LIMITING PROPERTIES OF THE EMPIRICAL PROBABILITY GENERATING FUNCTION OF STATIONARY RANDOM SEQUENCES AND PROCESSES

by

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ABSTRACT

Let $X_1, X_2, \ldots$ be a sequence of strictly stationary discrete random variables with probability generating function $\phi(t)$ and let $\hat{\phi}_n(t) = n^{-1}\sum_{i=1}^{n} t^{X_i}$ be the empirical probability generating function of the first n random variables. A strong law of large numbers and the weak convergence of $n^{1/2}(\hat{\phi}_n - \phi)$ to a Gaussian process are discussed. These results are extended to a sequence of stationary multivariate discrete random variables and to integer valued stationary stochastic processes. Applications to the statistical analysis of discrete data are discussed.

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1. INTRODUCTION. Let $\{P_k\}_{k\geq 0}$ be a discrete distribution with probability generating function

(1.1) $$\phi(t) = \sum_{k=0}^{\infty} t^k P_k \qquad |t| \leq 1$$

and let $X_1, X_2, \ldots$ be a sequence of nonnegative integer valued random variables with the same distribution $\{P_k\}_{k\geq 0}$. For each $n \geq 1$ let

(1.2) $$\hat{\phi}_n(t) = n^{-1} \sum_{j=1}^{n} t^{X_j} \qquad |t| \leq 1$$

be the empirical probability generating function (epgf) of the first n observations.

Under the assumption of $\{X_i\}_{i\geq 1}$ independent, the use of some sample generating functions in statistical inference is by now well known. Thus, for example, Press (1972), Heathcote (1972) and Feuerverger and Mureika (1977) have suggested the use of the empirical characteristic function $\hat{c}_n(t) = n^{-1}\sum_{j=1}^{n} exp(itX_j)$ to construct statistical procedures for stable laws, while Epps et al (1982) have proposed the empirical moment generating function $\hat{m}_n(t) = n^{-1}\sum_{j=1}^{n} exp(tX_j)$ to test hypotheses of goodeness of fit in the presence of nuisance parameters. The limiting behavior of $\hat{c}_n$ has been studied, among others, by Feuerverger and Mureika (1977) and Csörgo (1981a, 1981b), while Maĭboroda (1986) has proved a law of large numbers and a central limit theorem for $\hat{m}_n$. Recently, Kocherlakota and Kocherlakota (1986) and O'Reilly et al (1988) have used the empirical probability generating function $\hat{\phi}_n$ to construct goodness of fit tests for discrete random variables, leading to easier procedures than with the use of $\hat{c}_n$ or $\hat{m}_n$.

The purpose of this paper is to study the limiting behavior of the empirical probability generating function $\hat{\phi}_n$ of a stationary sequence of integer valued random variables $X_1, X_2, \ldots$. In Section 2 we prove a law of large numbers (Theorem 1) for $\hat{\phi}_n$ and a central limit theorem (Theorem 2) for the empirical probability generating process $Y_n(t) = n^{1/2}(\hat{\phi}_n(t) - \phi(t))$ in the Banach space of continuous functions on $[-1,1]$. In Section 3 we discuss extensions of Theorems 1 and 2 to a stationary sequence of multivariate discrete random variables and to integer valued stationary stochastic processes. Applications to the statistical analysis of discrete data are discussed in Section 4.

2. **THE EPGF OF A STATIONARY SEQUENCE.** Let $(\Omega, \mathcal{F}, P)$ be a probability space where there is defined a stationary sequence $\{X_n\}_{n\geq1}$ of nonnegative integer valued random variables with the same distribution $\{P_k\}_{k\geq0}$ and probability generating function $\phi(t)$. Let $\hat{\phi}_n(t)$ be the epgf given by (1.2) and

$$(2.1) \qquad Y_n(t) = n^{1/2}(\hat{\phi}_n(t) - \phi(t)) \qquad |t| \leq 1$$

be the epgf process. We shall denote by $C[-1,1]$ the Banach space of real valued continuous functions on the interval $[-1,1]$ with norm $\|f\| = \sup_{-1\leq t\leq1} |f(t)|$. Clearly ϕ_n, ϕ and Y_n belong to $C[-1,1]$.

The first result of this paper gives the $\|\cdot\|$-consistency of $\hat{\phi}_n$ as an estimator of ϕ.

THEOREM 1. Let $\{X_n\}_{n\geq1}$ be a strictly stationary ergodic integer valued random sequence. Then

$$(2.2) \qquad \|\hat{\phi}_n - \phi\| \to 0 \qquad \text{a.s.}$$

Proof. Define for $n \geq 1$

$$P_{k,n} = n^{-1}\sum_{j=1}^{n} 1_{\{X_j=k\}} \qquad k \geq 0 . \tag{2.3}$$

Then the epgf can be expressed as

$$\hat{\phi}_n(t) = \sum_{k=0}^{\infty} t^k P_{k,n} \qquad t \in [-1,1] \tag{2.4}$$

and therefore for $t \in [-1,1]$ and $n \geq 1$

$$|\hat{\phi}_n(t)-\phi(t)| \leq \sum_{k=0}^{\infty} |P_{k,n} - P_k| \leq 2 . \tag{2.5}$$

Since the stationary sequence $\{X_n\}_{n\geq 1}$ is ergodic, we have that for any $k \geq 0$

$$P_{k,n} = n^{-1}\sum_{i=1}^{n} 1_{\{k\}}(X_i) \underset{n\to\infty}{\longrightarrow} P(X_1=k) = P_k \quad \text{a.s.} \tag{2.6}$$

Then by the dominated convergence theorem, from (2.5) and (2.6) we obtain that

$$\sup_{-1\leq t\leq 1} |\hat{\phi}_n(t)-\phi(t)| \underset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s.} \tag{2.7}$$

and the proof is complete. ■

The next step is to study the central limit theorem or weak convergence of the processes $\{Y_n\}_{n\geq 1}$ in the Banach space $C[-1,1]$, under some mixing conditions on the sequence of random variables $\{X_n\}_{n\geq 1}$. Following Bradley (1986) we introduce some notation to be used in the remaining of the work: For two σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ of the basic probability space $(\Omega,\mathcal{F},P)$ define the dependence coefficients

$$\psi(\mathcal{F}_1,\mathcal{F}_2) := \sup_{A\in\mathcal{F}_1, B\in\mathcal{F}_2} |P(A\cap B)-P(A)P(B)|/(P(A)P(B)) \tag{2.8}$$

$$\varphi(\mathcal{F}_1,\mathcal{F}_2) := \sup_{A\in\mathcal{F}_1, B\in\mathcal{F}_2, P(A)>0} |P(B|A)-P(B)| \tag{2.9}$$

and let $\mathcal{F}_j^k = \sigma(X_i : j \leq i \leq k)$. For each $n = 1,2,3,\ldots$ define

(2.10) $$\psi(n) := \sup_{j\geq 1} \psi(\mathcal{F}_1^j, \mathcal{F}_j^\infty)$$

(2.11) $$\varphi(n) := \sup_{j\geq 1} \varphi(\mathcal{F}_1^j, \mathcal{F}_j^\infty)$$

We recall that the sequence of random variables $\{X_n\}_{n\geq 1}$ is said to be φ-mixing if $\lim_{n\to\infty} \varphi(n) = 0$ and ψ-mixing if $\lim_{n\to\infty} \psi(n) = 0$. Also the sequence $\{X_n\}_{n\geq 1}$ is φ-mixing (see Peligrad (1986)) if and only if

(2.12) $$|EXY-EXEY| \leq 2\phi^{1/2}(n)\|X\|_1\|Y\|_\infty$$

for every $X \in L_1(\mathcal{F}_1^m)$, $Y \in L_\infty(\mathcal{F}_{n+m}^\infty)$ and for every m; and is ψ-mixing (see Peligrad (1986)) if and only if

(2.13) $$|EXY-EXEY| \leq \psi(n)E|X|E|Y|$$

for every $X \in L_1(\mathcal{F}_1^m)$, $Y \in L_1(\mathcal{F}_{n+m}^\infty)$ and for every m.

Our second result of this work provides a central limit theorem for the processes $\{Y_n\}_{n\geq 1}$ under two mixing conditions on the sequence of random variables $\{X_n\}_{n\geq 1}$. As usual we denote weak convergence by $\Rightarrow$.

THEOREM 2. Let $\{X_n\}_{n\geq 1}$ be a strictly stationary random sequence with probability generating function $\phi(t)$ and $EX_1^2 < \infty$.

a) Assume $\{X_n\}_{n\geq 1}$ is a φ-mixing and $\sum_{i=1}^{\infty} \varphi^{1/2}(i) < \infty$. Then $Y_n \Rightarrow Y$ in $C[-1,1]$ where Y is a Gaussian process with mean zero and covariance

(2.14) $$K(s,t) = \phi(st)-\phi(s)\phi(t) + \sum_{k=1}^{\infty} r_k(s,t) + \sum_{k=1}^{\infty} r_k(t,s)$$

where $r_k(s,t) = COV(s^{X_1}, t^{X_{1+k}})$ $k \geq 0$.

b) Assume $\{X_n\}$ is ψ-mixing and $\sum_{i=1}^{\infty} \psi(i) < \infty$. Then $Y_n \Rightarrow Y$ in $C[-1,1]$ where Y is a Gaussian process, with mean zero and covariance (2.14).

Proof. We first observe that if we define the centered sequence of Banach space valued random variables $Z_i(t) = t^{X_i} - \phi(t)$ $|t| \le 1$ and $\mathcal{F}_k^j = \sigma(Z_i : k \le i \le j)$, then $\mathcal{F}_k^j \subset \mathcal{F}_k^j$ and if $\{X_n\}$ is ψ-mixing (repectively φ-mixing) then $\{Z_n\}$ is ψ-mixing (respectively φ-mixing) with the dependence coefficients for the sequence $\{Z_n\}$ being no greater that the corresponding ones for $\{X_n\}$.

Let $S_n(t) = \sum_{i=1}^{n} Z_i(t)$ and $\mathcal{P}_k(t) = E(Z_1(t)Z_{k+1}(t))$ $k = 0,1,2,\dots$.

Since the sequence $\{Z_i(t)_{n\ge1}\}$ is stationary

(2.15) $$\sigma_n^2(t) = VAR(S_n(t)) = n\mathcal{P}_0(t) + 2\sum_{k=1}^{n-1}(n-k)\mathcal{P}_k(t) .$$

On the other hand using (2.12) we have that

(2.16) $$\sum_{k=1}^{\infty}\mathcal{P}_k(t) \le 2\mathcal{P}_0(t)\sum_{k=1}^{\infty}\varphi^{1/2}(k) \qquad |t| \le 1$$

and using (2.13) we obtain

(2.17) $$\sum_{k=1}^{\infty}\mathcal{P}_k(t) \le \mathcal{P}_0(t)\sum_{k=1}^{\infty}\psi(k) \qquad |t| \le 1 .$$

Then

(2.18) $$n^{-1}\sigma_n^2(t) \to \sigma^2(t) \quad \text{as} \quad n \to \infty$$

where

(2.19) $$\sigma^2(t) = \mathcal{P}_0(t) + 2\sum_{k=1}^{\infty}\mathcal{P}_k(t) \qquad |t| \le 1 .$$

The weak convergence of the finite dimensional distributions of $Y_n(t) = n^{-1/2}S_n(t)$ to those of a Gaussian process follows using the Cramer-Wold device and, under assumption (a), Theorem 20.1 in Billingsley (1968), and under assumption (b), the main theorem in Nakhapetiyan (1981). To prove tightness of $\{Y_n(t)\}_{n\ge1}$ in $C[-1,1]$ we shall apply Theorem 12.3 in Billingsley (1968). Using (2.12) we have

(2.20) $$E(Y_n(t)-Y_n(s))^2 \le 2E(Z_1(t)-Z_1(s))^2\sum_{k=1}^{\infty}\varphi^{1/2}(k) \quad s,t \in [-1,1]$$

and using (2.13)

(2.21) $$E(Y_n(t)-Y_n(s))^2 \le E(Z_1(t)-Z_1(s))^2\sum_{k=1}^{\infty}\psi(k) \quad s,t \in [-1,1] .$$

Next it is easily shown that

$$(2.22)\qquad E(Z_1(t)-Z_1(s))^2 \le \theta(t-s)^2$$

where $\theta = \sum_{k=1}^{\infty} k^2 P_k < \infty$. Then by Theorem 12.3 in Billingsley (1968) $\{Y_n\}_{n\ge1}$ is tight in $C[-1,1]$ under assumptions a) or b). Finally by Theorem 8.1 of Billingsley $Y_n \Rightarrow Y$ in $C[-1,1]$. To find the covariance of Y we observe that for $n \ge 1$ and $s,t \in [-1,1]$

$$(2.23)\qquad E(Y_n(t)Y_n(s)) = r_0(s,t) + n^{-1}\sum_{k=1}^{n-1}(n-k)r_k(s,t) + n^{-1}\sum_{k=1}^{n-1}(n-k)r_k(t,s)$$

As in proving (2.16) and (2.17) we can show that

$$(2.24)\qquad \sum_{k=1}^{\infty} r_k(s,t) \le 2r_0(s,t)\sum_{k=1}^{\infty}\varphi^{1/2}(k) \qquad s,t \in [-1,1]$$

$$(2.25)\qquad \sum_{k=1}^{\infty} r_k(s,t) \le r_0(s,t)\sum_{k=1}^{\infty}\psi(k) \qquad s,t \in [-1,1]$$

and therefore $E(Y_n(t)Y_n(s)) \xrightarrow[n\to\infty]{} K(s,t)$ where K is given by (2.14) ∎

For the sake of completness we write the conclusions of Theorems 1 and 2 for a sequence of independent random variables.

THEOREM 3. Let $\{X_n\}_{n\ge1}$ be a sequence of independent integer valued random variables with p.g.f. $\phi(t)$ and $EX_1^2 < \infty$. Then

a) $\|\hat{\phi}_n-\phi\| \xrightarrow[n\to\infty]{} 0$ a.s.

b) $Y_n \Rightarrow Y$ in $C[-1,1]$ where $Y(t)$ is a zero mean Gaussian process with covariance

$$(2.26)\qquad K(s,t) = \phi(st)-\phi(s)\phi(t) \qquad s,t \in [-1,1]\ .$$

3. EXTENSIONS TO MULTIVARIATE DISTRIBUTIONS AND STATIONARY STOCHASTIC PROCESSES. We start this section by presenting extensios of Theorems 1 and 2 to multivariate discrete distributions. Let $\underline{X} = (X_1,\ldots,X_r)$ be an r-dimensional discrete random vector with probability generating function

$$(3.1) \qquad \phi(\underline{t}) = E(t_1^{X_1}\ldots t_r^{X_r}) \quad \underline{t} = (t_1,\ldots,t_r)$$

and let $\underline{X}^n = (X_1^n,\ldots,X_r^n)$ $n \geq 1$ be a multivariate stationary independent sequence of random variables with the same p.g.f. ϕ. Define the multivariate empirical probability generating function of the first n observations as

$$(3.2) \qquad \hat{\phi}_n(t) = \frac{1}{n}\sum_{i=1}^{n} t_1^{X_1^i}\ldots t_r^{X_r^i} \quad \underline{t} = (t_1,\ldots,t_r) \quad |t| \leq 1 .$$

Let $C([-1,1]^r)$ be the metric space of continuous functions from $[-1,1]^r$ to $\mathbb{R}$ with the norm

$$(3.3) \qquad |||f||| = \sup_{\underline{t}\in[-1,1]^r} |f(\underline{t})| .$$

We introduce the following notation: for $\underline{t},\underline{s} \in [-1,1]^r$ $\underline{t}\cdot\underline{s} = (t_1s_1,\ldots, t_rs_r)$.

THEOREM 4. If the sequence $\underline{X}^n$ $n \geq 1$ is ergodic $|||\hat{\phi}_n-\phi||| \to 0$ a.s. as $n \to \infty$.

Proof. Let $P_{k_1\ldots k_r} = P(X_1=k_1,\ldots,X_r=k_r)$ and

$$(3.4) \qquad \hat{P}^n_{k_1\ldots k_r} = \frac{1}{n}\sum_{i=1}^{n} 1\left\{X_1^i=k_1,\ldots,X_r^i=k_r\right\}.$$

Then

$$(3.5) \qquad \phi(\underline{t}) = \sum_{k_1\ldots k_r=0}^{\infty} t_1^{k_1}\ldots t_r^{k_r} P_{k_1\ldots k_r} \qquad |\underline{t}| \leq 1$$

and

$$(3.6) \qquad \hat{\phi}_n(\underline{t}) = \sum_{k_1\ldots k_r=0}^{\infty} t_1^{k_1}\ldots t_r^{k_r} \hat{P}_{k_1\ldots k_r} \qquad |\underline{t}| \leq 1 .$$

Since the sequence $\underline{X}^n$ is stationary and ergodic

(3.7) $$\hat{P}^n_{k_1 \ldots k_r} \xrightarrow[n\to\infty]{} P_{k_1 \ldots k_r} \quad \text{a.s.}$$

Then the proof of the theorem ends quite analogous to the proof Theorem 1. ∎

THEOREM 5. Let $Y_n(\underline{t}) = n^{1/2}(\hat{\phi}_n(\underline{t})-\phi(\underline{t}))$ and $EX_i^2 < \infty$ for $i=1,\ldots,r$. Let $\varphi(n)$ and $\psi(n)$ be the dependence coefficients for $\{\underline{X}^n\}_{n\geq 1}$ given by (2.10) and (2.11).

a) If $\{\underline{X}^n\}_{n\geq 1}$ is φ-mixing and $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ then $Y_n \Rightarrow Y$ in $C([-1,1]^r)$ where Y is a Gaussian random field with zero mean and covariance.

(3.8) $$K(\underline{s},\underline{t}) = \phi(\underline{s},\underline{t})-\phi(\underline{s})\phi(\underline{t}) + \sum_{k=1}^{\infty} r_k(\underline{s},\underline{t}) + \sum_{k=1}^{\infty} r_k(\underline{t},\underline{s}) \qquad \underline{s},\underline{t} \in [-1,1]^r$$

where $r_k(\underline{s},\underline{t}) = COV(s_1^{X_1^1}\ldots s_r^{X_r^1}, t_1^{X_1^{1+k}}\ldots t_r^{X_r^{1+k}})$

b) If $\{\underline{X}^n\}_{n\geq 1}$ is ψ-mixing and $\sum_{n=1}^{\infty} \psi(n) < \infty$ then $Y_n \Rightarrow Y$ in $C([-1,1]^r)$ where Y is a Gaussian random field with zero mean and covariance (3.8).

The proof of the theorem is quite analogous to the proof of Theorem 2 using the fact that

(3.9) $$E|Y_1(\underline{t})-Y_1(\underline{s})|^2 \leq \theta^2 e^2(\underline{s},\underline{t})$$

where $e(\underline{s},\underline{t}) = \max_{1\leq i\leq r} |t_i - s_i|$ and

(3.10) $$\theta^2 = \sum_{k=0}^{\infty} k^2(P_k(1)+\ldots+P_k(r)) < \infty .$$

Let us now consider the case of a stationary stochastic process. Let $\{X_\Delta\}_{\Delta\geq 0}$ be a measurable strictly stationary nonnegative integer valued stochastic process and let $\phi(t) = E(t^{X_s})$ $|t| \le 1$ be the probability generating function of X_s. For $S \ge 0$ define the empirical probability generating function

$$\hat{\phi}_S(t) = S^{-1}\int_0^S t^{X_s}ds \quad |t| \le 1 \tag{3.11}$$

and let

$$Y_S(t) = S^{1/2}(\hat{\phi}_S(t)-\phi(t)) \tag{3.12}$$

THEOREM 6.

a) If $\{X_s\}_{s\ge 0}$ is an ergodic process

$$\|\hat{\phi}_S-\phi\| \to 0 \quad \text{a.s.} \quad \text{as } S \to \infty .$$

b) If the process $\{X_s\}_{s\ge 0}$ is φ-mixing and $\int_0^\infty \varphi^{1/2}(s)ds < \infty$ then $Y_n \Rightarrow Y$ in $C[-1,1]$ where Y is a Gaussian process with zero mean and covariance

$$K(t_1,t_2) = \phi(t_1t_2)-\phi(t_1)\phi(t_2) + \int_0^\infty r_s(t_1,t_2)ds + \int_0^\infty r_s(t_2,t_1)ds \tag{3.13}$$

where $r_s(t_1,t_2) = COV(t_1^{X_0}t_2^{X_s})$.

c) If the process $\{X_s\}_{s\ge 0}$ is ψ-mixing and $\int_0^\infty \psi(s)ds < \infty$ then $Y_n \Rightarrow Y$ in $C[-1,1]$ where the process Y is as in (b).

The proof of the theorem is done using the device discussed in Kolmogorov (1931): First deduce the result for the stationary random sequence

$$\tilde{Y}_n(t) = \int_n^{n+1} Y_s(t)ds$$

and use the measurability condition.

4. **APPLICATIONS.** We now mention some applications to statistics. Let $X_1, X_2, \ldots$ be a strictly stationary sequence having a discrete distribution F_0 with probability generating function $\phi(t)$ $|t| \le 1$. The statistics

$$d = n \int_0^1 (\hat{\phi}_n(t) - \phi(t))^2 dt$$

can be used to test a goodness of fit for the specified distribution F_0. For example, if F_0 is the Poisson distribution with parameter $\lambda > 0$ then

$$d = n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} 1/(X_i + X_j + 1) + (2\lambda)^{-1} n[1 - exp(-2\lambda)]$$
$$- 2 \sum_{i=1}^{n} [(-1)^{X_i+1} X_i ! / (\lambda^{X_i+1}) exp(-\lambda)$$
$$+ \sum_{j=0}^{X_i} (-1)^j X_i (X_i - 1) \ldots (X_i - j + 1) / (\lambda^{X_i+1})]$$

In this case, under the assumption of independence and using Theorem 3, the distribution of d has been computed in O'Reilly et al (1988).

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