CODIMENSION TWO COMPLETE NON–COMPACT SUBMANIFOLDS WITH NON–NEGATIVE CURVATURE

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We study the topology of complete non-compact manifolds with non-negative sectional curvatures isometrically immersed in Euclidean spaces with codimension two. We investigate some conditions which imply that a such manifold is a topological product of a soul by a Euclidean space and this gives a complete topological description of this manifold.

1.INTRODUCTION

In [9], Sacksteder studied isometric immersions of manifolds with non-negative sectional curvatures in Euclidean spaces with codimension one,under non-degeneracy conditions about the curvature,namely, that at least one sectional curvature is positive at each point on the manifold. Under the same hypotheses we want to obtain a topological characterization of complete non-compact manifolds isometrically immersed in codimension two. This uses the existence of a compact soul in M, proved by Cheeger - Gromoll in [6]. Baldin - Noronha in [4], show some results along the same line. Basically, it is proved that if this manifold M^n is simply connected then M is diffeomorphic to $A^k \times \mathbb{R}^{n-k}$, where A is a k-dimensional soul of M. We obtain a similar conclusion without the simply connected condition and this allows us to know the topology of the manifold,as we know the topology of the compact soul by [2] and [3]. Our first result states:

Theorem 1 : Let $f: M^n \to \mathbb{R}^{n+2}$ be a substantial isometric immersion of a complete non-compact manifold with non-negative sectional curvatures, such that at least one of them is positive at each point x in M and let A^k be a k-dimensional soul of M. Then if $k \geq 2$, M is diffeomorphic to $A^k \times \mathbb{R}^{n-k}$ or $\pi_1(M)$ is finite. In the latter case M has the homotopy type of the Real Projective Space $\mathbb{R}P^2$ or k=3.

Remark. In the former case the possibilities for A^k follow from [2] and [3]. They are that A^k is homeomorphic to a sphere, or a product of two spheres, or a product of the circle S^1 by a homotopy sphere, or is diffeomorphic to the total space of a non-orientable fiber bundle over S^1 whose fibers are homotopy spheres.

This theorem is proved by showing that the two-dimensional flat torus and also the two-dimensional flat Klein bottle cannot be a soul for this manifold and when $k \geq 3$ we prove that, if $\pi_1(M) = \mathbb{Z}$, then the immersion is reducible along a soul A (see definition (2.7) below). This means that f reduces codimension when restricted to the soul.

Theorem 2 : Let $f: M^n \to \mathbb{R}^{n+2}$ be an isometric immersion with the same hypothesis of Theorem 1. If M is simply connected there exists an isometric immersion of the soul A in Euclidean space with codimension two.

This, together with Proposition 3.3 in [2], imply that the Complex Projective Space $\mathbb{C}P^2$ cannot be a soul for this manifold M.

Before we state our next result, we want to recall that the curvature tensor R at x in M can be regarded as an endomorphism $\Re$ of $T_xM \wedge T_xM$ which is symmetric with respect to the inner product defined by the Riemannian metric. The hypotheses of the above theorems imply that for each point x in M, there exist vectors U,V in T_xM such that $\Re(U\wedge V)\neq 0$. A two-form $\Re(U\wedge V)$ is defined to have rank 2p iff p is the largest integer such that $\Re(U\wedge V)\wedge \dots \wedge \Re(U\wedge V)$ (p times) $\neq 0$. Since we are studying codimension two, the two-form $\Re(U\wedge V)$ has rank at most 4.

<u>Theorem 3</u>: Let $f: M^n \to \mathbb{R}^{n+2}$ be a substantial isometric immersion of a complete non-compact manifold with non-negative sectional curvatures and such that for every point x in M there are vectors U,V in T_xM such that $\Re(U\wedge V)$ has rank 4. Let A^k be a k-dimensional soul of M, $k\neq 0$. Then $k\geq 2$ and M is diffeomorphic to $A^k\times\mathbb{R}^{n-k}$.

Moreover

i) If $k\geq 3$, then M is simply connected.

ii) If $k=2$, A is either the sphere S^2 or the real projective space $\mathbb{R}P^2$.

Finally, we will consider the index of relative nullity of f at a point x in M as

$$\nu_f(x) = \dim\{X\in T_xM : \alpha(X,Y)=0,\ \forall\, Y\in T_xM\}$$

where α is the second fundamental form. By Hartman [7], if M is not a cylinder, there exists a point x in M such that $\nu_f(x)=0$. If this point belongs to a soul we conclude:

<u>Theorem 4</u>: Let $f: M^n \to \mathbb{R}^{n+2}$ be a substantial isometric immersion of a complete non-compact manifold with non-negative sectional curvatures and k-dimensional soul A. If there is a point $x \in A$ such that $\nu_f(x)=0$ we have:

i) if $k \geq 3$, M^n is simply connected and diffeomorphic to $A^k \times \mathbb{R}^{n-k}$, where A^k is homeomorphic to the sphere S^k.

ii) if $k=2$ then M^n is diffeomorphic to $S^2 \times \mathbb{R}^{n-2}$ or has the homotopy type of $\mathbb{R}P^2$.

We want to observe that the results of Cheeger - Gromoll in [6] do not allow us to know the dimension of the soul. However, under our hypotheses, if the manifold has $\pi_1(M) = \mathbb{Z}$ and is not a topological product of a compact manifold by a Euclidean space, we can conclude that the soul is homeomorphic to the circle S^1.

2. SOME KNOWN RESULTS OF THE SOUL

It is a well known result of Weinstein [10], that if the codimension of a isometric immersion is two then the non-negativity of the sectional curvatures ($K \geq 0$) implies the non-negativity of the curvature operator($\mathfrak{R} \geq 0$).

For the case where M^n is complete non-compact manifold with $\mathfrak{R} \geq 0$, we now collect some properties of a soul A of M. We denote by A^k a k-dimensional soul of M, $0 \leq k < n$. We consider the splitting of the tangent bundle of M, $TM = TA \oplus TA^{\perp}$, where TA is the tangent bundle of A and $TA^{\perp}$ is the normal bundle of the inclusion $A \subset M$. We observe the following properties of a soul:

(2.1) If the soul is a point, then M is diffeomorphic to $\mathbb{R}^n$. (See[6])

(2.2) A soul A of M is a compact, totally convex submanifold of M without boundary and has $\mathfrak{R} \geq 0$. (See[6])

(2.3) The inclusion $i: A \to M$ is a homotopy equivalence and M is diffeomorphic to the total space of $TA^{\perp}$. (See [6]).

(2.4) If $X \in TA$ and $Y \in TA^{\perp}$, then $\mathfrak{R}(X \wedge Y) = 0$. Moreover, $\mathfrak{R}(\Lambda^2(TA)) \subset \Lambda^2(TA)$ and $\mathfrak{R}(\Lambda^2(TA^{\perp})) \subset \Lambda^2(TA^{\perp})$. (See [4], lemma 3.1) .

(2.5) The normal curvature tensor $R^{\perp}$ of the inclusion $i: A \to M$ vanishes (See [4], lemma 3.2).

From these properties we can state the following theorem (proved in [4])

(2.6) <u>Theorem</u>: If $\pi_1(M) = \{0\}$ and $\mathfrak{R} \geq 0$ then M is a topological product of a soul by a Euclidean space.

In order to prove Theorems 1 and 3 in the case that $\pi_1(M)\neq(0)$, we need an extrinsic property of the immersion, namely, reducibility along the soul.

(2.7)<u>Definition</u> : Let $f:M^n\to\mathbb{R}^{n+p}$, $p\geq 1$, be an isometric immersion of a complete, non-compact manifold M with $K\geq 0$, non-trivial k-dimensional soul A and second fundamental form α. We say that <u>f is reducible along A</u> if for $X\in TA$ and $Y\in TA^{\perp}$, $\alpha(X,Y)=0$.

(2.8)<u>Theorem</u> : If f is reducible along a soul A then M is diffeomorphic to $A^k\times\mathbb{R}^{n-k}$. (See [4], Proposition (5.4))

In the rest of this paper $\langle\ ,\ \rangle$, ∇ will denote the Riemannian metric and connection respectively. If ξ is a normal direction, A_ξ will denote the Weingarten operator and $\nabla^{\perp}\xi$ will be the normal connection.

3. BASIC LEMMAS. PROOF OF THEOREM 1

Consider $x\in A$. We want to investigate if f satisfies the reducibility condition at x. By abuse of notation, we will say "f is reducible at x." If for every $X\in T_xA$, $\alpha(X,X)=0$ or for every $Y\in T_xA^{\perp}$, $\alpha(Y,Y)=0$, by the Gauss equation, f is reducible at x, since $k\geq 0$ and $\Re(X\wedge Y)=0$.

To study the general case, let r(x) be the Lie algebra generated by the range of the curvature operator $\Re$ at the point x. If U is the orthogonal complement of the relative nullity subspace N(x), by Theorem 1 in [5] we have the following possibilities for r(x):

(3.1)
(a) $r(x) = \Lambda^2(U)$

(b) $r(x) = \Lambda^2(V) \oplus \Lambda^2(W)$, where $V \oplus W = U$

(c) $r(x) = u(2)$, the unitary algebra of some complex structure on U, if dimU=4.

Moreover, if (b) occurs with dimV>1 and dimW>1 then V and W are orthogonal to each other and $R^{\perp}(x)=0$, where $R^{\perp}$ is the normal curvature tensor of f.

<u>(3.2) Lemma :</u> If $\mathfrak{R}|_{\Lambda^2(T_x A)} \neq 0$ and $\mathfrak{R}|_{\Lambda^2(T_x A^{\perp})} \neq 0$ then f is reducible at x. Moreover, there is an orthonormal frame $\{\xi_1, \xi_2\}$ such that $A_{\xi_1}|_{T_x A} = 0$ and $A_{\xi_2}|_{T_x A^{\perp}} = 0$.

<u>Proof:</u> By (2.4), the only possibility is $r(x) = \Lambda^2(V) \oplus \Lambda^2(W)$ with dimV>1 and dimW>1 whence $R^{\perp}(x)=0$. Therefore, the lemma follows by Theorem D of [4].

(3.3) <u>Lemma :</u> (a) Suppose $\mathfrak{R}|_{\Lambda^2(T_x A)} \neq 0$ and $\mathfrak{R}|_{\Lambda^2(T_x A^{\perp})} = 0$. If $\alpha(Y,Y) \neq 0$ for some $Y \in T_x A^{\perp}$, there is an orthonormal frame $\{\xi_1, \xi_2\}$ in the normal space such that rank $A_{\xi_1} = 1$ and $A_{\xi_2}|_{T_x A^{\perp}} = 0$.

(b) If $\mathfrak{R}|_{\Lambda^2(T_x A)} = 0$ and $\mathfrak{R}|_{\Lambda^2(T_x A^{\perp})} \neq 0$ with $\alpha(X,X) \neq 0$ for some $X \in T_x A$, we have a similar conclusion with $A_{\xi_2}|_{T_x A} = 0$.

Proof : (a) Consider an orthonormal frame $\{X_1,\dots,X_n\}$ of T_xM such that $X_1,\dots,X_s \in N(x)$ and $Y \in \mathrm{Span}\{X_1,\dots,X_s,X_{s+1}\}$. We have

$$\Re(X_i \wedge X_j)=0, \qquad i=1,\dots,s \text{ and } j=1,\dots,n$$

Denoting by X' and X'' the orthogonal projection of the vector X onto T_xA and $T_xA^{\perp}$ respectively, by (2.4) we have:

$$\Re(Y \wedge X_j)= \Re(Y \wedge X'_j) + \Re(Y \wedge X''_j)=0, \quad j=1,\dots,n$$

Then the range of $\Re$ is contained in $\Lambda^2(W)$, where $W=\mathrm{Span}\{X_{s+2},\dots,X_n\}$, which implies

(3.4) $$r(x) \subseteq o(n-s-1)$$

where $o(n-s-1)$ is the orthogonal group. If $n-s=4$, $r(x)$ cannot be $u(2)$, since $u(2)$ is not contained in $o(3)$. Then $r(x)= \Lambda^2(V)\oplus\Lambda^2(W)$, where $\dim V=1$.

Then, following the proof of Theorem 1 in [5], there is one normal vector ξ_1 such that $\mathrm{rank} A_{\xi_1}=1$. If ξ_2 is a normal vector orthogonal to ξ_1 we have

(3.5) $$\Re= A_{\xi_2} \wedge A_{\xi_2}$$

We will prove that this basis $\{\xi_1,\xi_2\}$ satisfies the lemma. Consider $X,Y \in T_xM$. We have:

$$\Re(X \wedge Y)=(A_{\xi_2}X)' \wedge (A_{\xi_2}Y)'+(A_{\xi_2}X)' \wedge (A_{\xi_2}Y)''+(A_{\xi_2}X)'' \wedge (A_{\xi_2}Y)'+(A_{\xi_2}X)'' \wedge (A_{\xi_2}Y)''$$

where

$$\Omega = (A_{\xi_2}X)''\wedge(A_{\xi_2}Y)''=0$$

$$\omega = (A_{\xi_2}X)'\wedge(A_{\xi_2}Y)'' + (A_{\xi_2}X)''\wedge(A_{\xi_2}Y)' =0$$

since we are supposing $\mathfrak{R}|_{\Lambda^2(T_xA^\perp)}=0$. Let us suppose $(A_{\xi_2}X)'\neq 0$. Taking interior product of ω with $(A_{\xi_2}X)'$ we get

$$0= i((A_{\xi_2}X)')\omega = \|(A_{\xi_2}X)'\|^2 (A_{\xi_2}Y)'' - \langle (A_{\xi_2}Y)',(A_{\xi_2}X)'\rangle (A_{\xi_2}X)''$$

and therefore

$$(A_{\xi_2}Y)'' = \langle (A_{\xi_2}Y)',(A_{\xi_2}X)'\rangle \|(A_{\xi_2}X)'\|^{-2} (A_{\xi_2}X)''.$$

Taking interior product with $(A_{\xi_2}Y)'$ we get

$$0= i((A_{\xi_2}Y)')\,\omega = \langle (A_{\xi_2}X)',(A_{\xi_2}Y)'\rangle (A_{\xi_2}Y)'' - \|(A_{\xi_2}Y)'\|^2 (A_{\xi_2}X)''$$

$$= \|(A_{\xi_2}X)'\|^{-2}\{ \langle (A_{\xi_2}Y)', (A_{\xi_2}X)'\rangle^2 - \|(A_{\xi_2}X)'\|^2\|(A_{\xi_2}Y)'\|^2 \} (A_{\xi_2}X)''.$$

If $(A_{\xi_2}X)''\neq 0$ the above relation implies $(A_{\xi_2}Y)'=\lambda(A_{\xi_2}X)'$ and then

$$\mathfrak{R}(X\wedge Y) = (A_{\xi_2}X)' \wedge (A_{\xi_2}Y)' = 0.$$

Hence,

(3.6) if $\mathfrak{R}(X\wedge Y)\neq 0$ we have $(A_{\xi_2}X)'' = (A_{\xi_2}Y)'' = 0$.

Consider now the orthonormal basis $\{Z_1,\dots, Z_n\}$ which diagonalizes the operator A_{ξ_1} such that $A_{\xi_1}(Z_1) =\lambda Z_1$ and $A_{\xi_1}(Z_i)=0$, $i\geq 2$. Since $\mathfrak{R}\neq 0$ at x, there exist Z_i,Z_j such that $\mathfrak{R}(Z_i\wedge Z_j)\neq 0$. By (3.6), for every $Y\in T_xA^\perp$ we have $\langle\alpha(Z_i,Y), \xi_2\rangle = \langle\alpha(Z_j,Y),\xi_2\rangle = 0$. This implies $\alpha(Z_i,Y) = 0$, as we can suppose that $A_{\xi_1}(Z_i)=0$. In the Gauss equation this implies $\langle\alpha(Z_i,Z_i),\alpha(Y,Y)\rangle=0$, since $\mathfrak{R}(Y\wedge Z_i)=\mathfrak{R}(Y\wedge Z_i')+\mathfrak{R}(Y\wedge Z_i'')=0$. Because $\alpha(Z_i,Z_i)$ is orthogonal to ξ_1, we have that $\alpha(Y,Y)$ is orthogonal to ξ_2. Now, writing the Gauss equation for the sectional curvature of a plane spanned by $X\epsilon T_xA$ and $Y\epsilon T_xA^\perp$, we get

$$0=\langle A_{\xi_2}X,X\rangle\langle A_{\xi_2}Y,Y\rangle-\langle A_{\xi_2}Y,X\rangle^2=-\langle A_{\xi_2}Y,X\rangle^2.$$

This and (3.6) together imply $A_{\xi_2}|_{T_xA^\perp}=0$, concluding the proof of (a).

(b) This is proved in an analogous manner.

We observe that, under the hypotheses of lemma (3.3),in (a) there is only one vector $Y\epsilon T_xA^\perp$ such that $\alpha(Y,Y)\neq 0$ and in (b) only one vector $X\epsilon T_xA$ such that $\alpha(X,X)\neq 0$.

(3.7) Proposition: If $\dim A=k\geq 3$ and $\pi_1(M)=\mathbb{Z}$, then f is reducible along A.

PROOF: Let $\bar{f}=f|_A : A\to\mathbb{R}^{n+2}$, the isometric immersion of restricted to the soul. Since A is a totally geodesic submanifold of M the first normal space of $\bar{f}$ is at most-dimensional. We can easily generalize to $\bar{f}$, using the same arguments, Theorems (2.2) and (2.3) of [2], obtaining the same results, since they need only the fact of the first normal space be at most two-dimensional. We will denote by $\nu_{\bar{f}}(x)$ the index of relative nullity of the immersion $\bar{f}$.

Since A is compact, consider x A such that $\nu_{\bar{f}}(x)=0$. We claim that $\alpha(Y,Y)=0$, for every $Y\epsilon T_xA^\perp$. Otherwise, under the conditions of Lemma (3.2), all the sectional curvatures along planes tangent to A at x would be positive. Also, under the conditions of Lemma (3.3), the index of relative nullity would be n-k-1. Then in (3.4) we would have $r(x)=o(k)$. This implies that all the eigenvalues of $A_{\xi_2}|T_xA$ are non-null and then all the sectional curvatures along planes tangent to A at x would be positive.The slight generalization of Theorem (2.2) of [2] to this immersion $\bar{f}$ would imply that A and consequently M, is simply connected.

Now, we will prove the reducibility for $x \in A$ such that $\nu_{\bar{f}}(x)>0$. Let $N_\gamma(A)$ denote the set of points in A at which the index of relative nullity is γ. Since we know that f is reducible on the closure of $N_0(A)$, we will use the inductive argument used by Moore to prove Theorem 2 in [8]. Let $\gamma \geq 1$ and V be the open set

$$N_\gamma(A) - Cl\,[\,\cup\{\,N_\beta(A)\;/\;\beta<\gamma\}]$$

where Cl denotes closure, a set on which the index of relative nullity is equal to the constant γ.

We recall that if $\pi_1(A)=\mathbb{Z}$ by generalization of Theorem (2.3) of [2], x has a neighborhood isometric to an open subset of the product of the cyrcle S^1 by a (k-1)-dimensional homotopy sphere,which implies that there are two integrable and parallel distributions T_1 and T_2 such that $\dim T_1=1$ and $\dim T_2=k-1$. If some sectional curvature of A is positive at x and Z is tangent to T_1, Z must be relative nullity vector. Otherwise $r(x)=o(m-1)\oplus o(1)$ where $m=k-\gamma$. But in the proof of the Lemma (3.3) we see that $m+1=n-s$, which contradicts (3.4).

Now, consider $\sigma:(a,b)\rightarrow V$ a unit speed geodesic passing through x whose tangent vector $\sigma'(t)$ is the relative nullity vector $Z\in T_1$, for each $t\in(a,b)$. Assume that σ cannot be extended beyond the interval (a,b) without leaving V. Since A is compact, either $a>-\infty$ or $b<+\infty$. Suppose $b<+\infty$. By Theorem

(6.2) in [1], $\sigma(b)$ lies in the closure of $\cup\{ N_\beta(A) \,/\, \beta<\gamma \}$, a set on which f is reducible by the inductive hypothesis.

We will prove that if f is not reducible at x, f cannot be reducible at $\sigma(b)$, which will be a contradiction. If f is not reducible at x we can take the frame $\{\xi_1,\xi_2\}$ of the Lemma (3.3) such that $A_{\xi_2}|T_xA^\perp=0$. Let us denote by X and Y the unitary orthogonal projection of Z_1 (see the proof of Lemma (3.3)) onto T_xA and $T_xA^\perp$ respectively. Denoting by Z a vector in T_xA orthogonal to X and by $\nabla^\perp$ the normal connection, we can apply the Codazzi equation to X,Y,Z and ξ_2 to get

$$\langle\nabla_X^\perp\,\alpha(Z,Y),\,\xi_2\rangle - \langle\alpha(\nabla_X Z,Y),\,\xi_2\rangle - \langle\alpha(Z,\nabla_X Y),\,\xi_2\rangle=$$

$$=\langle\nabla_Z^\perp\,\alpha(X,Y),\,\xi_2\rangle - \langle\alpha(\nabla_Z X,Y),\,\xi_2\rangle - \langle\alpha(X,\nabla_Z Y),\,\xi_2\rangle$$

Since $\alpha(Z,Y)=0$, $A_{\xi_2}|T_xA^\perp=0$ and $\nabla_X Y\in T_xA^\perp$, the left hand side is equal to zero. The same reasons will imply that the right hand side is equal to

$$\langle \nabla_Z^\perp\alpha(X,Y),\,\xi_2\rangle = -\langle \alpha(X,Y),\,\nabla_Z^\perp\xi_2\rangle=0$$

As we are supposing that $\alpha(X,Y)\neq0$, we have

(3.8) $$\nabla_Z^\perp\xi_2 = \nabla_Z^\perp\xi_1=0$$

Now, with the same notation we will consider the vector fields X,Y and Z such that on $\sigma(t)$, $Z(t)=\sigma'(t)$. Applying the Codazzi equation to X,Y,Z and ξ_1 we get

$$\langle\nabla_X^\perp\alpha(Z,Y),\,\xi_1\rangle - \langle\alpha(\nabla_X Z,Y),\,\xi_1\rangle - \langle\alpha(Z,\nabla_X Y),\,\xi_1\rangle=$$

$$=\langle \nabla_Z^\perp\alpha(X,Y),\,\xi_1\rangle - \langle\alpha(\nabla_Z X,Y),\,\xi_1\rangle - \langle\alpha(X,\nabla_Z Y),\,\xi_1\rangle.$$

which implies

(3.9) $$\langle Z,\nabla_X X\rangle \langle \alpha(X,Y), \xi_1\rangle = Z(\langle \alpha(X,Y),\xi_1\rangle) - \langle \alpha(X,Y), \nabla_Z^{\perp} \xi_1\rangle$$

Since $X\epsilon T_2$, $Z\epsilon T_1$, and T_2 is parallel we have $\langle Z,\nabla_X X\rangle=0$ and by (3.8) we have in (3.9), $Z(\langle\alpha(X,Y),\xi_1\rangle)=0$ .This implies that $\langle\alpha(X,Y), \xi_1\rangle$ is constant on $\sigma(t)$ and then $\langle\alpha(X,Y), \xi_1\rangle\neq 0$ at $\sigma(b)$, which is the required contradiction.

Now, to complete this proof, we have to consider the case when all the sectional curvatures are null in a neighborhood of x. By our hypothesis there is a plane σ on T_xM such that $k(\sigma)>0$. From Lemma (3.3) we have ξ_2 such that $A_{\xi_2}=0$ and then $\langle R_{\bar{f}}^{\perp}(X,Z)\xi_1,\xi_2\rangle = \langle R^{\perp}(X,Z)\xi_1.\xi_2\rangle=0$ for every $X,Z\epsilon T_xA$. This implies reducibility at x, concluding the proof of the proposition.

In order to prove Theorem 1, first we observe that if the soul A is homeomorphic to the two-dimensional flat torus or flat Klein bottle, as $\mathfrak{R}(X\wedge Z)=0$ for every $X,Z\epsilon T_xA$ and every $x\epsilon A$, we would have in Lemma (3.3) $A_{\xi_2}|_{T_xA}=0$ and this would imply $\nu_{\bar{f}}(x)>0$, for each $x\epsilon A$. This is impossible, since A is compact.

Now, Theorem 1 follows from (2.6), (2.8), (3.7) and the generalization of Theorem (2.2) of [2].

4. PROOF OF THEOREMS 2,3 and 4

(4.1) Proof of Theorem 2.

Let us consider the normal bundle of M along A, denoted by $\upsilon M|_A$, and define a normal connection

$$\tilde{\nabla}^{\perp}: \mathfrak{X}(M) \times \Gamma(\upsilon M|_A) \to \Gamma(\upsilon M|_A)$$

by $\tilde{\nabla}_X^{\perp}\xi =$ orthogonal projection of $\nabla_X^{\perp}\xi$ onto $\upsilon M|_A$. Now, as the inclusion of A in M is totally geodesic, we define a second fundamental form α: $TA \oplus TA \to \upsilon M|_A$ by $\tilde{\alpha}(X,Y) = \alpha(X,Y)$ where α is the second fundamental form of f. It is clear that $\upsilon M|_A, \tilde{\nabla}^{\perp}$ and $\tilde{\alpha}$ verify the Gauss and Codazzi equations. Thus, we need to prove that the condition $\mathfrak{R}(x) \neq 0$ for all $x \in A$, implies the Ricci equation. Denoting by $\bar{\nabla}^{\perp}$ the normal connection for $\bar{f} = f|_A : A \to \mathbb{R}^{n+2}$ we have:

$$\bar{\nabla}_X^{\perp}\xi = \nabla_X^{\perp}\xi - \sum_{i=n-k+1}^{n} \langle \alpha(X,Z_i),\xi\rangle Z_i$$

where $\{Z_{n-k+1},\dots,Z_n\}$ is an orthonormal frame of $T_xA^{\perp}$. Thus, the Ricci equation for $\bar{f}$ is

$$\langle \bar{R}^{\perp}(X,Y)\xi,\eta\rangle = \langle R^{\perp}(X,Y)\xi,\eta\rangle - \sum_{i=n-k+1}^{n} \alpha(X,Z_i),\xi\rangle\langle\alpha(Y,Z_i),\eta\rangle = \langle [\tilde{A}_{\xi},\tilde{A}_{\eta}]X,Y\rangle$$

where $\tilde{A}_{\xi}: T_xA \to T_xA$ is given by $\tilde{A}_{\xi}(X) =$ orthogonal projection of $A_{\xi}X$ onto T_xA. Since by our definition of $\tilde{\nabla}_X^{\perp}\xi$, $\langle \tilde{R}^{\perp}(X,Y)\xi,\eta\rangle = \langle R^{\perp}(X,Y)\xi,\eta\rangle$, all we need is to prove is that

$$\sum_{i=n-k+1}^{n} \langle \alpha(X,Z_i),\xi\rangle \langle \alpha(Y,Z_i),\eta\rangle = 0.$$

But this follows directly from Lemmas (3.2) and (3.3).

This proves that the soul can be locally isometrically immersed in $\mathbb{R}^{k+2}$. Since A is simply connected, Theorem 2 follows.

(4.2) Proof of Theorem 3.

Consider $x\in A$. If f is not reducible at x, we will have by Lemma (3.3) a normal vector ξ_1 such that rank $A_{\xi_1}=1$. Then by the Gauss equation, $\mathfrak{R}(U\wedge V)$ cannot have rank 4. So f is reducible at x.

Now we consider x such that $\nu_{\bar{f}}(x)=0$. If $\alpha(Y,Y)=0$ for every $Y\in T_xA^\perp$, consider $U,V\in T_xA$ such that $\mathfrak{R}(U\wedge V)$ has rank 4. Denoting by U' and U'' the orthogonal projection onto T_xA and $T_xA^\perp$ respectively, we have $\mathfrak{R}(U\wedge V) = \mathfrak{R}(U'\wedge V')$. This implies $k\geq 4$. We claim that M is simpy connected. For otherwise, by the generalizations of Theorems (2.2) and (2.3) of [2], we would have $r(x)=O(k-1)\oplus O(1)$ and then there would be a normal vector ξ such that rank $A_\xi=1$, which contradicts $\mathfrak{R}(U\wedge V)\wedge\mathfrak{R}(U\wedge V)\neq 0$.

If there is $Y\in T_xA^\perp$ such that $\alpha(Y,Y)\neq 0$, we have the conditions of Lemma (3.2) and we can take that normal frame $\{\xi_1,\xi_2\}$. Then $k\geq 2$ (otherwise rank A_{ξ_1} would be one), and all the sectional curvatures along planes tangent to A are positive. If $k\geq 3$, M is simply connected. If $k=2$, A cannot be homeomorphic to flat torus or to the flat Klein Bottle.

Now Theorem (2.8) finishes the proof.

(4.3) <u>Proof of Theorem 4.</u>

Consider $x \in M$ such that $\nu_f(x)=0$. By (2.4) and (3.1) we have $r(x)=\Lambda^2(V)\oplus\Lambda^2(W)$ where $V=T_xA$ and $W=T_xA^{\perp}$. If $\dim W>1$ we are under the conditions of Lemma (3.2). If $\dim W=1$ we are under conditions of Lemma (3.3). In both cases, we have that all the sectional curvatures along planes tangent to A are positive at this point x . Again, if $k\geq 3$, M is simply connected and A^k homeomorphic to S^k. If $k=2$ then A is homeomorphic to S^2 or $\mathbb{R}P^2$. Now, applying Theorem (2.6) to the simply connected case we finish the theorem.

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