LINEARIZATION OF BOUNDED HOLOMORPHIC MAPPINGS ON BANACH SPACES

Jorge Mujica

RELATÓRIO TÉCNICO Nº 20/88

Abstract. The main result in this paper is the following linearization theorem. For each open set U in a complex Banach space E, there are a complex Banach space $G^\infty(U)$ and a bounded holomorphic mapping $g_U : U \to G^\infty(U)$ with the following universal property: For each complex Banach space F and each bounded holomorphic mapping $f : U \to F$, there is a unique continuous linear operator $T_f : G^\infty(U) \to F$ such that $T_f \circ g_U = f$. The correspondence $f \to T_f$ is an isometric isomorphism between the space $H^\infty(U;F)$ of all bounded holomorphic mappings from U into F, and the space $L(G^\infty(U); F)$ of all continuous linear operators from $G^\infty(U)$ into F. The rest of the paper is devoted to the study of some aspects of the interplay between the spaces $H^\infty(U; F)$ and $L(G^\infty(U); F)$.

Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
IMECC – UNICAMP
Caixa Postal 6065
13.081 – Campinas, SP
Brazil

Novembro – 1988

Linearization of Bounded Holomorphic Mappings on Banach Spaces

Jorge Mujica

Introduction

The main result in this paper is the following linearization theorem. For each open set U in a complex Banach space E, there are a complex Banach space $G^\infty(U)$ and a bounded holomorphic mapping $g_U : U \to G^\infty(U)$ with the following universal property: For each complex Banach space F and each bounded holomorphic mapping $f : U \to F$, there is a unique continuous linear operator $T_f : G^\infty(U) \to F$ such that $T_f \circ g_U = f$. The correspondence $f \to T_f$ is an isometric isomorphism between the space $H^\infty(U; F)$ of all bounded holomorphic mappings from U into F, and the space $L(G^\infty(U); F)$ of all continuous linear operators from $G^\infty(U)$ into F. The rest of the paper is devoted to the study of some aspects of the interplay between the spaces $H^\infty(U; F)$ and $L(G^\infty(U); F)$.

This paper is organized as follows. In Section 1 we establish our notation and terminology. In Section 2 we prove the aforementioned linearization theorem. In Section 3 we translate certain properties of a mapping $f \in H^\infty(U; F)$ into properties of the corresponding operator $T_f \in L(G^\infty(U); F)$. In Section 4 we give a seminorm description of the unique locally convex topology τ_γ on $H^\infty(U; F)$ such that the correspondence $f \to T_f$ is a topological isomorphism between the spaces $(H^\infty(U; F), \tau_\gamma)$ and $(L(G^\infty(U); F), \tau_c)$, where τ_c denotes the compact–open topology. Finally, in Section 5 we use the preceding results to establish necessary and sufficient conditions for the space $G^\infty(U)$ to have the approximation property. In particular we show that if U is a balanced, bounded, open set in a complex Banach space E, then $G^\infty(U)$ has the approximation property if and only if E has the approximation property.

1

1. Notation and Terminology

Throughout this paper the symbols $I\!R$ and $\mathbb{C}$ represent the field of all real numbers and the field of all complex numbers, respectively. The set of all positive integers is denoted by $I\!N$, whereas the set $I\!N \cup \{0\}$ is denoted by $I\!N_0$.

The letters E and F always represent complex Banach spaces. The letter U denotes a nonvoid open subset of E, whereas the symbol U_E represents the open unit ball of E.

$L(E;F)$ denotes the vector space of all continuous linear operators from E into F. Unless stated otherwise, $L(E;F)$ is endowed with its natural norm topology. We write E' instead of $L(E;\mathbb{C})$ for the dual of E.

$P(E;F)$ denotes the vector space of all continuous polynomials from E into F. For each $m \in I\!N_0$, $P(^mE;F)$ denotes the subspace of all m–homogeneous members of $P(E;F)$. Unless stated otherwise, $P(^mE;F)$ is endowed with its natural norm topology. When $F = \mathbb{C}$ we write $P(E)$ instead of $P(E;\mathbb{C})$, and $P(^mE)$ instead of $P(^mE;\mathbb{C})$.

$H(U;F)$ denotes the vector space of all holomorphic mappings from U into F, and $H^\infty(U;F)$ denotes the subspace of all bounded members of $H(U;F)$. Unless stated otherwise, $H^\infty(U;F)$ is endowed with the sup norm topology. When $F = \mathbb{C}$ we write $H(U)$ instead of $H(U;\mathbb{C})$, and $H^\infty(U)$ instead of $H^\infty(U;\mathbb{C})$.

For each $f \in H(U;F)$, $a \in U$ and $m \in I\!N_0$, $P^m f(a) \in P(^mE;F)$ denotes the mth term of the Taylor series of f at a.

If X is a locally convex space, then X'_b (resp. X'_c) denotes the dual X' of X, endowed with the topology of uniform convergence on the bounded (resp. compact) subsets of X.

If X is a topological space, then $C(X;F)$ denotes the vector space of all continuous mappings from X into F, and τ_c denotes the compact–open topology on $C(X;F)$, or on any subset of $C(X;F)$.

We refer to the books of S. Dineen [6] or the author [13] for the properties of polynomials and holomorphic mappings on infinite dimensional spaces; to the book of J. Lindenstrauss and L. Tzafriri [10] for the theory of Banach spaces; and to the books of J. Horváth [9] or H. H. Schaefer [16] for the theory of topological vector spaces.

2. Linearization of Bounded Holomorphic Mappings

Let U be an open subset of a Banach space E. As pointed out in S. Dineen's book [6, p. 417], a theorem of K. F. Ng [14] yields a Banach space $G^\infty(U)$ whose

dual is isometrically isomorphic to $H^\infty(U)$. A refinement of this idea leads to the linearization theorem announced in the introduction.

2.1. Theorem. Let U be an open subset of a Banach space E. Then there are a Banach space $G^\infty(U)$ and a mapping $g_U \in H^\infty(U; G^\infty(U))$ with the following universal property: For each Banach space F and each mapping $f \in H^\infty(U; F)$, there is a unique operator $T_f \in L(G^\infty(U); F)$ such that $T_f \circ g_U = f$. The correspondence

$$f \in H^\infty(U; F) \longrightarrow T_f \in L(G^\infty(U); F)$$

is an isometric isomorphism. These properties characterize $G^\infty(U)$ uniquely up to an isometric isomorphism.

Proof. Let B_U denote the closed unit ball of $H^\infty(U)$. By the Ascoli theorem B_U is a compact subset of $H^\infty(U)$ for the compact-open topology τ_c. Let $G^\infty(U)$ denote the closed subspace of all linear functionals $u \in H^\infty(U)'$ such that $u|(B_U, \tau_c)$ is continuous. Let

$$J_U : f \in H^\infty(U) \longrightarrow \hat{f} \in G^\infty(U)'$$

denote the evaluation mapping, that is, $\hat{f}(u) = u(f)$ for every $f \in H^\infty(U)$ and $u \in G^\infty(U)$. By the Ng theorem [14], J_U is an isometric isomorphism. Let

$$g_U : x \in U \longrightarrow \delta_x \in G^\infty(U)$$

denote the evaluation mapping, that is, $\delta_x(f) = f(x)$ for every $x \in U$ and $f \in H^\infty(U)$. Since

$$(2.1) \qquad\qquad (J_U f) \circ g_U(x) = f(x)$$

for every $x \in U$ and $f \in H^\infty(U)$, we see that g_U is weakly holomorphic, and therefore holomorphic (see [13, p. 65, Theorem 8.12]). Since $\|\delta_x\| = 1$ for every $x \in U$, we conclude that $g_U \in H^\infty(U; G^\infty(U))$.

Let ^{0} indicate polars with respect to the dual system $\langle G^\infty(U), G^\infty(U)' \rangle$. We claim that the closed unit ball $J_U(B_U)^0$ of $G^\infty(U)$ coincides with the closed, convex, balanced hull $\overline{\Gamma} g_U(U)$ of $g_U(U)$. In particular $g_U(U)$ generates a dense subspace of $G^\infty(U)$. Indeed, since J_U maps $H^\infty(U)$ onto $G^\infty(U)'$, one can readily check that

$$J_U(B_U) = g_U(U)^0,$$

and therefore

$$J_U(B_U)^0 = g_U(U)^{00} = \overline{\Gamma} g_U(U),$$

by the bipolar theorem, as asserted.

We claim that the pair $(G^\infty(U), g_U)$ has the required universal property. If $f \in H^\infty(U)$, then we define $T_f = J_U f \in G^\infty(U)'$ and the desired conclusion follows from (2.1). If $f \in H^\infty(U; F)$, then we define $T_f : G^\infty(U) \to F''$ by

$$(T_f u)(\psi) = u(\psi \circ f)$$

for every $u \in G^\infty(U)$ and $\psi \in F'$. Clearly $T_f \in L(G^\infty(U); F'')$ and $\|T_f\| = \|f\|$. Furthermore,

$$(T_f \delta_x)(\psi) = \psi \circ f(x)$$

for every $x \in U$ and $\psi \in F'$, and therefore $T_f \delta_x = f(x) \in F$ for every $x \in U$. Since the evaluations δ_x, with $x \in U$, generate a dense subspace of $G^\infty(U)$, it follows that $T_f \in L(G^\infty(U); F)$. The uniqueness of T_f follows also from the fact that $g_U(U)$ generates a dense subspace of $G^\infty(U)$.

Finally, the uniqueness of $G^\infty(U)$ up to an isometric isomorphism follows from the universal property, together with the isometry $\|T_f\| = \|f\|$. This completes the proof.

2.2. Remark. In the proof of Theorem 2.1 we have shown that the closed unit ball of $G^\infty(U)$ coincides with $\overline{\Gamma} g_U(U)$. The same proof shows that the closed unit ball of $G^\infty(U)$ coincides with $\overline{\Gamma} g_U(D)$ for each dense subset D of U. This shows in particular that $G^\infty(U)$ is separable whenever E is separable, a result noticed already by J. M. Ansemil and S. Dineen [1].

2.3. Proposition. Let E be a Banach space.

(a) If U is a bounded open set in E, then E is topologically isomorphic to a complemented subspace of $G^\infty(U)$.

(b) If U is the open unit ball of E, then E is isometrically isomorphic to a 1-complemented subspace of $G^\infty(U)$.

Proof. (a) By Theorem 2.1 there exists $T \in L(G^\infty(U); E)$ such that $T \circ g_U(x) = x$ for every $x \in U$. Fix $a \in U$ and let $S = P^1 g_U(a) \in L(E; G^\infty(U))$. Given $t \in E$, choose $r > 0$ such that $a + \varsigma t \in U$ for all $\varsigma \in \mathbb{C}$ with $|\varsigma| \leq r$. By the Cauchy integral formula

$$S(t) = \frac{1}{2\pi i} \int_{|\varsigma|=r} \frac{g_U(a + \varsigma t)}{\varsigma^2} \, d\varsigma$$

and therefore

$$T \circ S(t) = \frac{1}{2\pi i} \int_{|\varsigma|=r} \frac{a + \varsigma t}{\varsigma^2} \, d\varsigma = t.$$

4

(b) If U is the open unit ball of E, then it follows from Theorem 2.1 that $\|T\| = 1$. If we take $a = 0$ in (a) then it follows from the Cauchy integral formula that $\|S\| \leq 1$. Hence

$$\|t\| = \|T \circ S(t)\| \leq \|S(t)\| \leq \|t\|$$

for every $t \in E$, and thus S is an isometry.

By following the pattern of the proof of Theorem 2.1, we can also prove the following result, which has been previously obtained by R. Ryan [15] by using tensor product methods.

2.4. Theorem [15]. Let E be a Banach space and let $m \in \mathbb{N}$. Then there are a Banach space $Q(^m E)$ and a polynomial $q_m \in P(^m E; Q(^m E))$ with the following universal property: For each Banach space F and each polynomial $P \in P(^m E; F)$, there is a unique operator $T_P \in L(Q(^m E); F)$ such that $T_P \circ q_m = P$. The correspondence

$$P \in P(^m E; F) \longrightarrow T_P \in L(Q(^m E); F)$$

is an isometric isomorphism. These properties characterize $Q(^m E)$ uniquely up to an isometric isomorphism.

2.5. Remarks. Our proof of Theorem 2.4 yields also the following facts:

(a) If B_m denotes the closed unit ball of $P(^m E)$, then $Q(^m E)$ is the closed subspace of all $u \in P(^m E)'$ such that $u|(B_m, \tau_c)$ is continuous.

(b) $q_m : x \in E \to \delta_x \in Q(^m E)$ is the evaluation mapping, that is, $\delta_x(P) = P(x)$ for every $x \in E$ and $P \in P(^m E)$.

(c) The closed unit ball of $Q(^m E)$ coincides with the closed, convex, balanced hull $\overline{\Gamma} q_m(U_E)$ of $q_m(U_E)$. In particular $q_m(U_E)$ generates a dense subspace of $Q(^m E)$.

By using Theorem 2.4 we can also obtain the following result, which parallels Proposition 2.3.

2.6 Proposition. Let E be a Banach space and let $m \in \mathbb{N}$. Then E is isometrically isomorphic to a 1-complemented subspace of $Q(^m E)$.

We end this section with the following result, which shows the connection between the spaces $Q(^m E)$ and $G^\infty(U)$.

2.7. Proposition. Let E be a Banach space.

(a) If U is a bounded open set in E, then $Q(^m E)$ is topologically isomorphic to

a complemented subspace of $G^\infty(U)$.

(b) If U is the open unit ball of E, then $Q(^mE)$ is isometrically isomorphic to a 1–complemented subspace of $G^\infty(U)$.

Proof. (a) Fix $a \in U$. By Theorems 2.4 and 2.1 there are operators $S_m \in L(Q(^mE); G^\infty(U))$ and $T_m \in L(G^\infty(U); Q(^mE))$ such that the following diagrams are commutative.

$$
\begin{array}{ccc}
E & & \\
q_m \downarrow \quad \searrow^{P^m g_U(a)} & & \\
Q(^mE) \xrightarrow[S_m]{} G^\infty(U) & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
U & \hookrightarrow & E \\
g_U \downarrow & & \downarrow q_m \\
G^\infty(U) \xrightarrow[T_m]{} & & Q(^mE)
\end{array}
$$

Hence we have that

$$
T_m \circ S_m \circ q_m = T_m \circ [P^m g_U(a)] = P^m(T_m \circ g_U)(a)
$$
$$
= P^m q_m(a) = q_m.
$$

Since $q_m(E)$ generates a dense subspace of $Q(^mE)$, it follows that $T_m \circ S_m(u) = u$ for every $u \in Q(^mE)$.

(b) If U is the open unit ball of E, then the norm of q_m in $H^\infty(U; Q(^m(E))$ equals one, and hence $\|T_m\| = 1$. If we take $a = 0$ in (a) then it follows from the Cauchy integral formula that the norm of $P^m g_U(0)$ in $P(^mE; G^\infty(U))$ is not greater than one, and hence $\|S_m\| \leq 1$. Hence

$$
\|u\| = \|T_m \circ S_m(u)\| \leq \|S_m(u)\| \leq \|u\|
$$

for every $u \in Q(^mE)$, and thus S_m is an isometry.

3. Holomorphic Mappings with a Relatively Compact Range

In this section we translate certain properties of a mapping $f \in H^\infty(U; F)$ into properties of the corresponding operator $T_f \in L(G^\infty(U); F)$.

If X is a set and Y is a vector space, then a mapping $f : X \to Y$ is said to have finite rank if the subspace N of Y generated by $f(X)$ is finite dimensional. In that

6

case we define the rank of f to be the dimension of N.

Observe that the subspace of all finite rank mappings $f \in H^\infty(U; F)$ can be canonically identified with $H^\infty(U) \otimes F$, and likewise for other spaces of mappings.

Let $P_f(^m E; F)$ denote the subspace of all $P \in P(^m E; F)$ of the form $P(x) = \sum_{j=1}^m [\phi_j(x)]^m b_j$, with $\phi_j \in E'$ and $b_j \in F$. Let $P_f(E; F)$ denote the algebraic direct sum of the spaces $P_f(^m E; F)$, with $m \in \mathbb{N}_0$. The members of $P_f(E; F)$ are called continuous polynomials of finite type. Observe that each $P \in P_f(E; F)$ has finite rank.

As an immediate consequence of Theorems 2.1 and 2.4 we get the following result.

3.1. Proposition. Let E and F be Banach spaces, let U be an open set in E, and let $m \in \mathbb{N}$.

(a) A mapping $f \in H^\infty(U; F)$ has finite rank if and only if the corresponding operator $T_f \in L(G^\infty(U); F)$ has finite rank. In that case rank $f =$ rank T_f.

(b) A polynomial $P \in P(^m E; F)$ has finite rank if and only if the corresponding operator $T_P \in L(Q(^m E); F)$ has finite rank. In that case rank $P =$ rank T_P.

By a result of K. Floret [7], the operators of rank $\leq n$ in $L(G^\infty(U); F)$ form a closed subset of $L(G^\infty(U); F)$. Whence it follows that the mappings of rank $\leq n$ in $H^\infty(U; F)$ form a closed subset of $H^\infty(U; F)$. These results are true, not only for the norm topology, but for other suitable topologies as well. A. Chiacchio et al. [3] have used these results to prove that if F is a dual Banach space, then the mappings of rank $\leq n$ in $H^\infty(U; F)$ form a proximinal subset of $\ell^\infty(U; F)$, the space of all bounded mappings from U into F.

Following R. Aron and M. Schottenloher [2], and R. Ryan [15], respectively, we say that a mapping $f \in H(U; F)$ is compact (resp. weakly compact) if each $x \in U$ has a neighborhood $V_x \subset U$ such that $f(V_x)$ is relatively compact (resp. relatively weakly compact) in F. After examining the proofs of [2, Proposition 3.4] and [15, Proposition 4.5], we see that if U is connected, then $f \in H(U; F)$ is compact (resp. weakly compact) if and only if there is a nonvoid open set $V \subset U$ such that $f(V)$ is relatively compact (resp. relatively weakly compact) in F. Let $H_k(U; F)$ (resp. $H_{wk}(U; F)$) denote the subspace of all compact (resp. weakly compact) members of $H(U; F)$. We also write

$$H_k^\infty(U; F) = H^\infty(U; F) \cap H_k(U; F),$$
$$H_{wk}^\infty(U; F) = H^\infty(U; F) \cap H_{wk}(U; F),$$

and likewise for other spaces of holomorphic mappings.

Finally let $H_K^\infty(U; F)$ (resp. $H_{wK}^\infty(U; F)$) denote the subspace of all mappings $f \in$

$H^\infty(U; F)$ which have a relatively compact range (resp. relatively weakly compact range). Then clearly we have the inclusions

$$
\begin{array}{ccc}
H^\infty_K(U;F) & \subset & H^\infty_k(U;F) \\
\cap & & \cap \\
H^\infty_{wK}(U;F) & \subset & H^\infty_{wk}(U;F)
\end{array}
$$

and the following examples show that these inclusions are in general strict.

3.2. Example. Let Δ denote the open unit disc in $\mathbb{C}$, and let $f \in H^\infty(\Delta; c_0)$ be defined by

$$
f(\varsigma) = (\varsigma^m)_{m=1}^\infty = \sum_{m=1}^\infty \varsigma^m e_m,
$$

where e_m denotes the mth unit vector. Let $0 < r < 1$. Since $|\varsigma^m| < r^m$ for every $\varsigma \in r\Delta$ and $m \in \mathbb{N}$, we see that the set $f(r\Delta)$ is relatively compact in c_0 (see [5, p. 15, Exercise 6]). and thus $f \in H^\infty_k(\Delta; c_0)$. We claim that $f \notin H^\infty_{wK}(\Delta; c_0)$. Indeed, let (ς_j) be a sequence in Δ which converges to one. If the set $f(\Delta)$ were relatively weakly compact. then the sequence $(f(\varsigma_j))$ would have a weak cluster point in c_0. But this is impossible, for a glance at the coordinates of the terms of the sequence $(f(\varsigma_j))$ shows that the only candidate for a position as cluster point would be the point $(1, 1, 1, \ldots)$, which does not lie in c_0. This example shows that in general $H^\infty_K(U;F) \neq H^\infty_k(U;F)$ and $H^\infty_{wK}(U;F) \neq H^\infty_{wk}(U;F)$.

3.3. Example. Let E be a reflexive, infinite dimensional Banach space, let U be a bounded, open subset of E, and let $f(x) = x$ for every $x \in U$. Then it is clear that $f \in H^\infty_{wK}(U;E)$ and $f \notin H^\infty_k(U;E)$. This example shows that in general $H^\infty_K(U;F) \neq H^\infty_{wK}(U;F)$ and $H^\infty_k(U;F) \neq H^\infty_{wk}(U;F)$.

3.4. Proposition. Let E and F be Banach spaces, let U be an open set in E, and let $m \in \mathbb{N}$.

(a) [15] A polynomial $P \in P(^mE; F)$ is compact (resp. weakly compact) if and only if the corresponding operator $T_P \in L(Q(^mE); F)$ is compact (resp. weakly compact).

(b) A mapping $f \in H^\infty(U; F)$ has a relatively compact range (resp. relatively weakly compact range) if and only if the corresponding operator $T_f \in L(G^\infty(U); F)$ is compact (resp. weakly compact).

Proof. Since the closed unit ball of $Q(^mE)$ coincides with $\overline{\Gamma} q_m(U_E)$, (a) follows from the inclusions

$$
P(U_E) = T_P \circ q_m(U_E) \subset T_P[\overline{\Gamma} q_m(U_E)] \subset \overline{\Gamma} T_P \circ q_m(U_E) = \overline{\Gamma} P(U_E).
$$

8

Similarly, since the closed unit ball of $G^\infty(U)$ coincides with $\overline{\Gamma} g_U(U)$, (b) follows from the inclusions

$$f(U) = T_f \circ g_U(U) \subset T_f[\overline{\Gamma} g_U(U)] \subset \overline{\Gamma} T_f \circ g_U(U) = \overline{\Gamma} f(U).$$

Proposition 3.4 (a) is due to R. Ryan [15], but we have included it here explicitly for the convenience of the reader.

4. Other Topologies on $H^\infty(U;F)$

Theorem 2.1 tells us that the mapping

$$f \in H^\infty(U;F) \longrightarrow T_f \in L(G^\infty(U);F)$$

is an isometric isomorphism. In the next section we shall determine when the space $G^\infty(U)$ has the approximation property, and for that purpose it will be useful to have a seminorm description of the unique locally convex topology τ_γ on $H^\infty(U;F)$ such that the mapping

$$f \in (H^\infty(U;F); \tau_\gamma) \longrightarrow T_f \in (L(G^\infty(U);F), \tau_c)$$

is a topological isomorphism.

The corresponding problem for the space $P(^mE;F)$ has already been solved by R. Ryan [15]. His result is essentially the following.

4.1 Theorem [15]. Let E and F be Banach spaces, and let $m \in I\!N$. Then the mapping

$$P \in (P(^mE;F), \tau_c) \longrightarrow T_P \in (L(Q(^mE);F), \tau_c)$$

is a topological isomorphism.

R. Ryan [15] obtained Theorem 4.1 by using tensor product methods. We next give an alternative proof of Theorem 4.1, which will suggest how to tackle the corresponding problem for $H^\infty(U;F)$. The key tool in our proof of Ryan's theorem is the following result, established in [11], and which may be regarded as a generalization of the classical Banach–Dieudonné theorem.

4.2. Theorem [11]. Let E be a Banach space and let $m \in I\!N$. Then τ_c is the finest topology on $P(^mE)$ which coincides with τ_c on each norm bounded subset of $P(^mE)$.

With the aid of Theorem 4.2 we can prove the following result, which is also due to R. Ryan [15].

9

4.3. Proposition [15]. Let E be a Banach space, and let $m \in \mathbb{N}$. Then:

(a) $Q(^mE) = (P(^m(E), \tau_c)'_b$.

(b) The mapping

$$P \in (P(^mE), \tau_c) \longrightarrow T_P \in Q(^m(E)'_c$$

is a topological isomorphism.

(c) For each compact set $L \subset Q(^mE)$, there is a compact set $K \subset E$ such that $L \subset \bar{\Gamma} \, q_m(K)$.

Proof. (a) It follows from Theorem 4.2 that $Q(^mE) = (P(^mE), \tau_c)'$ algebraically. But since $P(^mE)$ and $(P(^mE), \tau_c)$ have the same bounded sets (see [13, p. 13, Theorem 2.6]), we conclude that $Q(^mE) = (P(^mE), \tau_c)'_b$.

(b) By Theorem 4.2 a set $N \subset (P(^mE), \tau_c)$ is open whenever $N \cap j \, B_m$ is open in $(j \, B_m, \tau_c)$ for every $j \in \mathbb{N}$. Hence an application of [12, Theorem 4.1] shows that the evaluation mapping

$$(P(^mE), \tau_c) \longrightarrow ((P(^mE), \tau_c)'_b)'_c$$

is a topological isomorphism. After combining this with (a), the desired conclusion follows.

(c) If L is a compact subset of $Q(^mE)$, then the polar L^0 is a 0-neighborhood in $Q(^mE)'_c$ and then, by (b), the set

$$\{P \in P(^mE) : T_P \in L^0\}$$

is a 0-neighborhood in $(P(^m(E), \tau_c)$. Hence there is a compact set $K \subset E$ such that

$$\{P \in P(^mE) : \sup_K |P| \leq 1\} \subset \{P \in P(^m(E) : \sup_L |T_P| \leq 1\}.$$

After writting $P = T_P \circ q_m$ we conclude that $q_m(K)^0 \subset L^0$ and therefore

$$L \subset L^{00} \subset q_m(K)^{00} = \bar{\Gamma} \, q_m(K).$$

This completes the proof of Proposition 4.3.

It follows from Proposition 4.3(c) that the mapping in Theorem 4.1 is continuous. And since the inverse mapping is clearly continuous, the proof of Theorem 4.1

is complete.

Motivated by Theorem 4.2 and by the definition of $G^\infty(U)$ we introduce the following auxiliary topology. Let τ_{bc} denote the finest locally convex topology on $H^\infty(U)$ which coincides with τ_c on B_U, or equivalently, on each scalar multiple of B_U. Clearly τ_{bc} exists: the convex, balanced sets $N \subset H^\infty(U)$ such that $N \cap \lambda B_U$ is a 0-neighborhood in $(\lambda B_U, \tau_c)$ for every $\lambda > 0$, form a 0-neighborhood base for τ_{bc}. We shall refer to τ_{bc} as the topology of bounded compact convergence. We shall prove the following theorem.

4.4. Theorem. Let U be an open subset of a Banach space E. Then τ_{bc} is the finest topology on $H^\infty(U)$ which coincides with τ_c on each norm bounded subset of $H^\infty(U)$.

The proof of Theorem 4.4 will also yield the following theorem.

4.5. Theorem. Let U be an open subset of a Banach space E. Then the topology of $(H^\infty(U), \tau_{bc})$ is generated by all the seminorms of the form

$$p(f) = \sup_j \alpha_j \, |f(x_j)|,$$

where (x_j) varies over all sequences in U, and (α_j) varies over all sequences of positive numbers tending to zero.

The key to the proofs of Theorems 4.4 and 4.5 is the following lemma.

4.6. Lemma. Let U be an open subset of a Banach space E. Let N be a subset of $H^\infty(U)$ containing the origin and such that $N \cap j B_U$ is open in $(j B_U, \tau_c)$ for every $j \in \mathbb{N}$. Then there exist $\delta > 0$ and a sequence $(A_j)_{j=0}^\infty$ of finite subsets of U such that

$$N(A_0; \delta) \cap \bigcap_{j=1}^\infty N(A_j; j) \subset N,$$

where, for each $A \subset U$ and $r > 0$, $N(A; r)$ denotes the set

$$N(A; r) = \{ f \in H^\infty(U) : \sup_A |f| \le r \}.$$

Proof. By hypothesis $N \cap B_U$ is open in B_U for the compact–open topology, and therefore for the norm topology. Whence there exists $\delta > 0$ such that

(4.1) $$\delta B_U \subset N.$$

11

We claim that there exists a finite set $A_0 \subset U$ such that

(4.2) $$B_U \cap N(A_0; \delta) \subset N.$$

Otherwise the sets $(B_U \setminus N) \cap N(A; \delta)$ would be nonvoid for every finite set $A \subset U$. These sets are closed in $(B_U \setminus N, \tau_c)$ and have the finite intersection property. Since the set $B_U \setminus N$ is a closed, and therefore compact, subset of (B_U, τ_c), we conclude that the intersection

$$\bigcap_A (B_U \setminus N) \cap N(A; \delta)$$

would be nonvoid. But if f belongs to this intersection, we see that $f \in \delta B_U \setminus N$, contradicting (4.1), and thus proving (4.2). Proceeding by induction we can show the existence of a sequence $(A_j)_{j=1}^\infty$ of finite subsets of U such that

(4.3) $$k B_U \cap N(A_0; \delta) \cap \bigcap_{j=1}^{k-1} N(A_j; j) \subset N$$

for every $k = 2, 3, \ldots$ Since $H^\infty(U) = \cup_{k=1}^\infty k B_U$, the desired conclusion follows.

Proof of Theorem 4.4. To begin with we claim that if $(K_j)_{j=1}^\infty$ is a sequence of compact subsets of U, and $(r_j)_{j=1}^\infty$ is a sequence of positive numbers tending to infinity, then the set

$$\bigcap_{j=1}^\infty N(K_j; r_j)$$

is a 0-neighborhood in $(H^\infty(U), \tau_{bc})$. Indeed, given $\lambda > 0$, choose $k \in \mathbb{N}$ such that $r_j > \lambda$ for every $j > k$. Then

$$\bigcap_{j=1}^\infty N(K_j; r_j) \cap \lambda B_U = \bigcap_{j=1}^k N(K_j; r_j) \cap \lambda B_U,$$

and our claim has been proved.

Now, let N be a subset of $H^\infty(U)$ such that $N \cap \lambda B_U$ is open in $(\lambda B_U, \tau_c)$ for every $\lambda > 0$. To prove the theorem it suffices to show that N is a neighborhood of f in $(H^\infty(U), \tau_{bc})$ for each $f \in N$. If $f = 0$ then the desired conclusion follows at once from Lemma 4.6 and the claim that we have just proved. Finally, the case where $f \in N$ is arbitrary can be reduced to the case $f = 0$ by proving that $(N - f) \cap \lambda B_U$ is open in $(\lambda B_U, \tau_c)$ for every $\lambda > 0$. Now, given $\lambda > 0$ choose $\mu > \lambda$ such that

$$\lambda B_U + f \subset \mu B_U.$$

By hypothesis there exists an open set M in $(H^\infty(U), \tau_c)$ such that

$$N \cap \mu B_U = M \cap \mu B_U.$$

12

Hence

$$(N - f) \cap \lambda B_U = N \cap (\lambda B_U + f) - f$$
$$= M \cap (\lambda B_U + f) - f = (M - f) \cap \lambda B_U,$$

proving that $(N - f) \cap \lambda B_U$ is open in $(\lambda B_U, \tau_c)$. This completes the proof of the theorem.

Proof of Theorem 4.5. In the course of the proof of Theorem 4.4 we have shown that the sets of the form

$$N = \bigcap_{j=1}^{\infty} N(A_j; r_j),$$

where (A_j) varies over all sequences of finite subsets of U, and (r_j) varies over all sequences of positive numbers tending to infinity, form a 0–neighborhood base in $(H^\infty(U), \tau_{bc})$. Since the Minkowski functional of N is the seminorm

$$p_N(f) = \sup_j (r_j^{-1} \sup_{A_j} |f|),$$

the desired conclusion follows at once.

We should remark that Theorem 4.4 follows also from the theory of Saks spaces (see [4, p. 48, Corollary 4.2]), but the proof that we have given here has the advantage of yielding, as a bonus, a proof of Theorem 4.5. In any case, what lies behind each of the proofs is the idea of the proof of the classical Banach–Dieudonné theorem.

4.7. Proposition. Let U be an open subset of a Banach space E. Then:

    (a) $H^\infty(U)$ and $(H^\infty(U), \tau_{bc})$ have the same bounded sets.

    (b) $G^\infty(U) = (H^\infty(U), \tau_{bc})'_b$

    (c) The mapping

$$f \in (H^\infty(U), \tau_{bc}) \longrightarrow T_f \in G^\infty(U)'_c$$

is a topological isomorphism.

    (d) For each compact set $L \subset G^\infty(U)$, there are a sequence $(x_j) \subset U$ and a sequence $(\alpha_j) \in c_0$ with $\alpha_j > 0$ such that

$$L \subset \overline{\Gamma}\{\alpha_j g_U(x_j) : j \in \mathbb{N}\}.$$

13

Proof. (a) It follows from the definitions of $G^\infty(U)$ and τ_{bc} that $G^\infty(U) = (H^\infty(U), \tau_{bc})'$ algebraically. Hence a set $(f_\alpha) \subset (H^\infty(U), \tau_{bc})$ is bounded if and only if it is $\sigma(H^\infty(U), G^\infty(U))$-bounded. But since $H^\infty(U)$ is isometrically isomorphic to $G^\infty(U)'$ and $G^\infty(U)$ is a Banach space, this means that (f_α) is norm bounded in $H^\infty(U)$.

(b) This follows at once from (a).

(c) By Theorem 4.4 a set $N \subset (H^\infty(U), \tau_{bc})$ is open whenever $N \cap j\, B_U$ is open in $j\, B_U$, for the induced topology, for every $j \in \mathbb{N}$. Hence an application of [12, Theorem 4.1] shows that the evaluation mapping

$$(H^\infty(U), \tau_{bc}) \longrightarrow ((H^\infty(U), \tau_{bc})'_b)'_c$$

is a topological isomorphism. After combining this with (b), we get the desired conclusion. Incidentally, [12, Theorem 4.1] gives another proof of (a).

(d) If L is a compact subset of $G^\infty(U)$, then the polar L^0 is a 0-neighborhood in $G^\infty(U)'_c$ and then, by (c), the set

$$\{f \in H^\infty(U) : T_f \in L^0\}$$

is a 0-neighborhood in $(H^\infty(U), \tau_{bc})$. Then by Theorem 4.5 there are a sequence $(x_j) \subset U$ and a sequence $(\alpha_j) \in c_0$ with $\alpha_j > 0$ such that

$$\{f \in H^\infty(U) : \sup_j \alpha_j |f(x_j)| \le 1\} \subset \{f \in H^\infty(U) : \sup_L |T_f| \le 1\}.$$

After writting $f = T_f \circ g_U$ we conclude that

$$\{\alpha_j g_U(x_j) : j \in \mathbb{N}\}^0 \subset L^0$$

and the desired conclusion follows from the bipolar theorem.

Now we can solve the problem posed at the beginning of this section.

4.8. Theorem. Let E and F be Banach spaces, and let U be an open subset of E. Let τ_γ denote the locally convex topology on $H^\infty(U; F)$ generated by all the seminorms of the form

$$p(f) = \sup_j \alpha_j \|f(x_j)\|,$$

where (x_j) varies over all sequences in U, and (α_j) varies over all sequences of positive numbers tending to zero. Then the mapping

$$f \in (H^\infty(U; F), \tau_\gamma) \longrightarrow T_f \in (L(G^\infty(U); F), \tau_c)$$

14

is a topological isomorphism.

Proof. If L is a compact subset of $G^\infty(U)$, then by Proposition 4.7 there are a sequence $(x_j) \subset U$ and a sequence $(\alpha_j) \in c_0$ with $\alpha_j > 0$ such that

$$L \subset \overline{\Gamma} \{\alpha_j g_U(x_j) : j \in I\!\!N\}.$$

Hence

$$\sup_{u \in L} \|T_f u\| \le \sup_j \|T_f(\alpha_j g_U(x_j))\| = \sup_j \alpha_j \|f(x_j)\|$$

for every $f \in H^\infty(U; F)$, and the mapping $f \to T_f$ is $\tau_\gamma - \tau_c$-continuous. Since the sequence $(\alpha_j g_U(x_j))$ tends to zero in $G^\infty(U)$, the inverse mapping is continuous as well. This completes the proof.

The topologies τ_γ, τ_c and τ_{bc} are related as follows.

4.9. Proposition. Let E and F be Banach spaces, and let U be an open set in E. Then:

  (a) $\tau_\gamma \ge \tau_c$ on $H^\infty(U; F)$.

  (b) τ_γ coincides with τ_c on each norm bounded subset of $H^\infty(U; F)$.

  (c) $\tau_\gamma = \tau_{bc}$ on $H^\infty(U)$.

  (d) If U is bounded, then τ_γ coincides with τ_c on $P(^m E; F)$, for each $m \in I\!\!N$.

Proof. (a) If K is a compact subset of U, then by Proposition 4.7 there are a sequence $(x_j) \subset U$ and a sequence $(\alpha_j) \in c_0$ with $\alpha_j > 0$ such that

$$g_U(K) \subset \overline{\Gamma} \{\alpha_j g_U(x_j) : j \in I\!\!N\}.$$

Hence it follows that

$$\sup_{x \in K} \|f(x)\| \le \sup_j \alpha_j \|f(x_j)\|$$

for every $f \in H^\infty(U; F)$.

  (b) Let $p(f) = \sup_j \alpha_j \|f(x_j)\|$, where $(x_j) \subset U$ and $(\alpha_j) \in c_0$, with $\alpha_j > 0$. Given $\lambda > 0$ and $\varepsilon > 0$ choose $k \in I\!\!N$ such that $2\alpha_j \lambda < \varepsilon$ for every $j > k$. Then for $f, g \in H^\infty(U; F)$ with $\|f\| < \lambda$ and $\|g\| < \lambda$ we have that $p(f - g) < \varepsilon$ whenever $\alpha_j \|f(x_j) - g(x_j)\| < \varepsilon$ for $j = 1, \ldots, k$. This shows that the restriction of p to each norm bounded subset of $H^\infty(U; F)$ is τ_c-continuous.

15

(c) This follows from Theorem 4.5 and the definition of τ_γ.

(d) Let $p(f) = \sup_j \alpha_j \|f(x_j)\|$, where $(x_j) \subset U$ and $(\alpha_j) \in c_0$, with $\alpha_j > 0$. Let $\beta_j = \sqrt[m]{\alpha_j}$ for every $j \in \mathbb{N}$. Then $p(P) = \sup_j \|P(\beta_j x_j)\|$ for every $P \in P(^m E; F)$, and thus $p|P(^m E; F)$ is τ_c-continuous.

From Theorem 4.8 and Proposition 4.9 we get at once the following corollary.

4.10. Corollary. Let E and F be Banach spaces, and let U be an open set in E. Then:

(a) The mapping

$$T \in (L(G^\infty(U); F), \tau_c) \longrightarrow T \circ g_U \in (H^\infty(U; F), \tau_c)$$

is continuous.

(b) The restriction of the inverse mapping

$$f \in (H^\infty(U; F), \tau_c) \longrightarrow T_f \in (L(G^\infty(U); F), \tau_c)$$

to each norm bounded subset of $H^\infty(U; F)$ is continuous.

We should remark that assertion (a) in Corollary 4.10 is indeed obvious, whereas assertion (b) is not obvious, but can be proved directly after Theorem 2.1, without recourse to the topologies τ_{bc} or τ_γ.

We end this section with an explicit description of the dual of $(H^\infty(U; F), \tau_\gamma)$.

4.11. Theorem. Let E and F be Banach spaces, and let U be an open set in E. Then the dual of $(H^\infty(U; F), \tau_\gamma)$ consists of all linear functionals of the form

$$u(f) = \sum_{j=1}^\infty \psi_j \circ f(x_j),$$

where (x_j) varies over all sequences in U, and (ψ_j) varies over all sequences in F' with $\sum_{j=1}^\infty \|\psi_j\| < \infty$.

Proof. We first assume that u has such a representation. Since $\sum_{j=1}^\infty \|\psi_j\| < \infty$, we can find a sequence (λ_j) of positive numbers tending to infinity such that $\sum_{j=1}^\infty \lambda_j \|\psi_j\| = c < \infty$. Then

$$|u(f)| \leq \sum_{j=1}^\infty \|\psi_j\| \, \|f(x_j)\| \leq c \sup_j \lambda_j^{-1} \|f(x_j)\|$$

16

for every $f \in H^\infty(U; F)$, proving that u is τ_γ-continuous.

Conversely, assume that u is a continuous-linear functional on $(H^\infty(U; F), \tau_\gamma)$. Then there are a sequence $(x_j) \subset U$ and a sequence $(\alpha_j) \in c_0$ with $\alpha_j > 0$ such that

$$|u(f)| \leq \sup_j \alpha_j \|f(x_j)\|$$

for every $f \in H^\infty(U; F)$. Now, it follows from the very definition of τ_γ that the linear mapping

$$v : f \in (H^\infty(U; F), \tau_\gamma) \longrightarrow (\alpha_j f(x_j))_{j=1}^\infty \in c_0(F)$$

is continuous. Since $|u(f)| \leq \|v(f)\|$ for every $f \in H^\infty(U; F)$; we may define a continuous linear functional w on $v(H^\infty(U; F))$ by $w(v(f)) = u(f)$ for every $f \in H^\infty(U; F)$. By the Hahn–Banach theorem w admits a continuous linear extension to $c_0(F)$. Since $c_0(F)' = \ell^1(F')$, there is a sequence $(\psi_j) \in \ell^1(F')$ such that

$$u(f) = w(v(f)) = \sum_{j=1}^\infty \alpha_j \psi_j \circ f(x_j)$$

for every $f \in H^\infty(U; F)$. This completes the proof.

4.12. Corollary. Let U be an open subset of a Banach space E. Then $G^\infty(U)$ consists of all linear functional $u \in H^\infty(U)'$ of the form

$$u(f) = \sum_{j=1}^\infty \alpha_j f(x_j),$$

where (x_j) varies over all sequences in U, and (α_j) varies over all absolutely summable sequences of complex numbers.

Proof. By Propositions 4.7 and 4.9, $G^\infty(U) = (H^\infty(U), \tau_\gamma)'$. Thus it suffices to apply Theorem 4.11.

5. The Approximation Property

To begin with we give necessary and sufficient conditions for the spaces $Q(^mE)$ and $P(^mE)$ to have the approximation property. These are polynomial analogues of classical results of A. Grothendieck [8, pp. 164-167, Propositions 35 et 36] (see also [10, pp. 32-33, Theorems 1.e.4 and 1.e.5]), which will be used without further reference. Our results in this section complement results of R. Aron and M. Schottenloher [2].

5.1. Theorem. Let E be a Banach space and let $m \in \mathbb{N}$. Then the following conditions are equivalent:

(a) E has the approximation property.

(b) For each Banach space F, $P_f({}^m E; F)$ is τ_c–dense in $P({}^m E; F)$.

(c) For each Banach space F, $P({}^m E) \otimes F$ is τ_c–dense in $P({}^m E; F)$.

(d) q_m lies in the τ_c–closure of $P({}^m E) \otimes Q({}^m E)$.

(e) $Q({}^m E)$ has the approximation property.

(f) For each Banach space F, $P({}^m F) \otimes E$ is τ_c–dense in $P({}^m F; E)$.

(g) For each Banach space F, $P({}^m F) \otimes E$ is norm–dense in $P_k({}^m F; E)$.

Proof. (a) $\Rightarrow$ (b): Let $P \in P({}^m E; F)$, let K be a compact subset of E, and let $\varepsilon > 0$. By using a standard compactness argument, we can find $\delta > 0$ such that $\|P(y) - P(x)\| < \varepsilon$ whenever $x \in K$ and $\|y - x\| < \delta$. Since E has the approximation property, there exists $T \in E' \otimes E$ such that $\|Tx - x\| < \delta$ for every $x \in K$. Then $P \circ T \in P_f({}^m E; F)$ and $\|P \circ T(x) - P(x)\| < \varepsilon$ for every $x \in K$.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d): obvious.

(d) $\Rightarrow$ (e): If follows from (d), Proposition 3.1 and Theorem 4.1 that the identity operator on $Q^m(E)$ lies in the τ_c–closure of $Q({}^m E)' \otimes Q({}^m E)$. Hence $Q({}^m E)$ has the approximation property.

(e) $\Rightarrow$ (a): This follows from Proposition 2.6.

(a) $\Rightarrow$ (f): Since E has the approximation property, $Q({}^m F)' \otimes E$ is τ_c–dense in $L(Q({}^m F); E)$ for each Banach space F. By applying Proposition 3.1 and Theorem 4.1 we get (f).

(f) $\Rightarrow$ (a): It follows from (f), Proposition 3.1 and Theorem 4.1 that $Q({}^m F)' \otimes E$ is τ_c–dense in $L(Q({}^m F); E)$ for each Banach space F. To show that E has the approximation property it suffices to prove that $F' \otimes E$ is τ_c–dense in $L(F; E)$ for each Banach space F. Let $A \in L(F; E)$ be given. By Proposition 2.6 there are operators $S \in L(F; Q({}^m F))$ and $T \in L(Q({}^m F); F)$ such that $T \circ S(y) = y$ for every $y \in F$. Then $A \circ T \in L(Q({}^m F); E)$ and therefore there is a net $(B_\alpha) \subset Q({}^m F)' \otimes E$ which

18

converges to $A \circ T$ for τ_c. Whence if follows that the net $(B_\alpha \circ S)$ lies in $F' \otimes E$ and converges to $A \circ T \circ S = A$ for τ_c.

(a) $\Rightarrow$ (g): Since E has the approximation property, $Q(^mF)' \otimes E$ is norm-dense in $L_k(Q(^mF); E)$ for each Banach space F. By applying Propositions 3.1 and 3.4, and Theorem 2.1, we get (g).

(g) $\Rightarrow$ (a): It follows from (g), Propositions 3.1 and 3.4, and Theorem 2.1, that $Q(^mF)' \otimes E$ is norm-dense in $L_k(Q(^mF); E)$ for each Banach space F. Then, by imitating the proof that (f) $\Rightarrow$ (a), we can show that $F' \otimes E$ is norm-dense in $L_k(F; E)$ for each Banach space F. This shows (a) and completes the proof of the theorem.

5.2. Proposition. Let E be a Banach space, and let $m \in \mathbb{N}$. Then $P(^mE)$ has the approximation property if and only if, for each Banach space F, $P(^mE) \otimes F$ is norm-dense in $P_k(^mE; F)$.

Proof. $Q(^mE)'$ has the approximation property if and only if, for each Banach space F, $Q(^mE)' \otimes F$ is norm-dense in $L_k(Q(^mE); F)$. Thus the desired conclusion follows from Propositions 3.1 and 3.4, and Theorem 2.1.

Before presenting the holomorphic analogues of Theorem 5.1 and Proposition 5.2, we give two auxiliary results.

5.3 Lemma. Let F be a Banach space, and let $f(\varsigma) = \sum_{m=0}^{\infty} \varsigma^m c_m$ be an F-valued holomorphic function on an open disc D centered at the origin. For each $m \in \mathbb{N}$ let

$$S_m f(\varsigma) = \sum_{k=0}^{m} \varsigma^k c_k$$

and

$$\sigma_m f(\varsigma) = \frac{1}{m+1} \sum_{k=0}^{m} S_k f(\varsigma) = \sum_{k=0}^{m} \frac{m+1-k}{m+1} \varsigma^k c_k.$$

Then for each $\varsigma \in D$ we have that

$$S_m f(\varsigma) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{it}\varsigma) D_m(t) dt$$

and

$$\sigma_m f(\varsigma) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{it}\varsigma) K_m(t) dt,$$

where $D_m(t)$ and $K_m(t)$ denote the classical Dirichlet kernel and Fejér kernel, respectively.

19

Proof. Since $f(re^{it}) = \sum_{m=0}^{\infty} r^m e^{imt} c_m$, we see that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{it}) e^{-ikt} dt = \begin{cases} r^k c_k & \text{for } k = 0, 1, 2, \dots \\ 0 & \text{for } k = -1, -2, \dots \end{cases}$$

Hence for $k = 1, 2, \dots$ we get that

$$\begin{aligned}
r^k e^{ik\theta} c_k &= \frac{e^{ik\theta}}{2\pi} \int_{-\pi}^{\pi} f(re^{it}) e^{-ikt} dt + \frac{e^{-ik\theta}}{2\pi} \int_{-\pi}^{\pi} f(re^{it}) e^{ikt} dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{it}) (e^{ik(\theta-t)} + e^{ik(t-\theta)}) dt \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(re^{it}) \cos k(t-\theta) dt \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(re^{i(t+\theta)}) \cos kt \, dt.
\end{aligned}$$

Thus

$$S_m f(re^{i\theta}) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(re^{i(t+\theta)})(\frac{1}{2} + \sum_{k=1}^{m} \cos kt) dt.$$

After recalling that

$$D_m(t) = \frac{1}{2} + \sum_{k=1}^{m} \cos kt \quad \text{and} \quad K_m(t) = \frac{1}{m+1} \sum_{k=0}^{m} D_m(t),$$

the desired conclusion follows.

5.4. Proposition. Let E and F be Banach spaces, and let U be a balanced open set in E. Let $f \in H(U; F)$ and let

$$S_m f(x) = \sum_{k=0}^{m} P^k f(0)(x)$$

and

$$\sigma_m f(x) = \frac{1}{m+1} \sum_{k=0}^{m} S_k f(x) = \sum_{k=0}^{m} \frac{m+1-k}{m+1} P^k f(0)(x).$$

Then:

(a) $S_m f \to f$ and $\sigma_m f \to f$ in $(H(U; F), \tau_c)$.

(b) For each $x \in U$ we have that

$$S_m f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{it} x) D_m(t) dt$$

20

and
$$\sigma_m f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{it}x) K_m(t) dt.$$

(c) If $\|f(x)\| \leq c$ for every $x \in U$, then $\|\sigma_m f(x)\| \leq c$ for every $x \in U$ and $m \in \mathbb{N}$, and $\sigma_m f \to f$ in $(H^\infty(U;F), \tau_\gamma)$.

Proof. (a) is clear. To show (b) fix $x \in U$ and apply the preceding lemma to the function
$$g(\varsigma) = f(\varsigma x) = \sum_{m=0}^{\infty} \varsigma^m P^m f(0)(x).$$

The first assertion in (c) follows from (b) and the facts that $K_m(t) \geq 0$ for every t and $\frac{1}{\pi} \int_{-\pi}^{\pi} K_m(t) dt = 1$ (see [17, p. 45]). Finally, the second assertion in (c) follows from (a) and Proposition 4.9.

5.5. Theorem. Let E be a Banach space, and let U be a balanced, bounded, open set in E. Then the following conditions are equivalent:

(a) E has the approximation property.

(b) For each Banach space F, $P_f(E;F)$ is τ_γ–dense in $H^\infty(U;F)$.

(c) For each Banach space F, $H^\infty(U) \otimes F$ is τ_γ–dense in $H^\infty(U;F)$.

(d) g_U lies in the τ_γ–closure of $H^\infty(U) \otimes G^\infty(U)$.

(e) $G^\infty(U)$ has the approximation property.

(f) For each Banach space F and each open set $V \subset F$, $H^\infty(V) \otimes E$ is τ_γ–dense in $H^\infty(V;E)$.

(g) For each Banach space F and each open set $V \subset F$, $H^\infty(V) \otimes E$ is norm–dense in $H_K^\infty(V;E)$.

(h) The identity mapping on U lies in the τ_γ–closure of $H^\infty(U) \otimes E$.

Proof. (a) $\Rightarrow$ (b): Let $f \in H^\infty(U;F)$ and let p be a continuous seminorm on $(H^\infty(U;F), \tau_\gamma)$. By Proposition 5.4 we can find $P \in P(E;F)$ such that $p(P - f) < 1/2$. Since, by Proposition 4.9, τ_γ coincides with τ_c on $P(^m E;F)$ for each $m \in \mathbb{N}$, repeated applications of Theorem 5.1 yield a polynomial $Q \in P_f(E;F)$ such that $p(Q - P) < 1/2$. Thus $p(Q - f) < 1$ and (b) has been proved.

21

The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a), (a) $\Longleftrightarrow$ (f) and (a) $\Longleftrightarrow$ (g) can be proved by following the footsteps of the proof of Theorem 5.1.

Finally it is clear that (c) $\Rightarrow$ (h) and to complete the proof of the theorem we shall prove that (h) $\Rightarrow$ (c). Let $f \in H^\infty(U;F)$ and let p be a continuous seminorm on $(H^\infty(U;F), \tau_\gamma)$. We wish to find $g \in H^\infty(U) \otimes F$ such that $p(g-f) < 1$. We may assume that

$$p(h) = \sup_j \alpha_j \|h(x_j)\|$$

for every $h \in H^\infty(U;F)$, where $(x_j) \subset U$ and $(\alpha_j) \in c_0$, with $\alpha_j > 0$. By Proposition 5.4 there exists $P \in P(E;F)$ such that

(5.1) $$p(P - f) < \frac{1}{2}.$$

Write $P = P_0 + \cdots + P_m$, with $P_k \in P(^kE;F)$. Certainly $P_0 \in H^\infty(U) \otimes F$. For each $k = 1, \ldots, m$ we shall find $u_k \in H^\infty(U) \otimes E$ such that

(5.2) $$p(P_k \circ u_k - P_k) < \frac{1}{2m}.$$

It will then follow that

$$P_0 + \sum_{k=1}^m P_k \circ u_k \in H^\infty(U) \otimes F$$

and

$$p\left(P_0 + \sum_{k=1}^m P_k \circ u_k - f\right) \leq \sum_{k=1}^m p(P_k \circ u_k - P_k) + p(P - f) < 1,$$

thus proving (c). Now, fix k with $1 \leq k \leq m$, let $\beta_j = \sqrt[k]{\alpha_j}$ for every $j \in \mathbb{N}$ and let $K = \{\beta_j x_j : j \in \mathbb{N}\} \cup \{0\}$. Since K is compact, we can find $\delta > 0$ such that

$$\|P_k(y) - P_k(x)\| < \frac{1}{2m}$$

whenever $x \in K$ and $\|y - x\| < \delta$. By (h) we can find $u_k \in H^\infty(U) \otimes E$ such that

$$\sup_j \beta_j \|u_k(x_j) - x_j\| < \delta.$$

Hence

$$p(P_k \circ u_k - P_k) = \sup_j \alpha_j \|P_k \circ u_k(x_j) - P_k(x_j)\|$$

$$= \sup_j \|P_k(\beta_j u_k(x_j)) - P_k(\beta_j x_j)\| < \frac{1}{2m}$$

22

and u_k satisfies (5.2). Thus (h) $\Rightarrow$ (c) and the proof of the theorem is complete.

By imitating the proof of Proposition 5.2 we can also obtain the following result.

5.6. Proposition. Let U be an open subset of a Banach space E. Then $H^\infty(U)$ has the approximation property if and only if, for each Banach space F, $H^\infty(U) \otimes F$ is norm–dense in $H_K^\infty(U; F)$.

Since it is still unknown whether $H^\infty(\Delta)$ has the approximation property, Proposition 5.6 may be of some use in this connection.

We next state two results on the metric approximation property, which are parallel to, and somewhat easier to prove than, Theorems 5.1 and 5.5.

5.7. Proposition. Let E be a Banach space and let $m \in \mathbb{N}$. Then the following conditions are equivalent:

(a) E has the metric approximation property.

(b) For each Banach space F, the polynomials of norm ≤ 1 in $P_f(^m E; F)$ are τ_c–dense in the closed unit ball of $P(^m E; F)$.

(c) For each Banach space F, the polynomials of norm ≤ 1 in $P(^m E) \otimes F$ are τ_c–dense in the closed unit ball of $P(^m E; F)$.

(d) q_m lies in the τ_c–closure of the polynomials of norm ≤ 1 in $P(^m E) \otimes Q(^m E)$.

(e) $Q(^m E)$ has the metric approximation property.

(f) For each Banach space F, the polynomials of norm ≤ 1 in $P(^m F) \otimes E$ are τ_c–dense in the closed unit ball of $P(^m F; E)$.

5.8. Proposition. Let E be a Banach space, and let U be the open unit ball of E. Then the following conditions are equivalent:

(a) E has the metric approximation property.

(b) For each Banach space F, the polynomials in $P_f(E; F)$ which are bounded by one on U, are τ_c–dense in the closed unit ball of $H^\infty(U; F)$.

(c) For each Banach space F, the mappings of norm ≤ 1 in $H^\infty(U) \otimes F$ are τ_c–dense in the closed unit ball of $H^\infty(U; F)$.

(d) g_U lies in the τ_c–closure of the mappings of norm ≤ 1 in $H^\infty(U) \otimes G^\infty(U)$.

(e) $G^\infty(U)$ has the metric approximation property.

(f) For each Banach space F and each open set $V \subset F$, the mappings of norm ≤ 1 in $H^\infty(V) \otimes E$ are τ_c–dense in the closed unit ball of $H^\infty(V;E)$.

(g) The identity mapping on U lies in the τ_c–closure of the mappings of norm ≤ 1 in $H^\infty(U) \otimes E$.

We should remark that the proof of Proposition 5.8 requires much less machinery than the proof of Theorem 5.5. Indeed, the proof of Theorem 5.5 relies heavily on Theorem 4.8, which was the ultimate goal of Section 4, whereas the proof of Proposition 5.8 relies only on Corollary 4.10, which, as we have remarked already, can be proved directly after Theorem 2.1, without recourse to the topologies τ_{bc} or τ_γ. A similar comment applies to Theorem 5.1 and Proposition 5.7.

By modifying the proof of Proposition 5.7 one can show that if a Banach space E has the λ–bounded approximation property for some $\lambda \geq 1$, then $Q(^m E)$ has the λ^m–bounded approximation property. Whence it follows that $Q(^m E)$ has the bounded approximation property if and only if E has the bounded approximation property. However, I have been unable to solve the following problem.

5.9. Problem. If E is a Banach space with the bounded approximation property, does $G^\infty(U_E)$ have the bounded approximation property?

If E is a Banach space with the bounded approximation property, J. Lindenstrauss and L. Tzafriri [10, p. 42, Problem 1.e.21] have asked whether there exists an equivalent norm $||| \cdot |||$ on E such that $(E, ||| \cdot |||)$ has the metric approximation property. In view of Proposition 5.8, this problem is equivalent to the following.

5.10. Problem. If E is a Banach space with the bounded approximation property, does there exist a convex, balanced, bounded, open set $U \subset E$ such that $G^\infty(U)$ has the metric approximation property?

References

[1] J. M. ANSEMIL – S. DINEEN, Locally determining sequences in infinite dimensional spaces. Note Mat. 7 (1987), 41-45.

[2] A. ARON – M. SCHOTTENLOHER, Compact holomorphic mappings on Banach spaces and the approximation property. J. Funct. Anal. 21 (1976), 7-30.

[3] A. CHIACCHIO – M. MATOS – S. ROVERSI, On best approximation by rational and holomorphic mappings between Banach spaces. J. Approx. Theory, to appear.

[4] J. B. COOPER, Saks Spaces and Applications to Functional Analysis, second edition. North–Holland Mathematics Studies, Vol. 139. North–Holland, Amsterdam, 1987.

[5] J. DIESTEL, Sequences and Series in Banach Spaces. Graduate Texts in Mathematics, Vol. 92, Springer, Berlin, 1984.

[6] S. DINEEN, Complex Analysis in Locally Convex Spaces. North–Holland Mathematics Studies, Vol. 57, North–Holland, Amsterdam, 1981.

[7] K. FLORET, Elementos de posto finito em produtos tensoriais topológicos. Atas 24º Seminário Brasileiro de Análise, pp. 189-195. Sociedade Brasileira de Matemática, 1986.

[8] A. GROTHENDIECK, Produits Tensoriels Topologiques et Espaces Nucléaires. Memoires of the American Mathematical Society, number 16. American Mathematical Society, Providence, Rhode Island, 1955.

[9] J. HORVATH, Topological Vector Spaces and Distributions, Vol. I. Addison–Wesley, Reading, Massachusette, 1966.

[10] J. LINDENSTRAUSS – L. TZAFRIRI, Classical Banach Spaces I. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 92. Springer, Berlin, 1977.

[11] J. MUJICA, Complex homomorphisms of the algebras of holomorphic functions on Fréchet spaces. Math. Ann. 241 (1979), 73-82.

[12] J. MUJICA, A Banach–Dieudonné theorem for germs of holomorphic functions. J. Funct. Anal. 57 (1984), 31-48.

[13] J. MUJICA, Complex Analysis in Banach Spaces. North–Holland Mathematics Studies, Vol. 120. North–Holland, Amsterdam, 1986.

[14] K. F. NG, On a theorem of Dixmier. Math. Scand. 29 (1971), 279-280.

[15] R. RYAN, Applications of topological tensor products to infinite dimensional holomorphy. Ph.D. thesis, Trinity College Dublin, 1980.

[16] H. H. SCHAEFER, Topological Vector Spaces, third printing. Graduate Texts in Mathematics, Vol. 3. Springer, Berlin, 1971.

[17] A. ZYGMUND, Trigonometrical Series. Dover, New York, 1955.

Instituto de Matemática
Universidade Estadual de Campinas
Caixa Postal 6065
13.081 – Campinas, SP
Brazil

RELATÓRIOS TÉCNICOS — 1988