}$ for all $t \in T$ and by Theorem 3, W is dense in $C(T)$, since $\operatorname{dist}(f(t); W(t)) = 0$, for all $t \in T$.

Case (2). Then $W(t) = \mathbb{R}$ for all $t \neq 0$, and $W(0) = 0$. By Theorem 3, $\operatorname{dist}(f; W) = |f(0)|$ for all $f \in C(T)$. Therefore W is dense in $\{f \in C(T);\ f(0) = 0\}$, and the linear span of $\{1, t^{m_1}, t^{m_2}, \ldots, t^{m_k}, \ldots\}$ is then dense in $C(T)$. □

Theorems 1 and 4 can be combined together to get further bivariate or multivariate results. As an example we have the following.

Corollary 1. *Let $N = (n_k)$ and $M = (m_k)$ be two sets of strictly increasing positive integers such that $N + N \subset N$ and the m_k's are in arithmetic progression, with $m_{k+1} - m_k = r \quad (k = 1, 2, 3, \ldots)$.*

Let $S = [a, b] \subset \mathbb{R}$ and $T = [c, d] \subset \mathbb{R}$ be two closed bounded intervals such that:

(a) *if all the n_k's are even, then $S \subset \mathbb{R}_+$ or $S \subset \mathbb{R}_-$*

(b) *if r is even, then $T \subset \mathbb{R}_+$ or $T \subset \mathbb{R}_-$.*

Let G be the linear span of $\{1, s^{n_1}, s^{n_2}, \dots s^{n_k}, \dots\}$ is $C(S)$, and let H be the linear span of $\{t^{m_1}, t^{m_2}, \dots t^{m_k}, \dots\}$ in $C(T)$. Then

(1) *if $0 \notin T$, then $G \overline{\otimes} H = C(S \times T)$;*

(2) *if $0 \in T$, then $G \overline{\otimes} H = \{f \in C(S \times T);\ f(s,0) = 0$ for all $s \in S\}$, and $G \overline{\otimes} H_1 = C(S \times T)$ where H_1 is the linear span of $\{1, t^{m_1}, t^{m_2}, \dots, t^{m_k}, \dots\}$ in $C(T)$.*

§2. Proof of Theorem 1

Let us identify $C(S \times T)$ with $C(S; C(T))$ in the usual way: with each $f \in C(S \times T)$ associate $\hat{f} \in C(S; C(T))$ defined as follows: for each $s \in S$, $\hat{f}(s) = f_s$, where f_s is the section of f defined by $f_s(t) = f(s,t)$ for all $t \in T$. Now $f_s \in C(T)$, and the mapping $s \mapsto f_s$ is continuous. Moreover the mapping $f \mapsto \hat{f}$ is a linear isometry of $C(S \times T)$ onto $C(S; C(T))$.

Let $W = (G \otimes H)^{\sim}$. Notice that W is a G-module. Let $s \in S$, and let $[s]$ be its equivalence class (mod. G). Notice that every element of G is constant on $[s]$. Hence the same is true of every element of $\overline{G}$, the uniform closure of G.

Claim (1). *If $f \in \overline{G} \# \overline{H}$, then $\hat{f}$ is constant on $[s]$, and its constant value is f_s which belongs to $\overline{H}$.*

Proof. Let $x \in [s]$. Then

$$f_s(t) = f(s,t) = f_t(s) = f_t(x) = f(x,t) = f_x(t)$$

for each $t \in T$, because $f_t \in \overline{G}$. Hence

$$\hat{f}(x) = f_x = f_s = \hat{f}(s)$$

for all $x \in [s]$.

Claim (2). *If all $g \in G$ vanish on $[s]$, then $\hat{f}[s] = 0$, for all $f \in \overline{G} \# \overline{H}$.*

Proof. Assume $\hat{f}[s] \neq 0$. Since $\hat{f}[s] = f_s$, there exists some point $t \in T$ such that $f(s,t) \neq 0$. But $f_t \in \overline{G}$ and therefore f_t vanishes on $[s]$ too, i.e., $f(s,t) = 0$. This contradiction shows that $\tilde{f}[s] = 0$.

Claim (3). *If all $g \in G$ vanish on $[s]$, then $W[s] = \{0\}$. Otherwise $W[s] = H$.*

Proof. The first part of Claim (3) follows from $W \subset (\overline{G}\#\overline{H})^{\sim}$ and Claim (2). Suppose now that $g(s) \neq 0$ for some function $g \in G$. Let $h \in H$ be given. Then $f = v \otimes h$ belongs to $G \otimes H$, where $v = (g(s))^{-1}g$. Now $\tilde{f}[s] = h$. Hence $W[s] = H$.

We can now finish the proof of Theorem 1. Let $f \in \overline{G}\#\overline{H}$. By Theorem 3, applied to the G-module $W = (G \otimes H)^{\sim}$, there is some equivalence class $[s]$ (mod. G) such that

$$\mathrm{dist}(\tilde{f}; W) = \mathrm{dist}(\tilde{f}[s]; W[s]).$$

Case 1. *All $g \in G$ vanish on $[s]$.*

By Claim (3), $W[s] = \{0\}$, and by Claim (2), $\tilde{f}[s] = 0$. Hence $\mathrm{dist}(\tilde{f}[s]; W[s]) = 0$.

Case 2. *For some $g \in G$, $g(s) \neq 0$.*

By Claim (3), $W[s] = H$. On the other hand $\tilde{f}[s] = f$, belongs to $\overline{H}$. Hence $\mathrm{dist}(\tilde{f}[s]; W[s]) = 0$.

In both cases we conclude that $\mathrm{dist}(\tilde{f}, W) = 0$, and so $\tilde{f}$ belongs to the closure of $W = (G \otimes H)^{\sim}$. Since the mapping $f \mapsto \tilde{f}$ is an isometric isomorphism, f belongs to the closure of $G \otimes H$. Hence $\overline{G}\#\overline{H} \subset G \overline{\otimes} H$. □

§3. Grothendieck's Approximation Property

The following result shows that to find a closed vector subspace $G \subset C(S)$ such that $G \overline{\otimes} H$ is properly contained in $G\#H$ for some closed vector subspace $H \subset C(T)$ is equivalent to find such a G without Grothendieck's approximation property.

Theorem 5. *For a closed vector subspace $G \subset C(S)$, S compact, the following are equivalent:*

(a) *G has the approximation property;*

(b) *for every Banach space E, the space $G \otimes E$ is dense in*

$$\{f \in C(S; E);\ \varphi \circ f \in G, \quad \text{for all} \quad \varphi \in E^*\};$$

(c) *for every compact Hausdorff space T and every closed vector subspace $H \subset C(T)$, $(G \overline{\otimes} H)^{\sim} = \{f \in C(S; H);\ \varphi \circ f \in G,\ \text{for all}\ \varphi \in H^*\};$*

(d) *for every compact Hausdorff space T and every closed vector subspace $H \subset C(T)$, $G \overline{\otimes} H = G \# H$.*

For a proof see Grothendieck [5], Bierstedt [1] and Prolla [8].

Corollary 2. *Every closed subalgebra $G \subset C(S)$, S a compact Hausdorff space, has the approximation property.*

The proof of Corollary 2 depends on Theorem 5 whose proof uses several tools from Functional Analysis. Hence an elementary direct proof that any closed subalgebra $G \subset C(S)$ has the approximation property is desirable. Let us recall the definition of the *metric approximation property*. Let $\lambda \geq 1$. We say that a Banach space E has the *λ-bounded approximation property* (λ- b.a.p. for short) if, for every totally bounded subset $B \subset E$ and every $\varepsilon > 0$, there is a continuous linear operator $T : E \to E$ of finite rank such that $\|x - Tx\| < \varepsilon$, for all $x \in B$, and $\|T\| \leq \lambda$. We say that E has the *metric approximation property* if it has the λ- b.a.p. for $\lambda = 1$.

Lemma 1. *Let $A \subset C(S)$ be a closed subalgebra containing the constants, and let $x \in S$ be given. If $N(x)$ is an open neighborhood of $[x]$ (mod. A), there exists an open neighborhood $W(x)$ of $[x]$, contained in $N(x)$, and such that, for each $0 < \delta < 1$, there is $\varphi \in A$ such that*

(1) $0 \leq \varphi(s) \leq 1$, *for all $s \in S$;*

(2) $\varphi(t) < \delta$, *for all $t \notin N(x)$;*

(3) $\varphi(t) > 1 - \delta$, *for all $t \in W(x)$.*

Proof. The set $M = \{\varphi \in A;\ 0 \leq \varphi \leq 1\}$ is closed and has property V, i.e., $1 - \varphi$ and $\varphi\psi$ belong to M, whenever φ and ψ belong to M. The result follows from Lemma 1, Prolla [9]. □

Lemma 2. *Let A be as in Lemma 1. For each $x \in S$, let there be given an open neighborhood $N(x)$ of $[x]$ (mod. A). There exists a finite set $\{x_1, \ldots, x_m\} \subset S$ such that, given $0 < \delta < 1$, there are $\varphi_1, \ldots, \varphi_m \in A$ such that*

(1) $0 \leq \varphi_i \leq 1$, $i = 1, \ldots, m$;

(2) $\sum_{i=1}^{m} \varphi_i(x) = 1$, *for all $x \in S$;*

(3) $0 \leq \varphi_i(t) < \delta$, *if* $t \notin N(x_i)$, $i = 1, \ldots, m$.

Proof. Select $x_1 \in S$ arbitrarily. Let $K = S \setminus N(x_1)$. For each $x \in K$, select an open neighborhood $W(x)$ by Lemma 1. By compactness of K, there exists a finite set $x_2, \ldots, x_m$ in K such that $K \subset W(x_2) \cup \cdots \cup W(x_m)$. Let $0 < \delta < 1$ be given. By Lemma 1, there are $\psi_2, \ldots, \psi_m$ in A such that $0 \leq \psi_i \leq 1$ and $\psi_i(t) < \delta$ for all $t \notin N(x_i)$, and $\psi_i(t) > 1 - \delta$ for all $t \in W(x_i)$, $i = 2, \ldots, m$.

Define $\varphi_2 = \psi_2$, $\varphi_3 = (1 - \psi_2)\psi_3, \ldots, \varphi_m = (1 - \psi_2) \cdots (1 - \psi_{m-1})\psi_m$. Clearly, $\varphi_i \in A$ and $0 \leq \varphi_i \leq 1$ for all $i = 2, \ldots, m$. Since

$$\varphi_2 + \cdots + \varphi_m = 1 - (1 - \psi_2)(1 - \psi_3) \cdots (1 - \psi_m)$$

can be easily verified by induction, let us define $\varphi_1 = (1 - \psi_2)(1 - \psi_3) \cdots (1 - \psi_m)$. Then $\varphi_1 \in A$, $0 \leq \varphi_1 \leq 1$ and $\varphi_1 + \varphi_2 + \cdots + \varphi_m = 1$. Hence (1) and (2) are verified. To prove (3), note that for each $i = 2, \ldots, m$ we have $\varphi_i(t) \leq \psi_i(t) < \delta$ for all $t \notin N(x_i)$. On the other hand, if $t \notin N(x_1)$, then $t \in K$ and for some index $j = 2, \ldots, m$, we have $t \in W(x_j)$. Hence $\psi_j(t) > 1 - \delta$ and so $1 - \psi_j(t) < \delta$. Thus

$$\varphi_1(t) = (1 - \psi_j(t)) \prod_{i \neq j} (1 - \psi_i(t)) < \delta. \qquad \square$$

Theorem 6. *Let S be a compact Hausdorff space and let $A \subset C(S)$ be a closed subalgebra. Then A has the 2-bounded approximation property. If A contains the constants, it has the metric approximation property.*

Proof. Suppose A contains the constants. Let $\varepsilon > 0$ and $B \subset A$ a totally bounded subset be given. There is a finite set $F \subset B$ such that, given $f \in B$ there is some $g \in F$ with $|f(x) - g(x)| < \varepsilon/3$ for all $x \in S$. For each $x \in S$ define

$$N(x) = \{t \in S;\ |g(t) - g(x)| < \varepsilon/6, \quad \text{for all} \quad g \in F\}.$$

Since F is finite, $N(x)$ is open. Notice that if $t \in [x]$ (mod. A), then $g(t) = g(x)$. Hence $N(x)$ is an open neighborhood of $[x]$ (mod. A). There exists a finite set $x_1, \ldots, x_m \in S$ with the property stated in Lemma 2. Let $M = \max\{||g||;\ g \in F\}$ and choose $0 < \delta < 1$ so small that $12\, m\, M\, \delta < \varepsilon$. For this δ there are $\varphi_1, \ldots, \varphi_m \in A$ such that (1) - (3) of Lemma 2 are true. Define a linear operator $T : A \to A$ by setting

$$(*) \qquad (Tf)(x) = \sum_{i=1}^{m} \varphi_i(x) f(x_i)$$

for all $f \in A$, $x \in S$. Clearly, T is a finite rank operator, and by (1) and (2), $||T|| \leq 1$.

Let now $f \in B$. There exists some $g \in F$ such that $\|f - g\| < \varepsilon/3$. Hence, for any $x \in S$,

$$|f(x) - (Tf)(x)| = |\sum_{i=1}^{m} \varphi_i(x)(f(x) - f(x_i))|$$
$$\leq \sum_{i=1}^{m} \varphi_i(x)|f(x) - f(x_i)|$$
$$\leq \sum_{i=1}^{m} \varphi_i(x) \left[|f(x) - g(x)| + |g(x) - g(x_i)| + |g(x_i) - f(x_i)|\right]$$
$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{m} \varphi_i(x)|g(x) - g(x_i)|.$$

Let $I(x) = \{1 \leq i \leq m;\ x \in N(x_i)\}$ and $J(x) = \{1 \leq i \leq m;\ x \notin N(x_i)\}$. For $i \in I(x)$, we have $|g(x) - g(x_i)| < \varepsilon/6$, and therefore

(a) $$\textstyle\sum_{i\in I(x)} \varphi_i(x)|g(x) - g(x_i)| < \frac{\varepsilon}{6} \sum_{i\in I(x)} \varphi_i(x) \leq \frac{\varepsilon}{6}.$$

For $i \in J(x)$, we have $\varphi_i(x) < \delta$, and therefore

(b) $$\textstyle\sum_{i\in J(x)} \varphi_i(x)|g(x) - g(x_i)| < \delta \sum_{i\in J(x)} |g(x) - g(x_i)|$$
$$\leq \delta m 2 M < \varepsilon/6.$$

From (a) and (b), $\sum_{i=1}^{m} \varphi_i(x)|g(x) - g(x_i)| < \varepsilon/3$, and so $\|f - Tf\| < \varepsilon$.

Suppose now that A does not contains the non-zero constants. By the Stone-Weierstrass Theorem this is equivalent to say that $N \neq \emptyset$, where $N = \{x \in S;\ \varphi(x) = 0 \text{ for all } \varphi \in A\}$. Let $A_e = A \oplus \mathbb{R}$. Hence A_e is a closed subalgebra containing the constants. Let $\varepsilon > 0$ and $B \subset A$ a totally bounded subset be given. Clearly $B \subset A_e$. Apply the first part to $\varepsilon/2$ and B. Let T be the operator defined in (*), which maps A_e into A_e. Each φ_i is of the form $\varphi_i = \psi_i + \lambda_i$, where $\psi_i \in A$ and $0 \leq \lambda_i \leq 1$. Define $U : A \to A$ by setting

$$(Uf)(x) = \sum_{i=1}^{m} \psi_i(x) f(x_i)$$

for all $f \in A$ and $x \in S$. Since $\sum_{i=1}^{m} \varphi_i(x) = 1$ is true for all $x \in S$, choosing $x \in N$ we see that $\sum_{i=1}^{m} \lambda_i = 1$. Hence

$$|(Uf)(x)| \leq |(Tf)(x)| + |\sum_{i=1}^{m} \lambda_i f(x_i)| \leq \|Tf\| + \|f\| \sum_{i=1}^{m} \lambda_i \leq 2\|f\|.$$

Hence $||U|| \leq 2$.

Let now $f \in B$. By the definition of T we have $|f(x) - (Tf)(x)| < \varepsilon/2$ for all $x \in S$. Choosing $x \in N$ we see that $|\sum_{i=1}^{m} \lambda_i f(x_i)| < \varepsilon/2$. Hence $||f - Uf|| < \varepsilon$ for all $f \in B$. □

Remark. While for any *self-adjoint* closed subalgebra $G \subset C(S;\mathbb{C})$, Theorem 6 remains true, the situation for non-selfadjoint subalgebras is different. H. Milne [7] showed that there exists a *uniform algebra* $G \subset C(S;\mathbb{C})$ without the approximation property, where S is a compact subset of $\mathbb{C}^n$, $n \geq 1$.

§4. Semi-Algebras

To present further cases in which $G \overline{\otimes} H = \overline{G} \# \overline{H}$, let us recall the following generalized Bernstein approximation theorem. (For a proof see Prolla [9], Theorem 1.)

Theorem 7. *Let X and E be as in Theorem 3. Let W be a non-empty subset of $C(X;E)$ such that $W + W \subset W$, and let $A \subset C(X;[0,1])$ be a subset with property V and such that $AW \subset W$. Then, for each $f \in C(X;E)$, there is some equivalence class $[x]$ (mod. A) such that*

$$\operatorname{dist}(f;W) = \operatorname{dist}(f|[x]; W|[x]).$$

The definition of property V is as follows: a subset $A \subset C(X;[0,1])$ has *property* V if, and only if, for every φ and ψ in A, the functions $1-\varphi$ and $\varphi\psi$ belong to A.

Remark. Theorem 7 can be used to prove density of subsets $W \subset C[a,b]$ which are not vector subspaces. Take, for example, W to be the set of all polynomials with *rational* coefficients. Let $A = \{g \in W;\ 0 \leq g \leq 1\}$. Then A has property V and $AW \subset W$. Moreover $W + W \subset W$. Now it is easy to find $g \in A$ of the form $x \mapsto px + q$, with $p, q \in \mathbb{Q}$, and therefore each equivalence class $[x]$ (mod. A) is the singleton $\{x\}$. On the other hand, $\overline{W(x)} = \mathbb{R}$, for each $x \in [a,b]$. Hence $\operatorname{dist}(f;W) = 0$ for all $f \in C[a,b]$.

Let us recall the definition of a *semi-algebra* (Bonsall [2]): a subset $G \subset C(S)$ is a semi-algebra if $G + G \subset G$, $GG \subset G$ and $\lambda G \subset G$, for all $\lambda \geq 0$. Clearly, if $A \subset C(S)$ is a subalgebra, then the set A^+ is a semi-algebra where $A^+ = \{g \in A;\ g \geq 0\}$.

We shall say that a semi-algebra G is of *type* V, if $\{g \in G;\ 0 \leq g \leq 1\}$ has property V. For any non-empty subset $X \subset S$, the set G of all $g \in C(S)$ such that

$g(x) \geq 0$, for each $x \in X$, is an example of a closed semi-algebra of type V.

Theorem 8. *If $G \subset C^+(S)$ is a semi-algebra of type V, and H is a non-empty subset of $C(T)$, then*

$$G \overline{\otimes} H = \overline{G} \# \overline{H}.$$

Proof. As in the proof of Theorem 1, let $W = (G \otimes H)^{\sim} \subset C(S; C(T))$. Since G is a semi-algebra, W is a convex cone, and therefore $W + W \subset W$. Let now $A = \{g \in G;\ 0 \leq g \leq 1\}$. Then $AA \subset A$ and $AG \subset G$. Therefore $AW \subset W$. Since G is of type V, the set A has property V. Let $f \in \overline{G} \# \overline{H}$. By Theorem 7, there is some equivalence class $[s]$ (mod. A) such that $\mathrm{dist}(\tilde{f}; W) = \mathrm{dist}(\tilde{f}|[s];\ W|[s])$. The proof that $\mathrm{dist}(\tilde{f}; W) = 0$ proceeds now exactly as in the proof of Theorem 1, if Claims (1) - (3) remain true when $[s]$ is the equivalence class of s (mod. A).

Claim (1). Its proof depends on the fact that every element of $\overline{G}$ is constant on $[s]$. To prove this, it suffices to show that every element of G is constant on $[s]$. But given $g \in G$, with $g \neq 0$, then $h = ||g||^{-1}g$ belongs to A, because $G \subset C^+(S)$ and $\lambda G \subset G$ for all $\lambda \geq 0$. Hence h is constant on $[s]$, and therefore the same is true of $g = ||g||h$.

Claim (2). No change needed in its proof.

Claim (3). It suffices to notice that $v = (g(s))^{-1}g$ belongs to G, because $g(s) > 0$. □

If A is a *unital* subalgebra of $C(S)$ then $G = A^+ = \{g \in A;\ g \geq 0\}$ is a semi-algebra of type V. When B is a subalgebra of $C(S)$, and $G = B^+$, then $G \overline{\otimes} H = \overline{G} \# \overline{H}$ is still true. The proof in this case proceeds as follows. Consider $W = (G \otimes H)^{\sim} \subset C(S; C(T))$. Let B_1 be the unital subalgebra generated by B and 1, and let

$$A = \{g \in B_1;\ 0 \leq g \leq 1\}.$$

Then $AW \subset W$, and A has property V. As in the proof of Theorem 8 the only thing to check is Claim (1), and again it suffices to show that every element of G is constant on $[s]$ (mod. A). Take $g \in G$, $g \neq 0$. Then $h = ||g||^{-1}g$ is such that $0 \leq h \leq 1$ and $h \in G \subset B \subset B_1$. Hence $h \in A$ and h is constant on $[s]$ (mod. A). Therefore the same is true for $g = ||g||h$.

Let us give an example in which the equality $G \overline{\otimes} H = \overline{G} \# \overline{H}$ is true, but G is neither a subalgebra nor a semi-algebra. Take $S = [0, 1]$, and T an arbitrary compact Hausdorff space. Let G be the set of all polynomials in the variable s with integral coefficients. Then G is neither a subalgebra nor a semi-algebra, but $G \pm G \subset G$ and $GG \subset G$. Let $H \subset C(T)$ be any non-empty subset. Define

$$A = \{\varphi \in C(S; [0, 1]); \ \ \varphi f + (1 - \varphi) g \in G, \quad \text{for all} \quad f, g \in G\}.$$

Then $AG \subset G$ and A has property V. Moreover, for any $s \in S$, $[s]$ (mod. A) is the singleton set $\{s\}$, because A contains the mapping $\varphi(x) = x$, for all $x \in S$. Now identify $C(S \times T)$ with $C(S; C(T))$ and let $W = (G \otimes \tilde{H})^{\sim}$. Then $W + W \subset W$ and $AW \subset W$. Let now $f \in \overline{G} \# \overline{H}$. By Theorem 7, there is some $s \in S$ such that

$$\operatorname{dist}(\tilde{f}; W) = \operatorname{dist}(\tilde{f}(s); W(s)).$$

Now $\tilde{f}(s) = f_s \in \overline{H}$. Hence, for any $\varepsilon > 0$ then is some $h \in H$ such that $\|f_s - h\|_{C(T)} < \varepsilon$. Let $u = 1 \otimes h$. Then $u \in G \otimes H$, and $\tilde{u} \in W$. Now $\tilde{u}(x)$ is the map $t \to u(x, t) = h(t)$, for each $x \in S$. In particular, $\tilde{u}(s) = h$. Hence $\|f_s - \tilde{u}(s)\| < \varepsilon$, and $\tilde{f}(s) \in \overline{W(s)}$. This shows that $\operatorname{dist}(\tilde{f}(s), W(s)) = 0$. Hence $\tilde{f}$ belongs to the closure of W, and f belongs to $G \overline{\otimes} H$.

§5. Grothendieck Spaces

Let V be a vector subspace of $C(X; E)$. The set G_V is by definition the set of all pairs (x, y) such that either

(1) $f(x) = f(y) = 0$ for all $f \in V$; or

(2) there exists $t \in \mathbb{R}$, $t \neq 0$, such that $f(x) = t f(y)$ for all $f \in V$ and $g(x) \neq 0$ for some $g \in V$.

The set G_V is an equivalence relation for X. Define a map $\gamma_V : G_V \to \mathbb{R}$ as follows: $\gamma_V(x, y) = 0$ if (1) is true, and $\gamma_V(x, y) = t$ if (2) is true. The subsets KS_V and WS_V of all pairs $(x, y) \in G_V$ such that $\gamma_V(x, y) \geq 0$ and $\gamma_V(x, y) \in \{0, 1\}$, respectively, are likewise equivalence relations for X. (The letters G, KS and WS stand for Grothendieck, Kakutani–Stone and Weierstrass–Stone, respectively.) The vector subspace

$$\Delta(V) = \{f \in C(X; E); \ f(x) = \gamma_V(x, y) f(y), \quad \text{for all} \quad (x, y) \in \Delta_V\}$$

where $\Delta \in \{G, KS, WS\}$, is called the *Δ-hull* of V. Notice that $\Delta(V)$ is a closed subspace of $C(X; E)$ containig V, and V is called a *Δ-subspace*, if $\Delta(V) = \overline{V}$. (See Blatter [2], for the study of these spaces in Approximation Theory.)

Let $V_{\mathbb{R}}$ denote linear span of the set $\{\varphi \circ f; \ \ f \in V, \ \ \varphi \in E^*\}$ in $C(X)$. The equivalence relations Δ_V and Δ_{V_E} are the same and the corresponding γ_V and γ_{V_E} coincide too. In particular, if $V \subset C(X)$ and $L = V \otimes E \subset C(X;E)$, then $L_E = V$ and $\Delta_V = \Delta_L$.

If $\Delta \in \{G, KS, WS\}$, we denote by $A(\Delta_V)$ the subalgebra of $C(X)$ of all functions $\varphi \in C(X)$ that are constant on the equivalence classes modulo Δ_V, where $V \in C(X;E)$ is given. When no confusion is feared we write simply $A(\Delta) = A(\Delta_V)$.

Theorem 9. *Let V be a Δ-subspace of $C(S)$ such that each equivalence class $[x]$ (mod. $A(\Delta)$) is contained in $[x]$ (mod. Δ_V). Then*

$$V \overline{\otimes} H = \overline{V} \# \overline{H}$$

for all non-empty subsets $H \subset C(T)$.

Proof. Identify $C(S \times T)$ with $C(S; C(T))$ as in the proof of Theorem 1. Let $L = (V \overline{\otimes} H)^{\sim}$. Note the Δ_V and Δ_L are the same, and L is an $A(\Delta)$-module. To simplify notation, for each $x \in S$, let $[x]_V = [x]$ (mod. Δ_V), $[x]_L = [x]$ (mod. Δ_L), and $[x] = [x]_A = [x]$ (mod. $A(\Delta)$).

Claim (1). *If $f \in \overline{V} \# \overline{H}$, then $\tilde{f} \in \Delta(L)$.*

Proof. Take $s \in [x]_V$. Then

$$\tilde{f}(s)(t) = f(s,t) = f_t(s) = \gamma_V(s,x) f_t(x) = \gamma_V(s,x) f(x,t) = \gamma_V(s,x)\tilde{f}(x)(t)$$

for all $t \in T$, because $f_t \in \overline{V} = \Delta(V)$. Since $[x]_V = [x]_L$ and $\gamma_V = \gamma_L$, we see that $\tilde{f}(s) = \gamma_L(s,x)\tilde{f}(x)$ for all $s \in [x]_L$, i.e., $\tilde{f} \in \Delta(L)$.

Let $f \in \overline{V} \# \overline{H}$. By Theorem 3, there is some equivalence class $[x]$ (mod. $A(\Delta)$) such that

$$\text{dist}(\tilde{f}, L) = \text{dist}(\tilde{f}[x]; L[x]).$$

By hypothesis, $[x] \subset [x]_V = [x]_L$.

Case 1. $\tilde{f}(x) = 0$.

Let $t \in [x]$. Since, by Claim 1, $\tilde{f} \in \Delta(L)$, then $\tilde{f}(t) = \gamma_L(t,x)\tilde{f}(x) = 0$, because $[x] \subset [x]_L$. Hence $\tilde{f}[x] = 0$ and so $\tilde{f}[x]$ belongs to $L[x]$.

Case 2. $\tilde{f}(x) \neq 0$.

We claim that $h(x) \neq 0$ for some $h \in V$. If not, $\gamma_V(x,x) = 0$ and so

$$\tilde{f}(x) = \gamma_L(x,x)\tilde{f}(x) = \gamma_V(x,x)\tilde{f}(x) = 0.$$

Hence $h(x) = 1$ for some $h \in V$. Let $g = h \otimes f(x)$.

Then $g \in V \otimes \overline{H} \subset \overline{V} \otimes \overline{H} \subset \overline{V} \overline{\otimes} \overline{H}$. Now $\overline{V} \otimes \overline{H}$ and $V \otimes H$ have the same closure in $C(S \times T)$ and so $\tilde{g} \in (V \overline{\otimes} H)^{\sim} = L$. For every $t \in [x]$ we have

$$\tilde{g}(t) = (h \otimes \tilde{f}(x))^{\sim}(t) = h(t)\tilde{f}(x) = \gamma_V(t,x)h(x)\tilde{f}(x) = \gamma_V(t,x)\tilde{f}(x) = \tilde{f}(t)$$

where the last equality follows from Claim (1) and $[x] \subset [x]_L$. Hence $\tilde{g}[x] = \tilde{f}[x]$ with $\tilde{g} \in L$, and therefore $\tilde{f}[x] \in L[x]$. In both cases, $\text{dist}(\tilde{f}[x];\ L[x]) = 0$. Hence $\text{dist}(\tilde{f}, L) = 0$ and so $f \in V \overline{\otimes} H$. □

Let X be a topological space and R an equivalence relation for X, and let $Y = X/R$ be the quotient topological space and $P : X \to Y$ the quotient mapping. The following are equivalent:

(a) P is a closed mapping,
(b) for every $x \in X$, and every open set $A \supset [x]$ (mod. R) there is an open set A' such that $A \supset A' \supset [x]$ (mod. R) and $A' = \cup\{[t] \text{ (mod. } R);\ t \in A'\}$.

When (b) is satisfied one says that R is upper semicontinuous.

Lemma 3. *Let S be a compact Hausdorff space. Let V be a Δ-subspace of $C(S)$ such that Δ_V is an upper semicontinuous equivalence relation for S. Let $A(\Delta)$ be the subalgebra of all $\varphi \in C(S)$ that are constant on each equivalence class $[x]$ (mod. Δ_V). Then each equivalence class $[x]$ (mod. $A(\Delta)$) is contained in $[x]$ (mod. Δ_V).*

Proof. Let Y be the quotient space S/Δ_V and let P be the quotient mapping. Let a and b be two distinct points of Y. Then $P^{-1}(a) = [s]$ and $P^{-1}(b) = [t]$ for some pair $s, t \in S$. Since S is Hausdorff, $\{s\}$ and $\{t\}$ are closed, and since P is a closed mapping, $[s]$ and $[t]$ are closed subsets of S. Now S is a normal space, hence there exists open sets A and B such that $A \cap B = \phi$, $[s] \subset A$ and $[t] \subset B$. Since Δ_V is upper semicontinuous, there are open saturated subsets A' and B' such that $[s] \subset A' \subset A$ and $[t] \subset B' \subset B$. Then $P(A')$ and $P(B')$ are two disjoint open sets in Y with $a \in P(A')$ and $b \in P(B')$. Thus Y is a Hausdorff space. Notice that as a continuous image (under P) of a compact space S, the space Y is compact. Hence Y is a compact Hausdorff space, and so $C(Y)$ separates the points of Y.

Consider now an equivalence class $[x]$ (mod. $A(\Delta)$). If it is not contained in $[x]$ (mod. Δ_V), then for some pair $s, t \in [x]$ (mod. $A(\Delta)$), we have $a \neq b$, if $a = P(s)$ and $b = P(t)$. Hence there exists $g \in C(Y)$, $0 \leq g \leq 1$, with $g(a) = 0$ and $g(b) = 1$. Let $f = g \circ P$. Then $f \in C(S)$, $0 \leq f \leq 1$, and $f \in A(\Delta)$. Moreover $f(t) = g(P(t)) = g(b) = 1$ and $f(s) = g(P(s)) = g(a) = 0$. Hence $s \not\equiv t$ (mod.

$A(\Delta)$), a contradiction. □

Theorem 10. *Let S be a compact Hausdorff space. Let V be a Δ-subspace of $C(S)$ such that Δ_V is an upper semicontinuous equivalence relation for S. Then $V \overline{\otimes} H = \overline{V} \# \overline{H}$ for all non-empty subsets $H \subset C(T)$.*

Proof. Apply Lemma 3 and Theorem 9.

Remarks (1). The importance in Approximation Theory, of those Δ-subspaces such that Δ_V is an upper semicontinuous equivalence relation for S was established by Blatter [2]. (See in particular Lemma 3.10 and Theorem 3.12 of [2].)

(2). When $\Delta = WS$, then Δ_V is the equivalence relation mod. V. Hence $V \subset A(\Delta)$ and therefore the hypothesis of Theorem 9 is verified in this case. Notice that, when G is a subalgebra of $C(S)$, then $\overline{G} = \Delta(G)$ by the Stone–Weierstrass theorem. Hence G is a WS-subspace and Theorem 9 generalizes Theorem 1.

(3). By a result of Lindenstrauss [6], the dual V^* of any closed Δ-subspace V is an abstract L_1-space, and therefore V^* has the metric approximation property. By a result of Grothendieck [5] V itself has then the metric approximation property. By the equivalence (a) $\iff$ (d) of Theorem 5 it follows that $V \overline{\otimes} H = V \# H$ is true for *all* closed Δ-subspaces, when H is a closed vector subspace of $C(T)$. However, the interest of Theorem 9 remains because it shows that under some mild assumption one can get the equality $V \overline{\otimes} H = \overline{V} \# \overline{H}$ using only tools from Approximation Theory that do not rely on deep facts from Functional Analysis. In this light, it would be interesting to find a direct elementary proof of the fact each closed Δ-subspace has the metric approximation property, as Theorem 6 did for the case of WS-subspaces.

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João B. Prolla
Departamento de Matemática
IMECC - UNICAMP
13.081 - Campinas, SP, Brazil

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