PROPERTIES FOR SPHERICAL HARMONIC POLYNOMIALS

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Abstract: Properties for spherical harmonic polynomials in $\mathbb{R}_3$ and $\mathbb{R}_4$ are derived: raising and lowering operators and recurrence relations between contiguous polynomials are obtained.

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Properties For Spherical Harmonic Polynomials

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Properties for spherical harmonic polynomials in $\mathbb{R}_3$ and $\mathbb{R}_4$ are derived: raising and lowering operators and recurrence relations between contiguous polynomials are obtained.

1. Introduction

In this paper we present a systematic derivation of properties for the spherical harmonic polynomials in $\mathbb{R}_3$ and in $\mathbb{R}_4$.

The principal polynomials in $\mathbb{R}_3$ are the spherical Gegenbauer polynomials and the well-known spherical harmonic polynomials, basis of the $\mathbb{R}_3$-angular momentum operator.

In $\mathbb{R}_4$ we have the hyperspherical harmonic polynomials, solutions of the angular part of the four-dimensional Laplace equation[1]. These polynomials are of special interest for applications to non-relativistic physical problems concerned with the Coulomb field, the well-know 0(4) symmetry of the hydrogen atom[2]. The study of representations for the 0(4) rotation group[3]; symmetry of scattering amplitudes[4] and relativistic quark models[5].

We present a systematic derivation of the properties of the spherical harmonic polynomials by the classical method of special functions analysis. We derive raising and lowering operators and recurrence relations between contiguous polynomials.

In Section 2 we derive the properties of the spherical Gegenbauer polynomials. In Section 3 we derive the properties of the spherical harmonic polynomials raising and lowering operators on the angular momentum quantum number and collected some peoperties to be used in Section 4 in order to determine properties for the $I\!R_4$-hyperspherical harmonic polynomials.

2. Spherical Gegenbeuer Polynomials

The spherical Gegenbauer polynomials are a complete and orthonormal set of $(n+1)$–polynomials, solutions of the following differential equation

$$\left\{\frac{1}{\sin^2\theta_1}\frac{d}{d\theta_1}\left(\sin^2\theta_1\frac{d}{d\theta_1}\right)+n(n+2)-\frac{\ell(\ell+1)}{\sin^2\theta_1}\right\}C_n^\ell(\cos\theta_1)=0 \qquad (2.1)$$

with $0\leq\theta_1\leq\pi,\;\; n=0,1,2\cdots,\;\; 0\leq\ell\leq n$ and

$$C_n^\ell(\cos\theta_1)=N(n,\,\ell)\sin^\ell\theta_1 C_{n-\ell}^{\ell+1}(\cos\theta_1) \qquad (2.2)$$

where $C_{n-\ell}^{\ell+1}(\cos\theta_1)$ are the usual Gegenbauer polynomials[1] and $N(n,\ell)$ is the following normalization factor

$$N(n,\ell)=\ell!\left\{\frac{2^{2\ell+1}}{\pi}(n+1)\frac{\Gamma(n-\ell+1)}{\Gamma(n+\ell+2)}\right\}^{1/2} \qquad (2.3)$$

In order to obtain recurrence relations and raising and lowering operators for the $C_n^\ell(\cos\theta_1)$ polynomials we use the following relations for the usual Gegenbauer polynomials[1]

$$(n+1)C_{n+1}^{\lambda}(x) + (n+2\lambda-1)C_{n-1}^{\lambda}(x) = 2(n+\lambda)\, x\, C_n^{\ell}(x) \tag{2.4}$$

$$\frac{d}{dx}C_n^{\lambda}(x) = 2\lambda C_{n-1}^{\lambda+1}(x) \tag{2.5}$$

$$(1-x^2)\frac{d}{dx}C_n^{\lambda}(x) = (n+2\lambda-1)C_{n-1}^{\lambda}(x) - n\, x\, C_n^{\lambda}(x) \tag{2.6}$$

$$= (n+2\lambda)\, x\, C_n^{\lambda}(x) - (n+1)C_{n+1}^{\lambda}(x).$$

From Eq. (2.4) we have the following recurrence relation between contiguous $\mathcal{C}_n^{\ell}(\cos\theta_1)$ on the n-quantum number

$$2(n+1)\cos\theta_1 \mathcal{C}_n^{\ell}(\cos\theta_1) = A_1(n,\ell)\mathcal{C}_{n+1}^{\ell}(\cos\theta_1) + B_1(n,\ell)\mathcal{C}_{n-1}^{\ell}(\cos\theta_1) \tag{2.7}$$

where

$$A_1(n,\ell) = \left\{\frac{n+1}{n+2}\,(n+\ell+2)(n-\ell+1)\right\}^{1/2}$$

$$B_1(n,\ell) = \left\{\frac{n+1}{n}\,(n-\ell)(n+\ell+1)\right\}^{1/2} \tag{2.8}$$

Raising and lowering operators on the n-quantum number can be derived from Eqs. (2.6), and we have

$$\mathcal{F}_+^n = \sin\theta_1\,\frac{d}{d\theta_1} + (n+2)\cos\theta_1 \tag{2.9}$$

$$\mathcal{F}_-^n = -\sin\theta_1\,\frac{d}{d\theta_1} + n\,\cos\theta_1 \tag{2.10}$$

where

$$\mathcal{F}_+^n C_n^\ell(\cos\theta_1) = A_1(n,\ell) C_{n+1}^\ell(\cos\theta_1) \tag{2.11}$$

$$\mathcal{F}_-^n C_n^\ell(\cos\theta_1) = B_1(n,\ell) C_{n-1}^\ell(\cos\theta_1) \tag{2.12}$$

equation (2.12) olds only for $n \geq 1$.

The commutation relation between $\mathcal{F}_+^n$ and $\mathcal{F}_-^n$ is given by

$$[\mathcal{F}_+^n, \mathcal{F}_-^n] = -2(n+1)\sin^2\theta_1 \tag{2.13}$$

These operators relate the set of polynomials $\{C_n^\ell(\cos\theta_1)\}$ with $\{C_{n\pm 1}^\ell(\cos\theta_1)\}$. The $\mathcal{F}_-^n$ operator can not reproduce all polynomials of the set $\{C_{n+1}^\ell(\cos\theta_1)\}$, because the maximum value of ℓ is $\ell = n$. We need also to define a raising operator on the ℓ-quantum numbers. However, from Eq. (2.12), we have $\mathcal{F}_-^n C_n^{\ell=n}(\cos\theta_1) = 0$, then the $\mathcal{F}_-^n$ operator can reproduce all polynomials of the set $\{C_{n-1}^\ell(\cos\theta_1)\}$.

The contiguous relation given in Eq.(2.7) can be written in terms of raising and lowering operators in the following way

$$2(n+1)\cos\theta_1 C_n^\ell(\cos\theta_1) = (\mathcal{F}_+^n + \mathcal{F}_-^n) C_n^\ell(\cos\theta_1) \tag{2.14}$$

By means of Eq.(2.5) we can derive raising and lowering operators for $C_n^\ell(\cos\theta_1)$ on the ℓ-quantum number, and we have

$$\mathcal{F}_+^\ell = -\frac{d}{d\theta_1} + \ell\cot\theta_1 \tag{2.15}$$

$$\mathcal{F}_-^\ell = \frac{d}{d\theta_1} + (\ell+1)\cot\theta_1 \tag{2.16}$$

where

$$\mathcal{F}_-^\ell C_n^\ell(\cos\theta_1) = C_1(n,\ell) C_n^{\ell+1}(\cos\theta_1) \tag{2.17}$$

$$\mathcal{F}_-^\ell C_n^\ell(\cos\theta_1) = D_1(n,\ell) C_n^{\ell-1}(\cos\theta_1) \tag{2.18}$$

with

$$C_1(n,\ell) = \{(n-\ell)(n+\ell+2)\}^{1/2} \tag{2.19}$$

$$D_1(n,\ell) = \{(n-\ell+1)(n+\ell+1)\}^{1/2} \tag{2.20}$$

equation (2.18) olds only for $\ell \geq 1$.

The commutation relation between $\mathcal{F}_+^\ell$ and $\mathcal{F}_-^\ell$ is given by

$$[\mathcal{F}_+^\ell, \mathcal{F}_-^\ell] = \frac{2\ell+1}{\sin^2\theta_1}. \tag{2.21}$$

These operators relate elements of the same set of polynomials $\{C_n^\ell(\cos\theta_1)\}$ for a given value of n. Both operators depend on the value of ℓ, and this fact must be taked into account when we construct all elements of the set, starting from a definite value of ℓ.

We can construct all polynomials, starting from $\ell = 0$, using Eq. (2.17). This equation is consistent with the fact that there is a maximum value of ℓ; $\ell = n$.

We can also see that Eq(2.18) is consistent with the fact that there is a minimum value of ℓ; $\ell = 0$.

Commutation relations between $\mathcal{F}_\pm^n$ and $\mathcal{F}_\pm^\ell$ can be derived and we have

$$\begin{aligned}
[\mathcal{F}_+^n, \mathcal{F}_+^\ell] &= \cos\theta_1 \frac{d}{d\theta_1} - \frac{\ell}{\sin\theta_1} - (n+2)\sin\theta_1 \\
[\mathcal{F}_+^n, \mathcal{F}_-^\ell] &= -\cos\theta_1 \frac{d}{d\theta_1} - \frac{\ell+1}{\sin\theta_1} + (n+2)\sin\theta_1 \\
[\mathcal{F}_-^n, \mathcal{F}_-^\ell] &= \cos\theta_1 \frac{d}{d\theta_1} + \frac{\ell+1}{\sin\theta_1} + n\sin\theta_1 \\
[\mathcal{F}_-^n, \mathcal{F}_+^\ell] &= -\cos\theta_1 \frac{d}{d\theta_1} + \frac{\ell}{\sin\theta_1} - n\sin\theta_1
\end{aligned} \tag{2.22}$$

Using these commutation relations we can derive the following contiguous relations

$$2(n+1)\sin\theta_1 C_n^{\ell}(\cos\theta_1) = -E_1(n,\ell)C_{n+1}^{\ell-1}(\cos\theta_1) + F_1(n,\ell)C_{n-1}^{\ell-1}(\cos\theta_1) \qquad (2.23)$$

$$2(n+1)\sin\theta_1 C_n^{\ell}(\cos\theta_1) = G_1(n,\ell)C_{n+1}^{\ell+1}(\cos\theta_1) - H_1(n,\ell)C_{n-1}^{\ell+1}(\cos\theta_1) \qquad (2.24)$$

where

$$E_1(n,\ell) = \left\{\frac{n+1}{n+2}(n-\ell+1)(n-\ell+2)\right\}^{1/2} \quad F_1(n,\ell) = \left\{\frac{n+1}{n}(n+\ell-1)(n+\ell)\right\}^{1/2} \qquad (2.5)$$

$$G_1(n,\ell) = \left\{\frac{n+1}{n+2}(n+\ell+2)(n+\ell+3)\right\}^{1/2} \quad H_1(n,\ell) = \left\{\frac{n-1}{n}(n-\ell)(n-\ell-1)\right\}^{1/2}$$

If we multiply both sides of Eq. (2.11) by $\mathcal{F}_-^{n+1}$ or Eq. (2.12) by $\mathcal{F}_+^{n-1}$ we reproduce the differential equation for $C_n^{\ell}(\cos\theta_1)$ given by Eq. (2.1). We can also reproduce Eq. (2.1) by means of $\mathcal{F}_+^{\ell}$ and $\mathcal{F}_-^{\ell}$ multiplying Eq. (2.17) by $\mathcal{F}_-^{\ell+1}$ or Eq. (2.18) by $\mathcal{F}_+^{\ell-1}$.

3. Spherical Harmonic Polynomials

The properties of the $Y_\ell^m(\theta_2,\phi)$ spherical harmonic polynomials in $\mathbb{R}_3$ are well-known. We collect here only the most important properties. The $Y_\ell^m(\theta_2,\phi)$ polynomials define a complete and orthonormal set of $(2\ell+1)$-polynomials and are solutions of the angular part of the Laplace equation in $\mathbb{R}_3$-spherical polar coordinates.

In terms of the $\mathbb{R}_3$-angular momentum operators the differential equations for $Y_\ell^m(\theta_2,\phi)$ are

$$L^2(\theta_2,\phi)Y_\ell^m(\theta_2,\phi) = \ell(\ell+1)Y_\ell^m(\theta_2,\phi) \tag{3.1}$$

$$L_z(\theta_2,\phi)Y_\ell^m(\theta_2,\phi) = m\,Y_\ell^m(\theta_2,\phi) \tag{3.2}$$

Raising and lowering operators on the m-quantum numbers are $L_+(\theta_2,\phi)$ and $L_-(\theta_2,\phi)$ respectively, where

$$L_+(\theta_2,\phi)Y_\ell^m(\theta_2,\phi) = \{(\ell-m)(\ell+m+1)\}^{1/2}Y_\ell^{m+1}(\theta_2,\phi) \tag{3.3}$$

$$L_-(\theta_2,\phi)Y_\ell^m(\theta_2,\phi) = \{(\ell+m)(\ell-m+1)\}^{1/2}Y_\ell^{m-1}(\theta_2,\phi). \tag{3.4}$$

Important properties of $Y_\ell^m(\theta_2,\phi)$ are the followings[1]:

$$\cos\theta_2 Y_\ell^m(\theta_2,\phi) = A_2(\ell,m)Y_{\ell+1}^m(\theta_2,\phi) + B_2(\ell,m)Y_{\ell-1}^m(\theta_2,\phi) \tag{3.5}$$

$$\exp(i\phi)\sin\theta_2 Y_\ell^m(\theta_2,\phi) = -C_2(\ell,m)Y_{\ell+1}^{m+1}(\theta_2,\phi) + D_2(\ell,m)Y_{\ell-1}^{m+1}(\theta_2,\phi) \tag{3.6}$$

$$\exp(-i\phi)\sin\theta_2 Y_\ell^m(\theta_2,\phi) = E_2(\ell,m)Y_{\ell+1}^{m-1}(\theta_2,\phi) - F_2(\ell,m)Y_{\ell-1}^{m-1}(\theta_2,\phi) \tag{3.7}$$

$$\sin\theta_2\frac{\partial}{\partial\theta_2}Y_\ell^m(\theta_2,\phi) = \ell A_2(\ell,m)Y_{\ell+1}^m(\theta_2,\phi) - (\ell+1)B_2(\ell,m)Y_{\ell-1}^m(\theta_2,\phi) \tag{3.8}$$

with

$$A_2(\ell,m) = \left\{\frac{(\ell-m+1)(\ell+m+1)}{(2\ell+1)(2\ell+3)}\right\}^{1/2} \quad B_2(\ell,m) = \left\{\frac{(\ell-m)(\ell+m)}{(2\ell-1)(2\ell+1)}\right\}^{1/2}$$

$$C_2(\ell,m) = \left\{\frac{(\ell+m+1)(\ell+m+2)}{(2\ell+1)(2\ell+3)}\right\}^{1/2} \quad D_2(\ell,m) = \left\{\frac{(\ell-m)(\ell-m-1)}{(2\ell-1)(2\ell+1)}\right\}^{1/2}$$

$$E_2(\ell,m) = \left\{\frac{(\ell-m+1)(\ell-m+2)}{(2\ell+1)(2\ell+3)}\right\}^{1/2} \quad F_2(\ell,m) = \left\{\frac{(\ell+m)(\ell+m-1)}{(2\ell-1)(2\ell+1)}\right\}^{1/2}$$

The corresponding raising and lowering operator on the ℓ-quantum number can be derived from the derivative property of $Y_\ell^m(\theta_2,\phi)$ and are given by

$$\mathcal{H}_+^\ell = \sin\theta_2 \frac{\partial}{\partial\theta_2} + (\ell+1)\cos\theta_2 \tag{3.9}$$

$$\mathcal{H}_-^\ell = -\sin\theta_2 \frac{\partial}{\partial\theta_2} + \ell\cos\theta_2 \tag{3.10}$$

where

$$\mathcal{H}_-^\ell Y_\ell^m(\theta_2,\phi) = G_2(\ell,m) Y_{\ell+1}^m(\theta_2,\phi) \tag{3.11}$$

$$\mathcal{H}_-^\ell Y_\ell^m(\theta_2,\phi) = H_2(\ell,m) Y_{\ell-1}^m(\theta_2,\phi) \tag{3.12}$$

and

$$G_2(\ell,m) = \left\{\frac{2\ell+1}{2\ell+3}(\ell+m+1)(\ell-m+1)\right\}^{1/2}$$

$$H_2(\ell,m) = \left\{\frac{2\ell+1}{2\ell-1}(\ell-m)(\ell+m)\right\}^{1/2}$$

The commutation relation is given by

$$[\mathcal{H}_+^\ell, \mathcal{H}_-^\ell] = -(2\ell-1)\sin^2\theta_2 \tag{3.13}$$

As in the case of the spherical Gegenbauer polynomials these operators relate the set of polynomials $\{Y_\ell^m(\theta_2,\phi)\}$ with the set $\{Y_{\ell\pm1}^m(\theta_2,\phi)\}$ and the same conclusions hold here.

The differential equation for $Y_\ell^m(\theta_2,\phi)$ can be reproduced by the application on $Y_\ell^m(\theta_2,\phi)$ of one of the following combinations of diagonal operators:

$$\mathcal{H}_-^{\ell+1}\mathcal{H}_+^\ell + L_z^2 \quad \text{or} \quad \mathcal{H}_+^{\ell-1}\mathcal{H}_-^\ell + L_z^2$$

4. Hypersherical Harmonic Polynomials in $\mathbb{R}_4$

The hypershperical harmonic polynomials in $\mathbb{R}_4$ are solutions of the angular part of the Laplace equation in four-dimensions

$$\mathcal{L}^2 \mathcal{Y}_{n,\ell}^m(\theta_1,\theta_2,\phi) = n(n+2)\mathcal{Y}_{n,\ell}^m(\theta_1,\theta_2,\phi) \tag{4.1}$$

with

$$\mathcal{L}^2 = -\frac{1}{\sin^2\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin^2\theta_1\frac{\partial}{\partial\theta_1}\right) + \frac{1}{\sin^2\theta_1}L^2(\theta_2,\phi) \tag{4.2}$$

where

$$\mathcal{Y}_{n,\ell}^m(\theta_1,\theta_2,\phi) = \mathcal{C}_n^\ell(\cos\theta_1)Y_\ell^m(\theta_1,\phi) \tag{4.3}$$

and $0 \le \theta_1 \le \pi$; $0 \le \theta_2 \le \pi$; $0 \le \phi \le 2\pi$; $n = 0,1,2,\ldots$; $0 \le \ell \le n$ and $-\ell \le m \le \ell$.

The $\mathcal{Y}_{n,\ell}^m(\theta_1,\theta_2,\phi)$ form a complete and orthonormal set of $(n+1)^2$-polynomials:

$$\int_\Omega \mathcal{Y}_{n',\ell'}^{*m'}(\theta_1,\theta_2,\phi)\mathcal{Y}_{n,\ell}^m(\theta_1,\theta_2,\phi)d\Omega = \delta_{n,n'}\,\delta_{\ell,\ell'}\,\delta_{m,m'} \tag{4.4}$$

and

$$\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\sum_{m=-\ell}^{\ell} \mathcal{Y}_{n,\ell}^{*m}(\theta_1',\theta_2',\phi')\mathcal{Y}_{n,\ell}^m(\theta_1,\theta_2,\phi) =$$

$$= \frac{1}{\sin^2\theta_1}\delta(\cos\theta_1 - \cos\theta_1')\delta(\cos\theta_2 - \cos\theta_2')\delta(\phi-\phi') \tag{4.5}$$

The raising and lowering operators on the n-quantum number are the partial differential operators defined by Eqs. (2.9) and (2.10):

$$\mathcal{F}_+^n \mathcal{Y}_{n,\ell}^m(\theta_1,\theta_2,\phi) = A_1(n,\ell)\mathcal{Y}_{n+1,\ell}^m(\theta_1,\theta_2,\phi)$$

$$\mathcal{F}_-^n \mathcal{Y}_{n,\ell}^m(\theta_1,\theta_2,\phi) = B_1(n,\ell)\mathcal{Y}_{n-1,\ell}^m(\theta_1,\theta_2,\phi) \tag{4.6}$$

and for the ℓ-quantum number we have the following combinations

$$\mathcal{F}_+^\ell \mathcal{H}_-^\ell \mathcal{Y}_{n,\ell}^m(\theta_1,\theta_2,\phi) = C_1(n,\ell)G_2(\ell,m)\mathcal{Y}_{n,\ell+1}^m(\theta_1,\theta_2,\phi) \tag{4.7}$$

$$\mathcal{F}_-^\ell \mathcal{H}_-^\ell \mathcal{Y}_{n,\ell}^m(\theta_1,\theta_2,\phi) = D_1(n,\ell)H_2(n,\ell)\mathcal{Y}_{n,\ell-1}^m(\theta_1,\theta_2,\phi) \tag{4.8}$$

The differential equation (4.1) can be reproduced by the application of one the following combinations of diagonal operators on $\mathcal{Y}_{n,\ell}^m(\theta_1,\theta_2,\phi)$

$$\mathcal{F}_-^{\ell-1}\mathcal{F}_+^\ell + \frac{1}{\sin^2\theta_1 \sin^2\theta_2}(\mathcal{H}_-^{\ell+1}\mathcal{H}_+^\ell + L_z^2)$$

$$\mathcal{F}_+^{\ell-1}\mathcal{F}_-^\ell + \frac{1}{\sin^2\theta_1 \sin^2\theta_2}(\mathcal{H}_+^{\ell-1}, \mathcal{H}_-^\ell + L_z^2) \tag{4.9}$$

or

$$\frac{1}{\sin^2\theta_1}(\mathcal{F}_-^{n+1}\mathcal{F}_+^n + L^2)$$

$$\frac{1}{\sin^2\theta_1}(\mathcal{F}_+^{n-1}\mathcal{F}_-^n + L^2) \tag{4.10}$$

The recurrence relations between contiguous $\mathcal{Y}_{n,\ell}^m(\theta_1,\theta_2,\phi)$ can be derived by appropriated combinations of properties of $C_n^\ell(\cos\theta_1)$ and of $Y_\ell^m(\theta_2,\phi)$ polynomials.

From Eq. (2.7) we derive a recurrence relation between contiguous $\mathcal{Y}^m_{n,\ell}(\theta_1,\theta_2,\phi)$ on the n-quantum number

$$
\begin{aligned}
&2(n+1)\cos\theta_1\mathcal{Y}^m_{n,\ell}(\theta_1,\theta_2,\phi) = \\
&= A_1(n,\ell)\mathcal{Y}^m_{n+1,\ell}(\theta_1,\theta_2,\phi) + B_1(n,\ell)\mathcal{Y}^m_{n-1,\ell}(\theta_1,\theta_2,\phi).
\end{aligned}
$$

Using Eqs. (2.23) and (2.24) for $C^\ell_n(\cos\theta_1)$ and Eqs. (3.5), (3.6) and (3.7) for $Y^m_\ell(\theta_2,\phi)$ we obtain

$$
\begin{aligned}
&2(n+1)\sin\theta_1\cos\theta_2\mathcal{Y}^m_{n,\ell}(\theta_1,\theta_2,\phi) = \\
&= A_2(\ell,m)G_1(n,\ell)\mathcal{Y}^m_{n+1,\ell+1}(\theta_1,\theta_2,\phi) - A_2(\ell,m)H_1(n,\ell)\mathcal{Y}^m_{n-1,\ell+1}(\theta_1,\theta_2,\phi) - \\
&- B_2(\ell,m)E_1(n,\ell)\mathcal{Y}^m_{n+1,\ell-1}(\theta_1,\theta_2,\phi) + B_2(\ell,m)F_1(n,\ell)\mathcal{Y}^m_{n-1,\ell-1}(\theta_1,\theta_2,\phi)
\end{aligned}
$$

$$
\begin{aligned}
&\exp(i\,\phi)\sin\theta_1\sin\theta_2\mathcal{Y}^m_{n,\ell}(\theta_1,\theta_2,\phi) = \\
&= -C_2(\ell,m)G_1(n,\ell)\mathcal{Y}^{m+1}_{n+1,\ell-1}(\theta_1,\theta_2,\phi) + C_2(\ell,m)H_1(n,\ell)\mathcal{Y}^{m+1}_{n-1,\ell+1}(\theta_1,\theta_2,\phi) - \\
&- D_2(\ell,m)E_1(n,\ell)\mathcal{Y}^{m+1}_{n+1,\ell-1}(\theta_1,\theta_2,\phi) + D_2(\ell,m)F_1(n,\ell)\mathcal{Y}^{m+1}_{n-1,\ell-1}(\theta_1,\theta_2,\phi)
\end{aligned}
$$

and

$$
\begin{aligned}
&\exp(-i\,\phi)\sin\theta_1\sin\theta_2\mathcal{Y}^m_{n,\ell}(\theta_1,\theta_2,\phi) = \\
&= E_2(\ell,m)G_1(n,\ell)\mathcal{Y}^{m-1}_{n+1,\ell+1}(\theta_1,\theta_2,\phi) - E_2(\ell,m)H_1(n,\ell)\mathcal{Y}^{m-1}_{n-1,\ell+1}(\theta_1,\theta_2,\phi) + \\
&+ F_2(\ell,m)E_1(n,\ell)\mathcal{Y}^{m-1}_{n+1,\ell-1}(\theta_1,\theta_2,\phi) - F_2(\ell,m)F_1(n,\ell)\mathcal{Y}^{m-1}_{n-1,\ell-1}(\theta_1,\theta_2,\phi).
\end{aligned}
$$

5. Conclusions

We have here derived properties for the spherical harmonic polynomials in $\mathbb{R}_3$ and $\mathbb{R}_4$ spaces: raising and lowering operators and recurrence relations between contiguous polynomials have also been obtained.

It is well known that operational methods are of particular use to specialists dealings with applied mathematicals. With such methods it is possible to reduce differential problems to algebraic problems. Ours raising and lowering operators are anticommutative and can be applied to solve non-homogeneous linear differential equation with constant coefficients using the Heavside method and operator calculus[6]. Some interesting applications of these operators follows these will be published elsewhere.

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