A FAMILY OF QUASI–NEWTON METHODS WITH DIRECT SECANT UPDATES OF MATRIX FACTORIZATIONS

José Mário Martínez

RELATÓRIO TÉCNICO Nº 04/88

ABSTRACT. We introduce a new family of least-change secant methods for solving systems of nonlinear algebraic equations. At each iteration of the methods of this family the matrix B_k is represented as a product of the form $A_k^{-1} R_k$, where the matrices A_k and R_k have a determined structure and A_{k+1}, R_{k+1} are chosen using a least-change secant rule. The first and second methods of Broyden, Schubert's method, Johnson–Austria's and Chadee's method are members of the family. We prove local and superlinear convergence results. We introduce particular members of the family which seem to be of practical usefulness.

Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
IMECC – UNICAMP
Caixa Postal 6065
13.081 - Campinas, SP
BRASIL

Março – 1988

A FAMILY OF QUASI-NEWTON METHODS WITH DIRECT SECANT UPDATES OF MATRIX FACTORIZATIONS

José Mario Martínez

Dept. of Applied Mathematics
IMECC - UNICAMP
CP - 6065
13081 - Campinas - SP
BRASIL

ABSTRACT. We introduce a new family of least-change secant methods for solving systems of nonlinear algebraic equations. At each iteration of the methods of this family the matrix B_k is represented as a product of the form $A_k^{-1}R_k$, where the matrices A_k and R_k have a determined structure and A_{k+1}, R_{k+1} are chosen using a least-change secant rule. The first and second methods of Broyden, Schubert's method, Johnson-Austria's and Chadee's method are members of the family. We prove local and superlinear convergence results. We introduce particular members of the family which seem to be of practical usefulness.

1. INTRODUCTION

We wish to consider the problem of finding a solution of the system of algebraic nonlinear equations:

$$F(x) = 0 \qquad (1.1)$$

$$F = (f_1, \ldots, f_n)^T$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a C^1 function and its Jacobian matrix is denoted by $J(x)$.

Least change secant update (or Quasi-Newton) methods (see [1,2,3,7-10,18]) are the most successful algorithms designed for solving (1.1). They are based on the Newton-like iteration:

$$x^{k+1} = x^k - B_k^{-1} F(x^k) \tag{1.2}$$

where, for each k, B_{k+1} is the "closest" matrix to B_k which satisfies

$$B_{k+1} \; s = y$$

$$s = x^{k+1} - x^k \;, \; y = F(x^{k+1}) - F(x^k) \tag{1.3}$$

and, perhaps, some additional conditions, such as sparsity patterns or symmetry, using some matrix norm. Equation (1.3) is known as the "secant equation" (or Fundamental Equation of Quasi-Newton methods).

The first method of Broyden ("Broyden's good method") is obtained using the Frobenius norm, and yields (see [9]):

$$B_{k+1} = B_k + \frac{(y - B_k \; s) s^T}{s^T s} \tag{1.4}$$

Very efficient algorithms have been obtained using formula (1.4) (see [18]). The best-known implementations of (1.4) store the Q - R factorization of B_k, and obtain the Q-R factorization of B_{k+1} using $O(n^2)$ arithmetic operations.

It is generally accepted that the "first" method of Broyden is more efficient than the "second" (Broyden's " bad " method), which is based on the formula :

$$B_{k+1} = B_k + \frac{(y - B_k s)\, y^T B_k}{y^T B_k s}. \tag{1.5}$$

This formula is equivalent to:

$$B_{k+1}^{-1} = B_k^{-1} + \frac{(s - B_k^{-1} y) y^T}{y^T y} \tag{1.6}$$

It is easy to verify that in (1.6), B_{k+1}^{-1} is the solution of the problem:

$$\underset{H}{\text{Minimize}} \; \|H - B_k^{-1}\|_F$$

$$\text{s.t. } Hy = s \tag{1.7}$$

The reasons why (1.4) should be better than (1.5)-(1.6) are not yet absolutely clear (see [9]).

Now, when n is large, and J(x) is sparse, it is desirable that the B_k's keep the same pattern of sparsity as J(x). Formula (1.4) doesn't have this property, so it is not well suited for large and sparse problems. This observation gave origin to the "sparse Broyden update" (or "Schubert's update") [2,21,16]. In Schubert's formula, B_{k+1} is chosen as the matrix with the same pattern of sparsity as J(x), satisfying (1.3), which minimizes $\|B - B_k\|_F$.

Unlike the correction (1.4), Schubert's correction is not represented by a rank-one matrix, so it is not easy to obtain a factorization of B_{k+1} using the factorization of B_k, as it so happens to be with (1.4).

This observation motivated Dennis and Marwil [6] to introduce the first Quasi-Newton method with a direct update of the L - U factorization of B_k. Essentially, the idea of Dennis-Marwil's method is the following: Set $B_k = L_k U_k$ and keep L_k fixed throughout the process, choosing U_{k+1}, for $k=0,1,2,\ldots$, as the matrix which minimizes $\|U - U_k\|_F$, having the same sparsity pattern as U_k and satisfying (1.3) (if possible). Unhappily, Dennis-Marwil's method doesn't seem to have the local convergence properties which are typical of other least-change methods. In fact, local and superlinear convergence is only obtained if the method is restarted with $B_k = F'(x^k)$ when k is a multiple of a fixed integer m. Related methods were introduced by Martínez [13,14]. These methods don't need periodic restarts for obtaining local linear convergence, but restarts seem to be necessary in practical computations.

Johnson and Austria [12] were the first to introduce a Quasi-Newton method with direct updates of factorizations which do have local and superlinear convergence without restarts. The idea of the method of Johnson and Austria is the following: Set $B_k = M_k^{-1} U_k$, U_k upper-triangular, M_k lower-triangular with 1's on the diagonal. Now, choose M_{k+1} and U_{k+1} with the same structure of M_k, U_k, satisfying (1.3), and minimizing $\|M - M_k + U - U_k\|_F$. Numerical experiments reported in [12] seem to indicate that this method is competitive with a standard implementation of Broyden's "good" method.

Chadee [4] adapted the method of Johnson-Austria to the sparse case. So, he included the condition that M_{k+1} and U_{k+1} keep the same sparsity pattern of M_k and U_k. Of course, M_k turns out to be L_k^{-1} where $B_k = L_k U_k$ is the L-U factorization of B_k. The obvious drawback of Chadee's method is that L_k^{-1} is usually less sparse than L_k. On the other hand, the convergence proofs impose that the same pivoting strategy is valid for B_0 and $F'(x^*)$, a restriction which suggests that restarts are strongly recommended in practice.

In this paper we introduce a family of Quasi-Newton methods which represent a generalization of the method of Johnson and Austria. The idea is the following: Suppose that the matrices of the form:

$$\bar{J} = \int_0^1 J(x + t(z - x))\,dt$$

may be factorized as $\bar{J} = A^{-1}R$, where for all x,z in the domain of definition of F, A and R belong to the linear manifolds S_A and S_R respectively. For example, in the case of the method of Johnson-Austria:

$$S_A = \{(a_{ij}) \in \mathbb{R}^{n\times n} \;/\; a_{ii} = 1 \;,\; a_{ij} = 0 \text{ for } j > i,\; i=1,\dots,n\}$$

$$S_R = \{(u_{ij}) \in \mathbb{R}^{n\times n} \;/\; u_{ij} = 0 \;,\; \text{for } j < i \;,\; i = 1,\dots,n\}.$$

Assume that $\|\cdot\|_a$ and $\|\cdot\|_b$ are two norms, derived from scalar products $\mathbb{R}^{n\times n}$. In most cases $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_F$. Set $A_o \in S_A$, $R_o \in S_R$. For all $k = 0,1,2,\dots$, define $B_k = A_k^{-1}R_k$, and choose (A_{k+1}, R_{k+1}) as the solution of the problem:

$$\text{Minimize} \quad \alpha\|A - A_k\|_a^2 + (1-\alpha)\|R - R_k\|_b^2 \quad (0 < \alpha < 1)$$

$$\text{s.t. } Rs - Ay = 0 \;,\; A \in S_A \;,\; R \in S_R \,.$$

Under mild assumptions we prove the local and superlinear convergence of the methods of this family. The methods of Johnson-Austria and Chadee, the method of Broyden-Schubert, the first and second methods of Broyden and the sparse symmetric secant method of Marwil-Toint [15,25,26] are particular cases of this family.

Moreover, the method of Dennis-Marwil and some variants of it may be interpreted as "limit cases" of the family ($\alpha = 0$ or $\alpha = 1$).

Some "new members" of the family may be of potential usefulness. In the dense case, a method with A dense and R upper triangular is suggested. For convergence we need a Q-R factorization of the initial B_o, but the orthogonality of A is not preserved (and not needed) in the process. In large and sparse situations, we think that some ad hoc factorizations which take advantage of the special structure of the Jacobian, may be used to generate potentially useful methods in the new family.

NOTATION

$\|\cdot\|$ will always denote the 2-norm of vectors or matrices.

$\|\cdot\|_F$ will denote the Frobenius norm of matrices

$\|\cdot\|_a$ ($\|\cdot\|_b$) will denote the norm in $\mathbb{R}^{n\times n}$ induced by the scalar product $\langle\ ,\ \rangle_a$ ($\langle\ ,\ \rangle_b$). So, $\|X\|_a^2 = \langle X,X\rangle_a$ ($\|X\|_b^2 = \langle X,X\rangle_b$) for all $X \in \mathbb{R}^{n\times n}$.

2. THE NEW FAMILY

Assume $F : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $0 < \alpha < 1$. S_A and S_R two linear manifolds in $\mathbb{R}^{n\times n}$, $x^o \in \Omega$, $A_o \in S_A$, $R_o \in S_R$ two nonsingular $n \times n$ matrices.

For all $k = 0,1,2,\ldots$, define

$$x^{k+1} = x^k - B_k^{-1}F(x^k), \tag{2.1}$$

$$B_k = A_k^{-1} R_k$$

and suppose that (A_{k+1}, R_{k+1}) is obtained from (A_k, R_k) as the solution of the problem:

$$\text{Minimize } \alpha \|A - A_k\|_a^2 + (1-\alpha)\|R - R_k\|_b^2 \tag{2.2}$$

$$\text{s.t.} \quad Rs - Ay = 0 \tag{2.3}$$

$$s = x^{k+1} - x^k, \quad y = F(x^{k+1}) - F(x^k)$$

$$A \in S_A, \; R \in S_R \tag{2.4}$$

Before setting the hypothesis which makes the method well-defined in a neighborhood of a solution, we shall exhibit some particular, and well-known, members of the family. For all these methods, $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_F$.

Different methods in the above defined family diverge according to the way in which the manifolds S_A and S_R are chosen.

In most cases, S_A (S_R) is defined by a possibly incomplete $n\times n$ matrix $\theta^A(\theta^R)$ and n sets of indexes $I_i^A(I_i^R) \subset \{1,\ldots,n\}$, such that

$$A = (a_{ij}) \in S_A \quad (R = (r_{ij}) \in S_R)$$

if and only if

$$a_{ij} = \theta_{ij}^A \quad (r_{ij} = \theta_{ij}^R) \text{ for all } j \notin I_i^A \; (j \notin I_i^R)$$

$$i = 1,\ldots,n .$$

This is the case of most the examples which we are going to exhibit in this section.

EXAMPLE 1.

Broyden's first method: In this method $A_k = I$ for all $k = 0,1,2,\ldots$. So,

$$I_i^A = \phi, \; i = 1,\ldots,n, \; \theta^A = I, \; I_i^R = \{1,\ldots,n\}, \; i = 1,\ldots,n .$$

EXAMPLE 2.

Broyden's second method: Contrary to Broyden's first method, $R_k = I$ for all $k = 0,1,2,\ldots$, $I_i^R = \phi$, $\theta^R = I$, $I_i^A = \{1,\ldots,n\}$, $i = 1,\ldots,n$.

EXAMPLE 3.

Schubert's method: As in Broyden's first method, $A_k \equiv I$, so $\theta^A = I$, $I_i^A = \phi$ for all $i = 1,\ldots,n$. But R_k needs to reflect the sparsity of J. So, let us define:

$$I_i' = \{j \in \{1,\ldots,n\} \;/\; \frac{\partial f_i}{\partial x_j}(x) = 0 \text{ for all } x \in \Omega\},\ i=1,\ldots,n\ .$$

S_R should contain the matrices with the sparsity pattern defined by I_i'. So,

$$I_i^R = \{1,\ldots,n\} - I_i'$$

$$\theta_{ij}^R = 0 \text{ for all } j \in I_i' \ ,\ i = 1,\ldots,n\ .$$

EXAMPLE 4.

Johnson-Austria's method: In this method $A_k^{-1}R_k$ should be the L-U factorization of B_k. So,

$$I_i^A = \{1,\ldots,i-1\}\ ,\ \theta_{ij}^A = 0 \text{ if } j > i\ ,\ \theta_{ii}^A = 1,\ i = 1,\ldots,n.$$

$$I_i^R = \{i,\ldots,n\}\ ,\ \theta_{ij}^R = 0 \text{ if } j < i\ ,\ i = 1,\ldots,n\ .$$

EXAMPLE 5.

Chadee's method: As in Johnson-Austria's method, $A_k^{-1}R_k$ should be the L-U factorization of B_k , but the sparsity patterns of A_k, R_k are to be preserved. So, suppose that, for all $x \in \Omega$, $J(x) = \mathcal{L}(x)^{-1}\,\mathcal{U}(x)$ where $\mathcal{L}(x)$ is lower triangular with diagonal entries

$\ell_{ii} = 1$, $i = 1,\ldots,n$, and, additionally $\ell_{ij} = 0$ for all $j \in I_i^L$, $i = 1,\ldots,n$, and $U(x) = (u_{ij})$ is upper triangular with $u_{ij} = 0$ for all $j \in I_i^U$, $i = 1,\ldots,n$. Thus, for $i=1,\ldots,n$,

$$I_i^A = \{1,\ldots,i-1\} - I_i^L$$

$$I_i^R = \{i,\ldots,n\} - I_i^U$$

$$\theta_{ij}^A = 0 \quad \text{if} \quad j > i \ , \ \theta_{ii}^A = 1 \ , \ \theta_{ij}^A = 0 \quad \text{if} \quad j \in I_i^L$$

$$\theta_{ij}^R = 0 \quad \text{if} \quad j < i \ , \ \theta_{ij}^R = 0 \quad \text{if} \quad j \in I_i^R \ .$$

Both in Johnson-Austria's as in Chadee's method, $\alpha = 1/2$.

EXAMPLE 6.

The "Quasi-Dennis-Marwil" method: Assume the hypothesis of Chadee's method. The solution (A,R) of (2.2)-(2-4) may be determined row by row as follows:

Set

$$R = \begin{pmatrix} r_1^T \\ \vdots \\ r_n^T \end{pmatrix} = (r_{ij}) \ , \quad A = \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix} = (a_{ij})$$

$$R_k = \begin{pmatrix} (r_1^k)^T \\ \vdots \\ (r_n^k)^T \end{pmatrix} = (r_{ij}^k) \ , \quad A_k = \begin{pmatrix} (a_1^k)^T \\ \vdots \\ (a_n^k)^T \end{pmatrix} = (a_{ij}^k)$$

Then, for $i = 1,\ldots,n$ we want to solve the problem:

$$\text{Minimize } \alpha \sum_{j=1}^{n} (a_{ij}-a_{ij}^{k})^2 + (1-\alpha) \sum_{j=1}^{n} (r_{ij}-r_{ij}^{k})^2$$

$$\text{s.t. } \sum_{j=1}^{n} r_{ij}s_j - \sum_{j=1}^{n} a_{ij}y_j = 0$$

$$r_{ij} = 0 \text{ if } i < j \text{ or } j \in I_i^U$$

$$a_{ij} = 0 \text{ if } j > i \text{ or } j \in I_i^L$$

$$a_{ii} = 1 .$$

So, dropping the index i, for simplicity of notation, we must solve:

$$\text{Minimize } \alpha \sum_{\substack{j=1 \\ j \notin I_i^L}}^{i-1} (a_j-a_j^k)^2 + (1-\alpha) \sum_{\substack{j=i \\ j \notin I_i^U}}^{n} (r_j-r_j^k)^2$$

$$\text{s.t. } \sum_{\substack{j=i \\ j \notin I_i^U}}^{n} r_j s_j - \sum_{\substack{j=1 \\ j \notin J_i^L}}^{i-1} a_j y_j - y_i = 0 .$$

Using standard Lagrange multiplier theory, we see that

$$\alpha(a_j-a_j^k) - \lambda y_j = 0 \; , \; j=1,\ldots,i-1 \; , \; j \notin I_i^L$$

$$(1-\alpha)(r_k-r_j^k) + \lambda s_j = 0 \; , \; j=1,\ldots,n \; , \; j \notin I_i^U$$

So,

$$-y_j(a_j-a_j^k) + \frac{\lambda}{\alpha} y_j^2 = 0 \; , \; j=1,\ldots,i-1 \; , \; j \notin I_i^L$$

$$s_j(r_j-r_j^k) + \frac{\lambda}{1-\alpha} s_j^2 = 0 \; , \; j=i,\ldots,n \; , \; j \notin I_i^U$$

Adding this set of equations, we obtain:

$$y_i + \sum_{\substack{j=i \\ j\notin I_i^U}}^{n} r_j^k s_j - \sum_{\substack{j=1 \\ j\notin I_i^L}}^{i-1} a_j^k y_j + \lambda \left[\frac{1}{\alpha} \sum_{\substack{j=1 \\ j\notin I_i^L}}^{i-1} y_j^2 + \frac{1}{1-\alpha} \sum_{\substack{j=i \\ j\notin I_i^U}}^{n} s_j^2 \right] = 0$$

Therefore:

$$\lambda = \left(y_i + \sum_{\substack{j=i \\ j\notin I_i^U}}^{n} r_j^k s_j - \sum_{\substack{j=1 \\ j\notin I_i^L}}^{i-1} a_j^k y_j\right) \Big/ \left(\frac{1}{\alpha} \sum_{\substack{j=1 \\ j\notin I_i^L}}^{i-1} y_j^2 + \frac{1}{1-\alpha} \sum_{\substack{j=i \\ j\notin I_i^U}}^{n} s_j^2\right)$$

and:

$$a_j = a_j^k + \frac{\lambda}{\alpha} y_j \quad , \; j=1,\ldots,i-1 \; , \quad j \notin I_i^L \tag{2.5}$$

$$r_j = r_j^k - \frac{\lambda}{1-\alpha} s_j \; , \; j=1,\ldots,n \quad , \quad j \notin I_i^U \tag{2.6}$$

Consider now $\lambda = \lambda(\alpha)$, $a_j = a_j(\alpha)$, $r_j = r_j(\alpha)$, and suppose $\sum_{\substack{j=1 \\ j\notin I_i^U}}^{n} s_j^2 \neq 0$.

In this case, it is easy to see that $L = \lim_{\alpha\to 1} \frac{\lambda(\alpha)}{1-\alpha}$ exists and

$$L = \left(y_i + \sum_{\substack{j=1 \\ j\notin I_i^U}}^{n} r_j^k s_j - \sum_{\substack{j=1 \\ j\notin I_i^L}}^{i-1} a_j^k y_j\right) \Big/ \sum_{\substack{j=1 \\ j\notin I_i^U}}^{n} s_j^2 \tag{2.7}$$

Similarly,

$$\lim_{\alpha\to 1} \frac{\lambda(\alpha)}{\alpha} = 0 \tag{2.8}$$

So, combining (2.5)-(2.7), we have

$$\tilde{a}_j = \lim_{\alpha\to 1^-} a_j(\alpha) = a_j^k \ , \ j=1,\ldots,i-1$$

$$\tilde{r}_j = \lim_{\alpha\to 1} r_j(\alpha) = r_j^k - Ls_j \ , \ j=1,\ldots,n \ , \ j \notin I_i^U \ .$$

But the expressions for $\tilde{a}_j$ and $\tilde{r}_j$ are exactly the ones which define the Dennis-Marwil method, which, in consequence, may be interpreted as a "limit case" of the new family. Members of the family with a close to 1 may be called "Quasi-Dennis-Marwil" methods. We will see, in the next section, that all these methods have local and superlinear convergence without restarts, a property which is not shared by their "limit cases".

EXAMPLE 7.

The Symmetric Secant Update and the Sparse Symmetric Secant Update ([21,15,25,26]): In this case, we know that $J(x)$ is symmetric for all $x \in \Omega$. Additionally, let us assume that $J(x)$ has some sparsity pattern, and, accordingly, define I_i', $i=1,\ldots,n$ as in Schubert's method. Also, S_A and S_R are defined as in Schubert's method by imposing the additional constraint of R being symmetric for all $R \in S_R$.

3. CONVERGENCE RESULTS

Assume that $F : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, Ω an open and convex set, $F \in C^1(\Omega)$, $x^* \in \Omega$, $J^* = J(x^*)$ nonsingular. Assume, further, that there are constants M, $p > 0$ such that

$$\|J(x) - J(x^*)\| \leq M\|x - x^*\|^p \qquad (3.1)$$

for all $x \in \Omega$.

This implies that, for all $x \in \Omega$, $i=1,\ldots,n$,

$$\|\nabla f_i(x) - \nabla f_i(x^*)\| \leq M\|x - x^*\|^p \qquad (3.2)$$

and (see [3]), for all $x,z \in \Omega$:

$$\|F(x) - F(z) - J(x^*)(x-z)\| \leq M\|x-z\| \, \sigma(x,z) \tag{3.3}$$

where

$$\sigma(x,z) = \max\{\|x-x^*\|^p , \|z-x^*\|^p\} \tag{3.4}$$

Let us suppose, without loss of generality, that $J(x)$ is nonsingular for all $x \in \Omega$.

Let $\mathcal{A},\mathcal{R} : W \subset \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$, W an open neighborhood of J^*, such that

(a) $\mathcal{A}(J^*)$, $\mathcal{R}(J^*)$ are nonsingular, and, for all $J \in W$, $J = \mathcal{A}(J)^{-1} \mathcal{R}(J)$.

(b) There exists $c > 0$ such that

$$\|\mathcal{A}(J) - \mathcal{A}(J^*)\|_a \leq c\,\|J - J^*\| , \tag{3.5}$$

$$\|\mathcal{R}(J) - \mathcal{R}(J^*)\|_b \leq c\,\|J - J^*\| \tag{3.6}$$

for all $J \in W$. In consequence, $\mathcal{A}$ and $\mathcal{R}$ are continuous in J^* and, without loss of generality, we may assume that $\mathcal{A}(J)$, $\mathcal{R}(J)$ are nonsingular for all $J \in W$.

(c) For all $x,z \in \Omega$, $\bar{J} = \int_0^1 J(x+t(z-x))\,dt \in W$,

$$\mathcal{A}(\bar{J}) \in S_A , \quad \text{and} \quad \mathcal{R}(\bar{J}) \in S_R . \tag{3.7}$$

Let $\alpha \in (0,1)$. We define the following scalar product, in the space $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$:

$$\langle (A,R), (A',R') \rangle_\alpha = \alpha \langle A,A' \rangle_a + (1-\alpha) \langle R,R' \rangle_b$$

The associated norm $\|\cdot\|_\alpha$ is given by:

$$\|(A,R)\|_\alpha^2 = \alpha\|A\|_a^2 + (1-\alpha)\|R\|_b^2$$

Let us call S the cartesian product $S_A \times S_R$. Let x and z be two arbitrary points in Ω. Define $s = z-x$, $y = F(z) - F(x)$.

Define:

$$V = V(x,z) = \{(A,R) \in S \;/\; Rs - Ay = 0\}.$$

Lemma 3.1 guarantees that problem (2.2)-(2.4) has a solution if $x^k, x^{k+1} \in \Omega$.

LEMMA 3.1. For all $x,z \in \Omega$, $V = V(x,z) \neq \phi$.

PROOF.

Let us write

$$y = F(z) - F(x) = [\int_0^1 J(x+t(z-x))\,dt]\,s = \tilde{J}s \qquad (3.8)$$

But, by (3.7) ,

$$\tilde{J} \in W \,,\; A(\tilde{J}) \in S_A \text{ and } R(\tilde{J}) \in S_R$$

Thus, $R(\tilde{J})s - A(\tilde{J})y = 0$ and $(A(\tilde{J}), R(\tilde{J})) \in S$. So $(A(J), R(J)) \in V$. ∎

Since the linear manifold V is nonvoid, it makes sense to define the orthogonal projection (related to $\|\cdot\|_\alpha$) of any pair of matrices on V. The following fundamental lemma gives a bound for α-distance of $(A(J^*), R(J^*))$ to V. Let us call $A_* = A(J^*)$, $R_* = R(J^*)$ and define:

$$(\hat{A},\hat{R}) = \text{Argmin}\{\|(A,R) - (A_*,R_*)\|_\alpha \ / \ (A,R) \in V\}$$

LEMMA 3.2.

$$\|(\hat{A},\hat{R}) - (A_*,R_*)\|_\alpha \leq c\ M\ \sigma(x,z) \tag{3.9}$$

PROOF. Let $\bar{J}$ defined as in (3.8), $\bar{A} = A(\bar{J})$, $\bar{R} = A(\bar{J})$. So,

$$\bar{J} \in V \text{ , and, by (3.2):}$$

$$\|\bar{J} - J^*\| \leq M\sigma(x,z) \ .$$

Therefore, by (3.5) and (3.6),

$$\|A(\bar{J}) - A(J^*)\|_a \leq c\ M\sigma(x,z)$$

$$\|R(\bar{J}) - R(J^*)\|_b \leq c\ M\sigma(x,z).$$

Thus, by elementary calculations:

$$\|(A(\bar{J}),R(\bar{J})) - (A_*,R_*)\|_\alpha \leq c\ M\sigma(x,z) \ .$$

But, by the definition of $(\hat{A},\hat{R})$,

$$\|(\hat{A},\hat{R}) - (A_*,R_*)\|_\alpha \leq \|(A(\bar{J}),\ R(\bar{J})) - (A_*,\ R_*)\|_\alpha \ ,$$

so the desired result is proved. ∎

We are now able to prove the following "Bounded Deterioration" result:

LEMMA 3.3. Assume x^k, x^{k+1} are defined by (2.1)-(2.4), and $x, x^{k+1} \in \Omega$. Then:

$$\|(A_{k+1},R_{k+1})-(A_*,R_*)\|_\alpha \leq \|(A_k,R_k)-(A_*,R_*)\|_\alpha + 2c\, M\sigma(x,z) \qquad (3.10)$$

PROOF.

$$\|(A_{k+1},R_{k+1})-(A_*,R_*)\|_\alpha \leq \|(A_{k+1},R_{k+1})-(\hat{A},\hat{R})\|_\alpha +$$
$$+ \|(\hat{A},\hat{R}) - (A_*,R_*)\|_\alpha . \qquad (3.11)$$

But (A_{k+1},R_{k+1}) is the α-projection of (A_k,R_k) on V, and $(\hat{A},\hat{R}) \in V$, so:

$$\|(A_{k+1},R_{k+1})-(\hat{A},\hat{R})\|_\alpha \leq \|(A_k,R_k)-(\hat{A},\hat{R})\|_\alpha \leq$$
$$\leq \|(A_k,R_k)-(A_*,R_*)\|_\alpha + \|(\hat{A}_*,\hat{R}_*)-(A,R)\|_\alpha \qquad (3.12)$$

So, by (3.11) and (3.12),

$$\|(A_{k+1},R_{k+1})-(A_*,R_*)\|_\alpha \leq \|(A_k,R_k) - (A_*,R_*)\|_\alpha +$$
$$+ 2\|(A_*,R_*) - (\hat{A},\hat{R})\|_\alpha ,$$

and the thesis follows by using Lemma 3.2. ∎

Let us assume now, in addition to the general hypothesis of this section, that $F(x^*) = 0$, $0 < r < 1$.

LEMMA 3.4. There exists Ω_1, a neighborhood of x^*, and N, a neighborhood of (A_*,R_*) such that, for all $x \in \Omega_1$, $(A,R) \in N$

$$\|x - R^{-1}AF(x) - x^*\| \leq r\|x - x^*\| ,$$

and $\|A\|$, $\|R\|$, $\|A^{-1}\|$, $\|R^{-1}\|$ are uniformly bounded.

PROOF. Using standard arguments (see, for example [10]), it follows that there exist neighborhoods N_1 and Ω_1 of J^* and x^* respectively such that, for all $B \in N_1$, $x \in \Omega_1$,

$$\|x - B^{-1}F(x) - x^*\| \leq r\|x - x^*\| . \tag{3.13}$$

Now consider the function $\varphi(A,R) = A^{-1}R$. This is a continuous function in (A_*,R_*), so there exists a neighborhood N of (A_*,R_*) such that $\varphi(N) \subset N_1$. Thus, by (3.13), for $x \in \Omega_1$, $(A,R) \in N$,

$$\|x - R^{-1}AF(x) - x^*\| \leq r\|x - x^*\|,$$

as desired.

Finally, if N is contained in a closed ball which is small enough, the boundedness of $\|A\|$, $\|R\|$, $\|A^{-1}\|$, $\|R^{-1}\|$ follows from elementary properties of continuous functions on compact sets. ∎

We are now ready to prove the local linear convergence theorem.

THEOREM 3.1. There exist neighborhoods B_ε, B_δ of x^* and (A_*,R_*) respectively, such that if $x^0 \in B_\varepsilon$, $(A_0,R_0) \in B_\delta$, then the algorithm defined by (2.1)-(2.4), with the hypothesis introduced in this section, is well-defined and for all k=0,1,2,...

$$\|x^{k+1} - x^*\| \leq r\|x^k - x^*\|$$

PROOF. Let δ_1 be such that

$$\mathcal{B}_{\delta_1} = \{(A,R) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \;/\; \|(A,R) - (A_*,R_*)\|_\alpha < \delta_1\} \subset N .$$

Suppose δ, ε small enough so that:

$$\delta + \frac{2\, c\, M\varepsilon^p}{1 - r^p} < \delta_1$$

and $\mathcal{B}_\varepsilon = \{x \in \mathbb{R}^n \;/\; \|x - x^*\| < \varepsilon\} \subset \Omega \cap \Omega_1$.

Suppose $x^o \in \mathcal{B}_\varepsilon$ and $(A_o,R_o) \in \mathcal{B}_\delta$. By induction on k, we will prove that:

(i) x^{k+1} is well-defined

(ii) $\|x^{k+1} - x^*\| \leq r\|x^k - x^*\|$

(iii) $\|x^{k+1} - x^*\| \leq r^{k+1}\varepsilon$

(iv) $\|(A_{k+1},R_{k+1}) - (A_*,R_*)\|_\alpha \leq \delta + 2\, cM\varepsilon^p \sum_{j=0}^{k} r^{pj} \leq \delta_1$.

For $k = 0$, the thesis follows directly from the definition of δ and ε, and the previous lemmas.

Let us suppose that (i)-(iv) are true for all $j=0,1,\ldots,k-1$. So, by (iv),

$$\|(A_k,R_k) - (A_*,R_*)\|_\alpha \leq \delta + 2\, cM\varepsilon^p \sum_{j=0}^{k-1} r^{pj}$$

$$\leq \delta + \frac{2\, cM\varepsilon^p}{1 - r^p} < \delta_1 .$$

Therefore,

$$(A_k,R_k) \in N . \tag{3.14}$$

But, by (iii),

$$\|x^k - x^*\| \leq r^k \varepsilon < \varepsilon \text{, so}$$

$$x^k \in \Omega_1 . \tag{3.15}$$

(3.14) and (3.15) imply, by Lemma 3.4, that x^{k+1} is well-defined, $\|x^{k+1} - x^*\| \leq r\|x^k - x^*\|$, and $\|x^{k+1} - x^*\| \leq r^{k+1}\varepsilon$. Finally, by Lemma 3.3,

$$\|(A_{k+1},R_{k+1}) - (A_*,R_*)\|_\alpha \leq \|(A_k,R_k) - (A_*,R_*)\|_\alpha$$

$$+ 2\,cM \max\{\|x^k - x^*\|^p, \|x^{k+1} - x^*\|^p\} \leq$$

$$\leq \delta + 2\,cM\varepsilon^p \sum_{j=0}^{k-1} r^{pj} + 2\,cM\cdot\|x^k - x^*\|^p \leq$$

$$\leq \delta + 2\,cM\varepsilon^p \sum_{j=0}^{k-1} r^{pj} + 2\,cM\varepsilon^p\, r^{pk} =$$

$$= \delta + 2\,cM\varepsilon^p \sum_{j=0}^{k} r^{pj} \; . \blacksquare$$

In Theorem 3.1 we proved the local linear convergence of the algorithm. We now want to address the problem of proving q-superlinear convergence. From now on, we will assume the hypothesis of Theorem 3.1.

LEMMA 3.5.

$$\lim_{k\to\infty} \|(A_{k+1},R_{k+1}) - (A_k,R_k)\|_\alpha = 0$$

PROOF. Suppose, by contradiction, that the thesis is not true. So, there exists $\gamma > 0$, and an infinite set of indexes $K_1 \subset \mathbb{N}$, such that

$$\|(A_{k+1},R_{k+1}) - (A_k,R_k)\|_\alpha \geq \gamma \tag{3.16}$$

for all $k \in K_1$

Now, the same reasoning of Theorem 3.1 leads to

$$\|(A_{k+j},R_{k+j})-(A_*,R_*)\|_\alpha \leq \|(A_k,R_k)-(A_*,R_*)\|_\alpha + K\|x^k-x^*\|^p \tag{3.17}$$

with

$$K = \frac{2\,cM}{1-r^p}, \quad \text{for all } k=0,1,2,\ldots, j \geq 0 .$$

Let $k \in K_1$. So, by the Pythagorian Theorem:

$$\|(A_{k+1},R_{k+1})-(A_*,R_*)\|_\alpha^2 = \|(A_{k+1},R_{k+1})-(\hat{A},\hat{R})\|_\alpha^2 + \|(\hat{A},\hat{B})-(A_*,B_*)\|_\alpha^2 =$$

$$= \|(A_k,R_k)-(\hat{A},\hat{R})\|_\alpha^2 - \|(A_{k+1},R_{k+1})-(A_k,R_k)\|_\alpha^2 + \|(A,R)-(A_*,R_*)\|_\alpha^2 .$$

Therefore, by (3.16) and Lemma 3.2,

$$\|(A_{k+1},R_{k+1})-(A_*,R_*)\|_\alpha^2 \leq [\|(A_k,R_k)-(A_*,R_*)\|_\alpha + \|(A_*,R_*)-(\hat{A},\hat{R})\|_\alpha]^2$$

$$- \gamma^2 + (cM)^2 \|x^k - x^*\|^{2p} \leq$$

$$\|(A_k,R_k) - (A_*,R_*)\|_\alpha^2 + 2\|(A_k,R_k)-(A_*,R_*)\|_\alpha \|(A_*,R_*)-(\hat{A},\hat{R})\|_\alpha$$

$$+ 2(cM)^2 \|x^k - x^*\|^{2p} - \gamma^2 \leq$$

$$\leq \|(A_k,R_k)-(A_*,R_*)\|_\alpha^2 + 2\delta_1 cM \|x^k - x^*\|^p$$

$$+ 2(c\,M)^2 \|x^k - x^*\|^{2p} - \gamma^2 \tag{3.18}$$

Thus, by (3.17) and (3.18), we have, for all $\ell \geq k+1$

$$\|(A_\ell,R_\ell)-(A_*,R_*)\|_\alpha^2 \leq \|(A_{k+1},R_{k+1})-(A_*,R_*)\|_\alpha^2 +$$

$$2K\|x^{k+1}-x^*\|^p \|(A_{k+1},R_{k+1})-(A_*,R_*)\|_\alpha + K^2\|x^{k+1}-x^*\|^{2p} \leq$$

$$\leq \|(A_{k+1},R_{k+1})-(A_*,R_*)\|_\alpha^2 + 2K\delta_1 \|x^k-x^*\|^p + K^2\|x^k-x^*\|^{2p} \leq$$

$$\leq \|(A_k,R_k)-(A_*,R_*)\|_\alpha^2 + (2\delta_1 cM + 2K\delta_1)\|x^k-x^*\|^p$$

$$+ (2(cM)^2 + K^2)\|x^k-x^*\|^{2p} - \gamma^2 .$$

But $\lim_{k\to\infty}\|x^k-x^*\| = 0$. So, there exists $k_o \in \mathbb{N}$ such that, for all $k \in K_1$, $k \geq k_o$, $\ell \geq k+1$, we have:

$$\|(A_\ell,R_\ell)-(A_*,R_*)\|_\alpha^2 \leq \|(A_k,R_k)-(A_*,R_*)\|_\alpha^2 - \frac{\gamma^2}{2} \tag{3.19}$$

Let us consider $\{\ell_o,\ell_1,\ell_2,\dots\} \subset K_1$, such that $k_o \leq \ell_o < \ell_1 < \ell_2 < \dots$. Inequality (3.19) implies that

$$\|(A_{\ell_j},R_{\ell_j})-(A_*,R_*)\|_\alpha^2 \leq \|(A_{\ell_o},R_{\ell_o})-(A_*,R_*)\|_\alpha^2 - \frac{j\gamma^2}{2} .$$

But this implies that $\|(A_{\ell_j},R_{\ell_j})-(A_*,R_*)\|_\alpha^2 < 0$ for large enough j, which is a contradiction. ∎

LEMMA 3.6.

$$\lim_{k\to\infty}\|B_{k+1} - B_k\| = 0 .$$

PROOF. By Lemma 3.5, we have:

$$\lim_{k\to\infty} \alpha \|A_{k+1} - A_k\|_a^2 + (1-\alpha)\|R_{k+1} - R_k\|_b^2 = 0 .$$

This implies that:

$$\lim_{k\to\infty} \|A_{k+1} - A_k\|_a^2 = \lim_{k\to\infty} \|R_{k+1} - R_k\|_b^2 = 0 .$$

And so,

$$\lim_{k\to\infty} \|A_{k+1} - A_k\| = \lim_{k\to\infty} \|R_{k+1} - R_k\| = 0 \tag{3.20}$$

Now

$$\|B_{k+1} - B_k\| = \|A_{k+1}^{-1} R_{k+1} - A_k^{-1} R_k\| \le$$

$$\le \|A_{k+1}^{-1} R_{k+1} - A_{k+1}^{-1} R_k\| + \|A_{k+1}^{-1} R_k - A_k^{-1} R_k\| \le$$

$$\le \|A_{k+1}^{-1}\| \|R_{k+1} - R_k\| + \|R_k\| \|A_{k+1}^{-1} - A_k^{-1}\| \le$$

$$\le \|A_{k+1}^{-1}\| \|R_{k+1} - R_k\| + \|A_k^{-1}\| \|A_{k+1}^{-1}\| \|A_{k+1} - A_k\| \|R_k\| ,$$

and the desired result follows from both (3.20) and the uniformly boundedness of $\|A_k^{-1}\|$, $\|R_k\|$ on N. ■

THEOREM 3.2. Under the hypothesis of Theorem 3.1, x^k converges q-superlinearly to x^*.

PROOF. By the Dennis-Moré condition [7], we only need to prove that:

$$\lim_{k\to\infty} \frac{\|(B_k - J^*)(x^{k+1} - x^k)\|}{\|x^{k+1} - x^k\|} = 0$$

Now,

$$\frac{\|(B_k - J^*)(x^{k+1} - x^k)\|}{\|x^{k+1} - x^k\|} \leq \frac{\|(B_k - B_{k+1})(x^{k+1} - x^k)\|}{\|x^{k+1} - x^k\|} +$$

$$+ \frac{\|(B_{k+1} - J^*)(x^{k+1} - x^k)\|}{\|x^{k+1} - x^k\|} \leq$$

$$\|B_k - B_{k+1}\| + \frac{\|F(x^{k+1}) - [F(x^k) + J^*(x^{k+1} - x^k)]\|}{\|x^{k+1} - x^k\|}$$

But the first term in the second member of this inequality tends to 0 by Lemma 3.6, and the second goes to zero by (3.3). So, the theorem is proved.

4. SOME NEW METHODS IN THE FAMILY

A. DENSE PROBLEMS

Consider first the dense case. Particular methods in the new family differ in the way the norms $\|\cdot\|_a$ and $\|\cdot\|_b$, the manifolds S_A and S_R and the parameter α are chosen. In Section 2 we analyzed three particular choices of S_A and S_R, which lead to Broyden's "good" method, Broyden's "bad" method and Johnson-Austria's method respectively. But, many other choices are of course, possible. Let us comment some of them.

(i) The "dense-dense" method: Here $S_A = S_R = \mathbb{R}^{n \times n}$. This method may be interpreted as a combination of Broyden's good and bad methods. When α is close to 1, it approaches Broyden's "good" method, and when α in close to 0, it is close to Broyden's "bad" method. See Figure 1 for a geometrical interpretation of this situation. We observe that the pair (A_{k+1}, R_{k+1}) is, in some natural way, closer to (A_k, R_k) when α is close to $\frac{1}{2}$. Thus, it may be conjectured that the "dense-dense" ($\alpha \approx \frac{1}{2}$) method is less subject

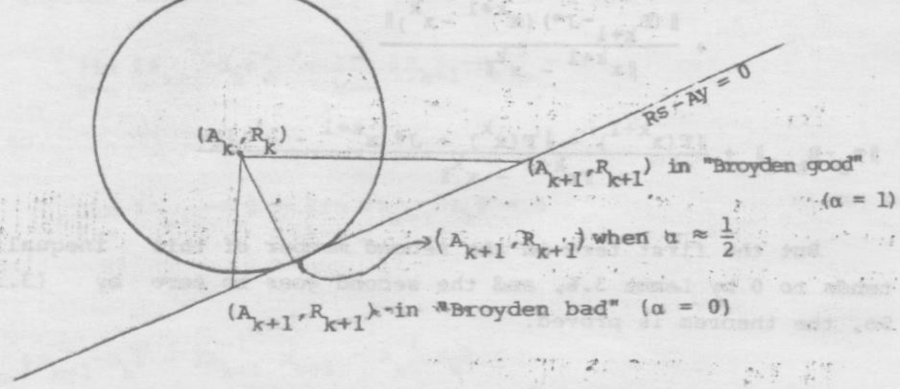


Figure 1: The "dense-dense" method

to deterioration, and thus more stable than both the first and second methods of Broyden.

The implementation of this method, however, should be more expensive than the previous ones. In fact, A_{k+1} may be obtained directly from A_k , as in Broyden's "bad" method, but a Q-R factorization of R_k should be stored, and updated, as in [18] in order to solve (1.2). In addition, we should need $O(2.5n^2)$ storage positions, in order to store A_k and the Q-R factorization of R_k. So, we don't know if this is a practical method, though is should be interesting to study its properties in order to understand the stability properties of other methods. It is curions that we are free to choose between different initial pairs, (A_o, R_o), corresponding to different initial factorizations of $J(x^o)$. For example, we may choose (a) $A = J(x^o)^{-1}$, $R = I$, (b) $A = I$, $R = J(x^o)$, (c) $A = L^{-1}$, $R = U$, so that $F'(x^o) = LU$, or (d) $A = Q^T$, $R = U$, so that QU is the "Q-R" factorization of $J(x^o)$. The validity of these choices, in relation to the convergence theorem, is justified in the following Lemma:

LEMMA 4.1. Let J^* be a nonsingular matrix, $J^* = L^*U^* = Q^*R^*$, the L-U and Q-R factorizations of J^* respectively. Let us write, in a neighborhood of J^*, $J = L(J)\,U(J) = Q(J)R(J)$. Thus, there exist $c_1, c_2, c_3 > 0$ such that, for J close enough to J^*,

$$\|J^{-1} - J^{*-1}\|_F \leq c_1 \|J - J^*\| \tag{4.1}$$

$$\|L(J) - L(J^*)\|_F \leq c_2 \|J - J^*\| \tag{4.2}$$

$$\|L^{-1}(J) - L^{-1}(J^*)\|_F \leq c_2 \|J - J^*\| \tag{4.3}$$

$$\|U(J) - U(J^*)\|_F \leq c_2 \|J - J^*\| \tag{4.4}$$

$$\|Q(J) - Q(J^*)\|_F \leq c_3 \|J - J^*\| \tag{4.5}$$

$$\|R(J) - R(J^*)\|_F \leq c_3 \|J - J^*\| \tag{4.6}$$

PROOF. See Appendix.

(ii) The "dense-triangular" method:

Define $S_A = \mathbb{R}^{n\times n}$, $S_R = \{(r_{ij}) \in \mathbb{R}^{n\times n} / r_{ij} = 0 \text{ if } i > j\}$.

As in the case of the "dense-dense" method, we way interpret that one iteration of this method approaches Broyden's "bad" method's iteration when α is close to 0. But, when $\alpha \to 1$, the method is related to Dennis-Marwil's method, in the sense that only R(the upper-triangular part of the factorization) tends to be changed, and the dense part tend to keep the same. All the "factorizations" which may be used to implement the first iteration of the "dense-dense" method may also be used for the "dense - triangular" method, except (b). But, of course, the most natural situation is to begin with the Q-R factorization of B_o. Unlike the "dense-dense" method, this seems to be a practical algorithm. In fact, A_{k+1} and R_{k+1} may be obtained from A_k and R_k using $O(1.5n^2)$ flops, and system (1.2) may be solved directly using backward substitution.

We implemented the "dense-triangular" method for $\alpha = 1/2$ and we ran it with some functions of the set presented in [17]. The results were encouraging. The situations of convergence and number of iterations were similar to Broyden's good method, and the computer times was much less because of the simplicity of the implementation, in comparison to the rather expensive updating of the Q-R factorizations in Broyden's good method.

(iii) The "dense-diagonal" method:

Set $S_A = \mathbb{R}^{n\times n}$, $S_R = \{(r_{ij}) \in \mathbb{R}^{n\times n} \ / \ r_{ij} = 0 \text{ if } i \neq j\}$.

Observe that this is Broyden's "bad" method, where some freedom was incorporated in the choice of R. In fact, in Broyden's "bad" method $S_R = \{I\}$. Possibly, the method improves the stability of Broyden's bad method.

(iv) Other "dense" methods in the family: Some other choices of S_A and S_R lead to methods which may be useful for particular problems, or the stability properties of which deserve attention. A reasonably complete list is obtained examining Table 1. For understanding this table, let us define the following sets in $\mathbb{R}^{n\times n}$:

$M = \mathbb{R}^{n\times n}$

$UT = \{(b_{ij}) \in \mathbb{R}^{n\times n} / b_{ij} = 0 \text{ for } i > j,\ i,j = 1,\dots,n\}$

$UT1 = UT \cap \{(b_{ij}) \in \mathbb{R}^{n\times n} / b_{ii} = 1,\ i=1,\dots,n\}$

$LT = \{B \in \mathbb{R}^{n\times n} / B = C^T \text{ with } C \in UT\}$

$LT1 = \{B \in \mathbb{R}^{n\times n} / B = C^T \text{ with } C \in UT1\}$

$D = \{(b_{ij}) \in \mathbb{R}^{n\times n} / b_{ij} = 0 \text{ if } i \neq j\}$

$I = \{I\}$,

So, different methods are obtained choosing S_A and S_R as different sets in the list above. Now, Table 1 may be interpreted:

	M	UT	UT1	LT	LT1	D	I
M	"dense-dense"	"dense-triang"	new	new	new	"dense-diag"	Broyden's bad
UT	new (not interesting)	↯	↯	new	new	↯	↯
UT1	new (not interesting)	↯	↯	new	↯	↯	↯
LT	new (not interesting)	new	new	↯	↯	↯	↯
LT1	new (not interesting)	Johnson-Austria	↯	↯	↯	↯	↯
D	new	↯	↯	↯	↯	↯	↯
I	Broyden good	↯	↯	↯	↯	↯	↯

Table 1: Methods in the "dense" family.

B. PARTIALLY LINEAR PROBLEMS

Suppose that $F(x) = \begin{pmatrix} Cx - b \\ F_2(x) \end{pmatrix}$. So, $F'(x) = \begin{pmatrix} C \\ F'_2(x) \end{pmatrix}$.

The first method of Broyden is not well suited for this type of problem because the whole Q-R factorization of B_k should be changed from one iteration to another without taking advantage of the linearity of the first components of F. In contrast, let us define a method in the new family which seems to be adequate for this situation. Suppose that $C \in \mathbb{R}^{m\times n}$, $q = n-m$. Let $L \in \mathbb{R}^{m\times m}$ such that $L = (\ell_{ij})$ is lower-triangular, $\ell_{ii} = 1$, $i = 1,\ldots,m$ and $LA = U = (u_{ij})$, such that $u_{ij} = 0$ if $i > j$.

Define:

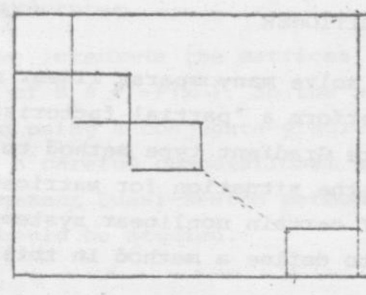
$$S_A = \{(a_{ij}) \in \mathbb{R}^{n\times n} \ / \ a_{ii} = 1 \ , \ a_{ij} = 0 \ \text{ if } \ i > j \ , \ a_{ij} = \ell_{ij} \ \text{ if } \ i < j \ , \ i,j = 1,\ldots,n\}$$

$$S_R = \{(r_{ij}) \in \mathbb{R}^{n\times n} \ / \ r_{ij} = u_{ij} \ \text{ if } \ i \le m \ , \ j = 1,\ldots,n\}.$$

S_A and S_R reflect the structure factorization of the L-U factorization of C. The whole factorization of R_k may be completed with a small number of operations if q is small as related to n.

C. SPECIAL STRUCTURES

Suppose that J(x) has the "dual block-angular structure" (see [5]):

$J(x) =$  $=$

$$= \begin{bmatrix} B_1(x) & & & \\ & \ddots & & C(x) \\ & & B_m(x) & \end{bmatrix}$$

where $B_i(x) \in \mathbb{R}^{m_i \times n_i}$, $C(x) \in \mathbb{R}^{n \times n_F}$, $\Sigma\, m_i = \Sigma\, n_i + n_F = n$. It is easy to verify that the Q-R factorization of $J(x)$ has the same structure of $J(x)$ in the following sense: The structure of Q is exactly the same structure as J, and R is an upper-triangular matrix such that $r_{ij} = 0$ whenever $\frac{\partial f_i}{\partial x_j}(x) = 0$. So it is natural to define S_A and S_R as the sets of matrices in $\mathbb{R}^{n \times n}$ which have the structure patterns of Q and R respectively.

A similar analysis may be done with respect to other particular structures, such as staircase, staircase with dense columns, etc.

D. UPDATING THE PRECONDITIONER

The best way to solve many sparse linear systems of equations $Cx = b$, is to perform a "partial factorization", $P^{-1}C = C'$, and to apply a Conjugate Gradient type method to $C'x = P^{-1}b$ (See [20],[22]). If this is the situation for matrices with the structure of the Jacobian of certain nonlinear systems $F(x) = 0$, it should be interesting to define a method in this new family where.

S_A = matrices with the structure of P

S_R = matrices with the structure of C'.

5. FINAL REMARKS

In this paper we introduced a new family of Quasi-Newton methods for solving nonlinear simultaneous equations, which turns out to be a generalization of the method of Johnson-Austria [12]. The new family includes some well-known methods, such as the first and second methods of Broyden, Schubert's method and, of course, Johnson-Austria's and Chadee's methods. So, the same convergence analysis is valid for all these methods, as they are seen as particular members of this new family.

We suggested some new methods which turn out to be "new members" of the family. For dense problems, the "dense-triangular" method, with an initial Q-R factorizations semms to be of practical interest. Other methods, such as the "dense-dense", the "dense-diagonal" and the "diagonal-dense" seem to be useful to study the stability properties of least-change methods. They deserve future research.

For particular sparse structures it is possible to introduce ad-hoc methods in the new family which may be of practical interest. This in the case, for instance, of block-angular, stair case, and band-structures.

Finally, we interprete the matrices A_k^{-1} as preconditioners for the solution of $B_k s = -F(x^k)$. So the solution of $R_k s = -A_k F(x^k)$ may be calculated using a conjugate-gradient method. This point of view deserves a careful computational and theoretical analysis. In particular, Inexact Quasi-Newton methods (see [24]) based on the new family should be studied.

APPENDIX. PROOF OF LEMMA 4.1.

(4.1) is a well-known result, based on Banach's Lemma [19]. (4.2), (4.3) and (4.4) were proved by Dennis and Marwil [6]. So, let us prove (4.5) and (4.6).

Let us write $Q = (q_{ij})$, $R = (r_{ij})$. Analyzing an algorithm for obtaining Q and R, it is easy to verify that

$$q_{ij} = q_{ij}(J)$$

$$r_{ij} = r_{ij}(J)$$

where q_{ij}, r_{ij} are compositions of C^1-functions. So, q_{ij} and r_{ij} are C_1-functions is a neighborhood of J^*. So, for J belonging to this neighborhood we have,

$$|q_{ij} - q^*_{ij}| \le \bar{C}_5 \|J - J^*\|$$

$$|r_{ij} - r^*_{ij}| \le \bar{C}_6 \|J - J^*\|.$$

Thus

$$\|Q - Q^*\|_F \leq \sqrt{n}\, \bar{C}_5 \, \|J - J^*\|$$

$$\|R - R^*\|_F \leq \sqrt{n}\, \bar{C}_6 \, \|J - J^*\|$$

and the thesis follows using the equivalence of norms in finite-dimensional linear spaces. ∎

ACKNOWLEDGEMENTS. The author is indebted to Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) and to Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for financial support given to his work.

REFERENCES

[1] C.G. BROYDEN (1965), A class of methods for solving nonlinear simultaneous equations, *Math. Comp.*, 19, pp.577-593.

[2] C.G. BROYDEN (1971), The convergence of an algorithm for solving sparse nonlinear systems, *Math. Comp.*, 25, pp.285--294.

[3] C.G. BROYDEN, J.E. DENNIS and J.J. MORÉ (1973), On the local and superlinear convergence of quasi-Newton methods, J. Inst. Math. Appl. 12, pp. 223-245.

[4] F.F. CHADEE (1985), Sparse Quasi-Newton methods and the continuation problem, T.R. SOL 85-8, Dept. of Operations Research, Stanford University.

[5] M.G. COX (1980), The least squares solution of overdetermined linear equations having band or augmented band structure, NPL Report DNACS 26/80, National Physical Laboratory, Teddington, Middlesex, UK.

[6] J.E. DENNIS and E.S. MARWIL (1982), Direct secant updates of matrix factorizations, *Math. Comp.* 38, pp. 459-476.

[7] J.E. DENNIS and J.J. MORÉ (1974), A characterization of superlinear convergence and its application to quasi-Newton methods, *Math. Comp.*, 28, pp. 549-560.

[8] J.E. DENNIS and R.B. SCHNABEL (1979), Léast change secant updates for quasi-Newton methods, SIAM Rev., 21, pp. 443 --459.

[9] J.E. DENNIS and R.B. SCHNABEL (1983), Numerical Methods for Unconstrained Optimization and Nonlinear Equations, Prentice-Hall, Englewood Cliffs, NJ.

[10] J.E. DENNIS and H.F. WALKER (1981), Convergence theorems for least-change secant update methods, SIAM J. Numer. Anal., 18, pp. 949-987.

[11] S.C. EISENSTAT and T. STEIHAUG (1982), Local convergence results for inexact quasi-Newton methods, Department of Mathematical Sciences, TR 82-7, Rice University.

[12] G.W. JOHNSON and N.H. AUSTRIA (1983), A quasi-Newton method employing direct secant updates of matrix factorizations, SIAM J. Numer. Anal., 20, pp. 315-325.

[13] J.M. MARTÍNEZ (1983), A Quasi-Newton method with a new updating for the LDU factorization of the approximate Jacobian, Math. Aplic. e Comput. 2, pp. 131-142.

[14] J.M. MARTÍNEZ (1987), Quasi-Newton methods with factorization scaling for solving sparse nonlinear systems of equations, Computing 38, pp. 131-141.

[15] E.S. MARWIL (1978), Exploiting Sparsity in Newton-type methods, Cornell Applied Mathematics PhD thesis.

[16] E.S. MARWIL (1979), Convergence results for Schubert's method for solving sparse nonlinear equations, SIAM J.Numer.Anal., 16, pp. 588-604.

[17] J.J. MORÉ, B.S. GARBOW and K.E. HILLSTROM (1981), Testing unconstrained minimization software, ACM TOMS 7, pp.17-41.

[18] J.J. MORÉ and J.A. TRANGENSTEIN (1976). On the global convergence of Broyden's method, Math. Comp. 30, pp. 523-540.

[19] J. M. ORTEGA and W.C. RHEINBOLDT (1970), Iterative Solution of Nonlinear Equations in Several Variables, Academic Press , New York.

[20] C.C. PAIGE and M.A. SAUNDERS (1982), LSQR, An Algorithm for sparse linear equations and sparse least-squares, ACM TOMS 8, pp. 43-71.

[21] M.J.D. POWELL (1970), A new algortihm for unconstrained optimization, in Nonlinear Programming, J.B.Rosen, O.L. Mangasarian and K.Ritter, eds., Academic Press, NY, pp. 31-65.

[22] M.A. SAUNDERS (1979), Sparse Least Squares by Conjugate Gradients: A comparison of Preconditioning methods, Proc. Computer Science and Statistics: 12th. Annual Symposium on the Interface, J.F. Gentleman editor, Univ. of Waterloo, Ontario, pp. 15-20.

[23] L.K. SCHUBERT (1970), Modification of a quasi-Newton method for nonlinear equations with a sparse Jacobian, Math. Comp., 24, pp. 27-30.

[24] T. STEIHAUG (1980), Quasi-Newton methods for large scale nonlinear problems, PhD Thesis, Scholl of Organization and Management, TR # 49, Yale University.

[25] PH.L. TOINT (1977), On sparse and symmetric matrix updating subject to a linear equation, Math. Comp. 31, pp. 954-961.

[26] PH.L. TOINT (1978), Some numerical results using a sparse matrix updating formula in unconstrained optimization, Math. Comp. 32, pp. 839-851.

RELATÓRIOS TÉCNICOS — 1988

01/88 – A Linear Continuous Transportation Problem – *Enrique D. Andjel, Tarcísio L. Lopes* and *José Mario Martínez.*

02/88 – A Splitting Theorem for Complete Manifolds With Non-Negative Curvature Operator – *Maria Helena Noronha.*

03/88 – Mathematical Physics of the Generalized Monopole without Strings – *W. A. Rodrigues Jr., M. A. Faria-Rosa, A. Maia Jr.* and *E. Recami.*