SOME REMARKS ABOUT HARMONIC MAPS INTO FLAG MANIFOLDS

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ABSTRACT. This paper is about harmonic maps from closed Riemann surfaces into flag manifolds. We also discuss the stability of certain harmonic maps, the Eells–Wood maps, with respect to a precise set of left-invariant metrics on $F(n) = \dfrac{U(n)}{\underbrace{U(1) \times \ldots \times U(1)}_{n\text{-times}}}$.

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§1. INTRODUCTION

This note is concerned with the problem of understanding harmonic maps from compact Riemann surfaces into flag manifolds. Since minimal surfaces are harmonic in some conformal structure, special cases are provided by minimal surfaces.

In the late 60's Chern [6] and Calabi [5] published several papers on minimal immersions into spheres or more generally real projective spaces, which are in the spirit of this investigation. Then Simons [17], Lawson [16], Hsiang and Lawson [15] published important papers in this direction.

The problem was reexamined by physicists Glaser and Stora [12] and Din-Zakrzewski [8]. They called the attention that the right problem should be to look at harmonic maps into complex projective spaces. Inspired by these ideas Eells and Wood [10] gave a complete classification for harmonic maps from $\mathbb{C}P^1$ to $\mathbb{C}P^n$, and some important partial results for the higher genus cases in terms of holomorphic data. A number of related results have appeared including Burstall and Wood [4], Chern and Wolfson [7], Uhlenbeck [20]. These authours have studied harmonic maps into other homogeneous symmetric spaces like Lie groups and Grassmannians.

On the other hand, very little is known about the homogeneous (non-symmetric) case. One reason that we want to understand the non-symmetric case, beside its own intrinsic interest, is that the finite dimensional flags model in finite dimensions the geometry of the Loop group; i.e., maps from S^1 to a compact Lie Group G. In a well-known paper Atiyah calls attention to the identification of holomorphic maps into the loop group and instantons [2]. See also Freed's paper [11] for more details.

Secondly the energy functional whose critical points are the harmonic maps is more tractable from the point of view of computations than for example the Yang-Mills functional, but appears to share some of their important properties. See Wolfson's paper [21] for more details.

If one wants to understand the problem of harmonic maps into a homogeneous (non-symmetric) space it is natural to start by understanding harmonic maps from closed Riemann surfaces to flag manifolds. These are by definition the quotient of a compact Lie group by any maximal torus. Some of the first work in this problem was done by Guest [14] using a entirelly different approach.

In §2 we discuss some basic geometric facts concerning to flag manifolds.

In §3 we develop our calculus in terms of projection operators. We derive the Euler-Lagrange equations and the expression for the holomorphic quadratic differential in terms of projection operators.

In §4 we describe the holomorphic map equations. We define the Eells-Wood maps and prove that these maps are the only ones that are holomorphic and harmonic when $F(n)$ has the normal Killing form metric.

We also prove several partial results that lead us to expect that any harmonic map from $\mathbb{C}P^1$ to $F(n)$ must be holomorphic with respect to some almost complex structure on $F(n)$.

Finally in §5 we analyse the effect on the index form if we perturb a Kähler metric on $F(n)$. The main results are:

THEOREM A. Let $\psi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow$ $\longrightarrow (F(n), g_{(a_1,\ldots,a_{n-1},a_1+a_2+\varepsilon_1,\ldots,a_1+\ldots+a_{n-1}+\varepsilon_\ell)})$ where $\varepsilon_1,\ldots,\varepsilon_\ell$ are non-negative and ψ is a Eells-Wood map. Then ψ is stable.

THEOREM B. Let $\psi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow$ $\longrightarrow (F(n), g_{(a_1,\ldots,a_{n-1},a_1+a_2-\varepsilon_1,\ldots,a_1+\ldots+a_{n-1}-\varepsilon_\rho)})$ be a full Eells-Wood map where $\varepsilon_1,\ldots,\varepsilon_\ell$ are positive and $a_1+a_2-\varepsilon_1 > 0$, $,\ldots,a_1+\ldots+a_{n-1}-\varepsilon_\ell > 0$. Then ψ is not stable.

COROLLARY. Let $\psi = (\pi_1,\dots,\pi_n) : M^2 \longrightarrow F(n)$ be a full Eells-Wood map where $F(n)$ is equipped with the Killing form metric. Then ψ is not stable.

Furthermore we give a lower bound for the index of the full Eells-Wood maps for any left-invariant metric that appears in Theorem B, and compute precisely the index of the Eells-Wood maps when $F(3)$ is equipped with the Killing form metric.

The contents of this paper are part of my doctoral thesis [17]. I want to express my gratitude to my thesis advisor Prof. Karen Uhlenbeck for her profound advise, criticism and encouragement.

§2. BASIC DEFINITIONS

We call a flag manifold the quotient of a compact Lie group by any maximal torus e $F(n) = \dfrac{U(n)}{\underbrace{U(1)\times\dots\times(U1)}_{n\text{-times}}}$

Let S^+ be the set of positive roots with respect to a choice of fundamental Weyl chamber in $\underbrace{u(1)\times\dots\times u(1)}_{n\text{-times}}$. According to [3], [14] or [17] we can see that $F(n)$ has $2^{|S^+|}$ almost complex structures from which $n!$ which is equal to the order of the Weyl group of $U(n)$ are integrable.

We define a family of left invariant inner products on $F(n) = \dfrac{U(n)}{U(1)\times\dots\times U(1)}$ where $u(n) = p \oplus u(1) \oplus \dots \oplus u(1)$. Let $A,B \in p$ and consider the inner product

$$\langle A,B\rangle_a = \mathrm{tr}(a^{ij}E_iAE_jB^*). \text{ where } E_i = \begin{array}{cc} & \begin{array}{cccc} & & i & \end{array} \\ i & \begin{bmatrix} 0 & & & \\ & \ddots & & 0 \\ & & 0 & \\ 0 & & 1 & \\ & & 0 & \\ & & & \ddots\ 0 \end{bmatrix} \end{array}, a = (a^{ij})$$

with $a^{ij} > 0$. Note that we are using the Einstein summation convention. We can see that for each $a = (a^{ij})$, $a^{ij} = a^{ji} > 0$, $\langle\ ,\ \rangle_a$ is left invariant.

If we restrict our almost complex structures in $F(n)$ to the integrable ones, we can see that $g_{(a_1,\ldots,a_n,a_1+a_2,\ldots,a_{n-2}+a_{n-1},\ldots,a_{n-3}+a_{n-2}+a_{n-1},\ldots,a_1+a_2+\ldots+a_{n-1})}$ give all left-invariant Kähler metrics on $F(n)$. (See V.11, Proposition in [17] for more details). It is worthwhile to point out that if we consider $F(n)$ with the normal metric induced from the natural bi-invariant metric on $U(n)$ such metric is not Kahler (See [17] for a proof of this fact). We will call such normal metric on $F(n)$ the Killing form metric.

§3. HARMONIC MAPS FROM COMPACT RIEMANN SURFACES TO FLAG MANIFOLDS AND A CONSERVATION OF ENERGY FORMULA

We recall that $F(n)$ is the set of n-tuples $(L_1,\ldots,L_n)$, where L_i is a 1-dimensional subspace of $\mathbb{C}^n$, L_i is perpendicular to L_j if $i \neq j$ and $L_1 \oplus \ldots \oplus L_n = \mathbb{C}^n$. Then the toutolologously defined vector bundles on $F(n)$ have as fibres over a flag $(L_1,\ldots,L_n)$ the vector spaces $L_1,\ldots,L_n$ respectively. We denote these bundles by the same letters $L_1,\ldots,L_n$.

As usual we identify a smooth map $\phi : M^2 \longrightarrow \mathbb{C}P^{n-1}$ with a subbundle $\underset{\sim}{\phi}$ of $\underline{\mathbb{C}}^n = M^2 \times \mathbb{C}^n$ of rank one which has fibre at $x \in M$ given by: $\underline{\phi}_x = T_{\phi(x)}$ where T is the tautological line bundle over $\mathbb{C}P^{n-1}$, i.e. $\underline{\phi} = \phi^*(T)$.

Any subbundle $\underline{\phi}$ of $\underline{\mathbb{C}}^n$ inherits a metric denoted by $\langle\ ,\ \rangle_\phi$ and connection denoted by D_ϕ, from the flat metric and connection ∂ on $\underline{\mathbb{C}}^n$. Explicitly: $\langle V,W\rangle_\phi = \langle V,W\rangle$ for any $V, W \in \phi_x$, $x \in M^2$ and $(D_\phi)_Z W = \pi_\phi(\partial_Z W)$, $W \in \Gamma(\underline{\phi})$, $Z \in D(M)^{(1,0)}$. Here $\pi_\phi : \underline{\mathbb{C}}^n \longrightarrow \underline{\phi}$ denotes the Hermitian projection on the subbundle $\underset{\sim}{\phi}$.

Note that we always describe $F(n)$ in terms of the natural embedding $F(n) \longrightarrow \underbrace{\mathbb{C}P^{n-1} \times \ldots \times \mathbb{C}P^{n-1}}_{n\text{-times}}$. So $\phi : M^2 \longrightarrow F(n)$ is described as $\phi = (\pi_1, \ldots, \pi_n)$ where $\pi_j : M^2 \longrightarrow \mathbb{C}P^{n-1}$ and $\pi_i \pi_j = \delta_{ij}\pi_i$.

Now consider $\phi = (\pi_1, \ldots, \pi_n) : M^2 \longrightarrow F(n)$ be a smooth map. ϕ determines the tautologously defined vector bundles $\pi_1, \ldots, \pi_n$ over M^2. Let $\frac{\partial \pi_i}{\partial x} = \frac{\partial \pi_i}{\partial x}$ be the covariant derivative of π_i with respect to x and A_x^i the projection of $\frac{\partial \pi_i}{\partial x}$ onto $\pi_i^{\perp}$. See [17] for more details.

3.1. DEFINITION. We call the partial second fundamental forms of $\phi = (\pi_1, \ldots, \pi_n) : M^2 \longrightarrow F(n)$ the maps: $A_x^{ij} = \pi_i (A_x^j) = \pi_i \frac{\partial \pi_j}{\partial x}$ if $i \neq j$.

Note that $A_x^{ij} \in \mathrm{Hom}(\pi_j, \pi_i)$ and $\sum_j A_x^{ij}$ is the second fundamental form of the span the π_i in $\mathbb{C}^n$.

Now if we think in M^2 as a complex one dimensional manifold, then we define $\frac{\partial \pi_i}{\partial z} = \frac{1}{2}\left(\frac{\partial \pi_i}{\partial x} - \sqrt{-1}\,\frac{\partial \pi_i}{\partial y}\right)$ and $\frac{\partial \pi_i}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial \pi_i}{\partial x} + \sqrt{-1}\,\frac{\partial \pi_i}{\partial y}\right)$. We also define $A_z^{ij} = \pi_i \frac{\partial \pi_j}{\partial z}$ and $A_{\bar{z}}^{ij} = \pi_i \frac{\partial \pi_j}{\partial \bar{z}}$. See [17] fore more details about some basic formal properties of the A_μ^{ij}'s where $\mu = z$ or $\bar{z}$.

The following formula will be very useful.

3.2. PROPOSITION. $\displaystyle\sum_{i,j=1}^{n} \left(\frac{\partial}{\partial z}(A_{\bar{z}}^{ij}) - \frac{\partial}{\partial \bar{z}}(A_z^{ij})\right) = 2 \sum_{\substack{i,j=1 \\ (i\neq j)}}^{n} [A_z^{ij}, A_{\bar{z}}^{ji}] +$

$$+ \sum_{p,i,j=1}^{n} [A_z^{ij}, A_{\bar{z}}^{jp} + A_{\bar{z}}^{pi}]$$

PROOF. Let $i \neq j$ and consider

$$\frac{\partial}{\partial z}(A^{ij}_{\bar z}) - \frac{\partial}{\partial \bar z}(A^{ij}_{z}) = \frac{\partial}{\partial z}(\pi_i \frac{\partial \pi_j}{\partial \bar z}) - \frac{\partial}{\partial \bar z}(\pi_i \frac{\partial \pi_j}{z}) =$$

$$= \frac{\partial \pi_i}{\partial z}\cdot\frac{\partial \pi_j}{\partial \bar z} - \frac{\partial \pi_i}{\partial \bar z}\cdot\frac{\partial \pi_j}{\partial z} \text{ . But } \frac{\partial \pi_i}{\partial z} = A^i_z + (A^i_z)^* =$$

$$= \sum_{k(\neq i)} A^{ki}_z - \sum_{\ell(\neq i)} A^{i\ell}_z, \text{ and } \frac{\partial \pi_j}{\partial \bar z} = \sum_{n(\neq j)} A^{mj}_{\bar z} - \sum_{p(\neq j)} A^{jp}_{\bar z} .$$

Hence it follows that

$$\frac{\partial \pi_i}{\partial z}\cdot\frac{\partial \pi_j}{\partial \bar z} = (\sum_k A^{ki}_z - \sum_\ell A^{i}_z)\ (\sum_m A^{mj}_{\bar z} - \sum_p A^{jp}_{\bar z}) =$$

$$= \sum_k A^{ki}_z\cdot A^{ij}_{\bar z} - \sum_\ell A^{i\ell}_z\cdot A^{\ell i}_{\bar z} + \sum_p A^{ij}_z\cdot A^{jp}_{\bar z} .$$

Similarly

$$\frac{\partial \pi_i}{\partial \bar z}\cdot\frac{\partial \pi_j}{\partial z} = \sum_k A^{ki}_{\bar z}\cdot A^{ij}_z - \sum_j A^{ij}_{\bar z}\cdot A^{ji}_z - \sum_\ell A^{i\ell}_{\bar z}\cdot A^{\ell j}_z + \sum_p A^{ij}_{\bar z}\cdot A^{jp}_z .$$

Then we get:

$$\sum_{i,j}(\frac{\partial}{\partial z}(A^{ij}_{\bar z}) - \frac{\partial}{\partial \bar z}(A^{ij}_z)) = \sum_{k,i,j} A^{ki}_z\cdot A^{ij}_{\bar z} - \sum_{k,i,j} A^{ki}_{\bar z}\cdot A^{ij}_z$$

$$- \sum_{\ell,i,j} A^{i\ell}_z\cdot A^{\ell j}_{\bar z} + \sum_{\ell,i,j} A^{i\ell}_{\bar z}\cdot A^{\ell j}_z + \sum_{p,i,j} A^{ij}_z\cdot A^{jp}_{\bar z} -$$

$$- \sum_{p,i,j} A^{ij}_{\bar z}\ A^{jn}_z = \sum_{k,i,j} A^{ki}_z\ A^{ij}_{\bar z} + \sum_{i,j} A^{ji}_z\ A^{ij}_{\bar z} -$$

$$- \sum_{k,i,j} A^{ki}_{\bar{z}} \cdot A^{ij}_{z} - \sum_{i,j} A^{ji}_{\bar{z}} \cdot A^{ij}_{z} - \sum_{\ell,i,j} A^{i\ell}_{z} \cdot A^{\ell j}_{\bar{z}} +$$

$$+ \sum_{\ell,i,j} A^{i\ell}_{\bar{z}} \cdot A^{\ell j}_{z} + \sum_{p,i,j} A^{ij}_{z} \cdot A^{jn}_{\bar{z}} + \sum_{i,j} A^{ij}_{z} \cdot A^{ji}_{\bar{z}} + \sum_{p,i,j} A^{ij}_{\bar{z}} \cdot A^{jp}_{z}$$

$$- \sum_{i,j} A^{ij}_{\bar{z}} \cdot A^{ji}_{z} = 2 \sum_{i,j} [A^{ij}_{z}, A^{ji}_{\bar{z}}] + \sum_{p,i,j} [A^{ij}_{z}, A^{jp}_{\bar{z}} + A^{pi}_{\bar{z}}].$$

Now let $\mu = z$ or $\bar{z}$ and

$$A_\mu = \begin{bmatrix} 0 & A^{12}_\mu & \cdots & A^{1n}_\mu \\ A^{21}_\mu & 0 & & A^{2n}_\mu \\ \vdots & & & \\ A^{n1}_\mu & A^{n2}_\mu & \cdots & 0 \end{bmatrix}$$

We can rewrite the formula above as:

3.3. COROLLARY. $\frac{\partial}{\partial z}(A_{\bar{z}}) - \frac{\partial}{\partial \bar{z}}(A_z) = [A_z, A_{\bar{z}}] + [A_z, A_{\bar{z}}]_\hbar$ where $[A_z, A_{\bar{z}}]_\hbar$ denotes the diagonal part of the matriz $[A_z, A_{\bar{z}}]$.

We now study the energy integral in terms of projection operators and write down exprecitly the Euler-Lagrange equations for our variational problem.

3.4. DEFINITION. Given a smooth map $\phi = (\pi_1, \ldots, \pi_n) : M^2 \longrightarrow F(n) = \frac{U(n)}{U(1) \times \ldots \times U(1)}$, we define the energy of ϕ as:

$$E(\phi) = \frac{1}{2} \sum_{i=1}^{n} \int_{M^2} \left(\left| \frac{\partial \pi_i}{\partial x} \right|^2 + \left| \frac{\partial \pi_i}{\partial y} \right|^2 \right) dx\, dy \ .$$

We now prove some formulas that come from the conservations laws associated with the invariance of E under the action of U(n) according to Noether's theorem. See [1] for more details.

We call $q : M^2 \longrightarrow \mu(n)$ a angular momentum map. Given $\phi =$ $= (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow F(n)$ let $[\pi_i,q] = \pi_i q^* + q^*\pi_i = \pi_i q - q\pi_i$. The map q gives rise naturally a to variation of ϕ, $\delta\phi(q) : M^2 \longrightarrow F(n)$ given by: $(\delta\phi)(q)(x) = (\frac{d}{dt}|_{t=0} \exp \operatorname{ad} e^{-tq(x)}\pi_1(x),\ldots,\frac{d}{dt}|_{t=0} \exp \operatorname{ad} e^{-tq(x)}\pi_n(x)) = ([\pi_1(x),q(x)],\ldots,$ $,[\pi_n(x),q(x)])$.

The we can compute the first variation of the energy for ϕ.

3.5. PROPOSITION. Let $\phi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow F(n)$ be a smooth map. Then:

$$(\delta E)(\delta\phi(q)) = - \sum_{i=1}^{n} \langle\langle [\pi_i,\Delta\pi_i],q \rangle\rangle \quad \text{where}$$

$\langle\langle\ ,\ \rangle\rangle$ is the L^2-Hilbert inner product on $C^0(M^2,F(n))$.

PROOF. $$(\delta E)(\delta\phi(q)) = \sum_{i=1}^{n} \int_{M^2} (\langle \frac{\partial\pi_i}{\partial x}, \frac{\partial}{\partial x}(\delta\pi_i(q))\rangle + \langle \frac{\partial\pi_i}{\partial y}, \frac{\partial}{\partial y}(\delta\pi_i(q))\rangle)\,dx\,dy.$$

But since the boundary of M^2 is empty if we integrate by parts we have:

$$(\delta E)(\delta\phi(q)) = \sum_{i=1}^{n} \int_{M^2} \langle -\Delta\pi_i, \delta\pi_i(q)\rangle\, dxdy = \sum_{i=1}^{n} \langle\langle -\Delta\pi_i, [\pi_i,q]\rangle\rangle .$$

But by using the cyclic property of trace we can easily see that $\langle\langle A,[B,C]\rangle\rangle = \langle\langle [B^*,A],C\rangle\rangle$. Then finally we can prove:

$$(\delta E)(\delta\phi(q)) = - \sum_{i=1}^{n} \langle\langle [\pi_i,\Delta\pi_i],q \rangle\rangle .$$

We know that $\phi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow F(n)$ is harmonic if and only if it is a critical point of the energy integral; i.e., for any variation $\delta\phi(q)$ of ϕ we have $(\delta E)(\delta\phi(q)) = 0$. Then by using the fundamental lemma of the calculus of variations we prove the useful harmonic map equations for our variational problem:

3.6. COROLLARY. Let $\phi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow F(n)$. ϕ is harmonic if and only if $\sum_{i=1}^{n} [\pi_i,\Delta\pi_i] = 0$. $\circ$

It will be also useful to write the harmonic map equations in a divergence free form.

3.7. COROLLARY. $\phi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow F(n)$ is harmonic if and only if $\sum_{i,j} (\frac{\partial}{\partial z} (A^{ij}_{\bar z}) + \frac{\partial}{\partial \bar z}(A^{ij}_z)) = 0$.

We will close this paragraph by proving a conservation law formula for harmonic maps $\phi : \mathbb{C}P^1 \longrightarrow F(n)$ using our calculus, which is the minimal map equation (see [9]).

3.8. THEOREM. Let $\mu = z$ or $\bar z$ and

$$A_\mu = \begin{bmatrix} 0 & A^{12}_\mu & \ldots & A^{1n}_\mu \\ A^{21}_\mu & 0 & \ldots & A^{2n}_\mu \\ \vdots & & & \\ A^{n1}_\mu & A^{n2}_\mu & \ldots & 0 \end{bmatrix}$$

where A^{ij} are the partial second fundamental forms of a harmonic map $\phi = (\pi_1,\ldots,\pi_n) : \mathbb{C}P^1 \longrightarrow F(n)$. Then:

$$0 = \mathrm{tr}(A_\mu) = \mathrm{tr}(A_\mu^2)$$

PROOF. Clearly $\mathrm{tr}(A_\mu) = 0$. According to 3.3 Corollary $\frac{\partial}{\partial \bar z}(A_z) - \frac{\partial}{\partial z}(A_{\bar z}) = [A_z,A_{\bar z}] + [A_z,A_{\bar z}]_{\hbar}$. We will prove the proposition for

$\mu = z$ but, of course the same proof works for $\mu = \bar{z}$. Since is harmonic 3.7 Corollary tells us that $\frac{\partial}{\partial \bar{z}}(A_z) + \frac{\partial}{\partial z}(A_{\bar{z}}) = 0$, therefore $\frac{\partial}{\partial \bar{z}}(A_z) = \frac{1}{2}[A_z, A_{\bar{z}}] + \frac{1}{2}[A_z, A_{\bar{z}}]_{\hbar}$.

But $\frac{\partial}{\partial \bar{z}}(\mathrm{tr}(A_z^2)) = \mathrm{tr}(\frac{\partial}{\partial \bar{z}}(A_z^2)) =$

$$= \mathrm{tr}(\tfrac{1}{2} A_z [A_z, A_{\bar{z}}] + \tfrac{1}{2} A_z [A_z, A_{\bar{z}}]) + \mathrm{tr}([A_z, A_{\bar{z}}] \tfrac{1}{2} A_z +$$

$$+ [A_z, A_{\bar{z}}]_{\hbar} A_z) = \mathrm{tr}([A_z^2, A_{\bar{z}}]) + \mathrm{tr}(A_z [A_z, A_{\bar{z}}]_{\hbar}) = 0$$

since A_z has only zeros in the main diagonal and $[A_z, A_{\bar{z}}]$ is a diagonal matrix.

Therefore $\mathrm{tr}(A_z^2)dz^2$ is a holomorphic 2-form over $\mathbb{C}P^1$. But then according to Riemann-Rock we have that $\mathrm{tr}(A_z^2) = 0$.

4. HOLOMORPHIC MAPS FROM COMPACT RIEMANN SURFACES TO FLAG MANIFOLDS

In this paragraph we consider $F(n)$ equipped with the Killing form metric. M^2 will always denote a compact Riemann surface without boundary and all maps will be smooth.

Let $\phi = (\pi_1, \ldots, \pi_n) : M^2 \longrightarrow F(n)$ then:

$$E(\phi) = \sum_{i,j=1}^{n} \int_{M^2} |A_\mu^{ij}|^2 V_g = - \sum_{i,j=1}^{m} \int_{M^2} \mathrm{tr}(A_\mu^{ij} A_{\bar{\mu}}^{ji}) V_g$$

where $\mu = z$ or $\bar{z}$ and A^{ij} are the partial second fundamental associated to ϕ.

Now let $[1,n] = \{x \in \mathbb{Z} ; 1 \leq x \leq n\}$. Consider $D = \{(i,i) ; 1 \leq i \leq n\}$ and S^+ to be a partition of $[1,n] \times [1,n] - D$ containg $\frac{(n^2-n)}{2}$ elements such that if $(i,j) \in S^+$ then $(j,i) \notin S^+$. Let S^- to be the complement of S^+ in $[1,n] \times [1,n] - D$. We call S^+ by a positive system in $[1,n]$.

4.1. DEFINITION. Let E^o and $\overline{E}$ denote the ∂ and $\overline{\partial}$-energy respectively, defined by

$$E^o_{S^+}(\phi) = \sum_{(i,j)\in S^+} \int_{M^2} |A_z^{ij}|^2 V_g \quad \text{and}$$

$$\overline{E}_{S^+}(\phi) = \sum_{(i,j)\in S^-} \int_{M^2} |A_z^{ij}|^2 V_g = \sum_{(i,j)\in S^+} \int_{M^2} |A_{\overline{z}}^{ij}|^2 V_g$$

where S^+ is a positive system in $[1,n]$.

Therefore $\phi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow F(n)$ is holomorphic with respect to the almost complex structure determined by the positive system S^+ if and only if $A_{\overline{z}}^{ij} = 0$, $\forall (i,j) \in S^+$.

As corollary we can prove:

4.2. PROPOSITION. Let $\phi = (\pi_1,\pi_2,\pi_3) : \mathbb{C}P^1 \longrightarrow F(3)$ be a harmonic map such that $\pi_1 : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2$ is holomorphic or antiholomorphic. Then ϕ is holomorphic with respect to some almost complex on $F(3)$.

PROOF. By above $\pi_1 : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2$ is $\pm$-holomorphic if and only if $A_\mu^{12} A_\mu^{21} = A_\mu^{13} A_\mu^{31} = 0$. Now according to 3.8 Theorem we have that $0 = tr(A_\mu^2) = A_\mu^{12} A_\mu^{21} + A_\mu^{13} A_\mu^{31} + A_\mu^{23} A_\mu^{32}$. Theorefore $A_\mu^{ij} A_\mu^{ji} = 0$, $\forall (i,j)$ such that $1 \leq i,j \leq 3$. Therefore ϕ is holomorphic with respect to some almost complex structure on $F(3)$.

Now we prove a very useful formula, namely:

4.3. LEMMA. $\pi_i \frac{\partial}{\partial \overline{\mu}}(A_\mu)\pi_j = \frac{\partial}{\partial \overline{\mu}}(A_\mu^{ij}) + [A_\mu^{ij}, A_{\overline{\mu}}] + \pi_i [A_{\overline{\mu}}, A_\mu]\pi_j$, where $\mu = z$ or $\overline{z}$.

PROOF. $\frac{\partial}{\partial \overline{\mu}}(A_\mu^{ij}) = \frac{\partial}{\partial \overline{\mu}}(\pi_i A_\mu \pi_j) = \frac{\partial \pi_i}{\partial \overline{\mu}} A_\mu \pi_j +$

$$+ \pi_i \frac{\partial(A_\mu)}{\partial\bar\mu}\pi_j + \pi_i A_{\bar\mu}\frac{\partial\pi_j}{\partial\bar\mu} = (\sum_k A^{ki}_{\bar\mu} - \sum_\ell A^{i\ell}_{\bar\mu}).$$

$$\cdot(\sum_m A^{mj}_\mu) + \sum_k A^{ik}_\mu(\sum_\ell A^{\ell j}_{\bar\mu} - \sum_m A^{jm}_{\bar\mu}) + \pi_i \frac{\partial(A_\mu)}{\partial\bar\mu}\pi_j =$$

$$= \pi_i[A_\mu, A_{\bar\mu}]\pi_j + [A_{\bar\mu}, A^{ij}_\mu] + \pi_i \frac{\partial(A_\mu)}{\partial\bar\mu}\pi_j \,.$$

Hence: $$\pi_i \frac{\partial(A_\mu)}{\partial\bar\mu}\pi_j = \frac{\partial}{\partial\bar\mu}(A^{ij}_\mu) + [A^{ij}_\mu, A_{\bar\mu}] + \pi_i[A_{\bar\mu}, A_\mu]\pi_j \,.$$

We will make use of the formula above. We start by defining:

4.4. DEFINITION. Let $\phi = (\pi_1,\dots,\pi_n) : M^2 \longrightarrow F(n)$ be a harmonic map. ϕ is called totally isotropic if $[A_z, A_{\bar z}]_p = 0$ where $[A_z, A_{\bar z}]_p$ denotes the off diagonal part of the $n\times n$-matrix $[A_z, A_{\bar z}]$.

Our result is that totally isotropic maps behave like maps into symmetric spaces as suggests 3.3 Corollary. In fact we conjecture that any harmonic map $\phi = (\pi_1,\dots,\pi_n) : \mathbb{C}P^1 \longrightarrow F(n)$ is totally isotropic.

4.5. THEOREM. Let $\phi = (\pi_1,\dots,\pi_n) : M^2 \longrightarrow F(n)$ be a totally isotropic map. Then $A^{ij} \in \mathrm{Hom}(\pi_j,\pi_i) \cong \pi_j^* \otimes \pi_i$ is a holomorphic section of the line bundle $\pi_j^* \otimes \pi_i$ over M^2 when such bundle has a suitable complex structure.

PROOF. Let L be the canonical line bundle over $\mathbb{C}P^{n-1}$ and we also denote by π_i the pull-back of L_i via $\pi_i : M^2 \longrightarrow \mathbb{C}P^{n-1}$.

Now let us define the connection

$\nabla^{ji} : D(M)^{(1,0)} \times \Gamma(\pi_j^* \otimes \pi_i) \longrightarrow \Gamma(\pi_j^* \otimes \pi_i)$ by:

$\nabla^{ji}_{\frac{\partial}{\partial z}}(Q) = \frac{\partial Q}{\partial z} + [Q, A_z]$ for any $Q \in \Gamma(\mathrm{Hom}(\pi_j,\pi_i))$. Similarly, we define $\nabla^{ji}_{\frac{\partial}{\partial \bar{z}}}(Q) = \frac{\partial Q}{\partial \bar{z}} + [Q, A_{\bar{z}}]$.

On the other hand 4.3 Proposition tells that $\pi_i \frac{\partial (A_z)}{\partial \bar{z}} \pi_j =$

$$= \frac{\partial}{\partial \bar{z}}(A^{ij}_z) + [A^{ij}_z, A_{\bar{z}}] + \pi_i[A_{\bar{z}}, A_z]\pi_j = \nabla^{ji}_{\frac{\partial}{\partial \bar{z}}}(A^{ij}_z) + \pi_i[A_{\bar{z}}, A_z]\pi_j \quad (*).$$

But since ϕ is totally isotropic we have that $\pi_i[A_z, A_{\bar{z}}]\pi_j = 0$ if $i \neq j$ since $[A_z, A_{\bar{z}}]_p = 0$.

On the other hand, since ϕ is harmonic we must have:

$$\frac{\partial}{\partial \bar{z}}(A_z) = [A_{\bar{z}}, A_z] - \frac{1}{2}[A_{\bar{z}}, A_z]_p = [A_{\bar{z}}, A_z].$$ Therefore

$\pi_i \frac{\partial}{\partial \bar{z}}(A_{\bar{z}})\pi_j = 0$ if $i \neq j$. Hence by (*) we have that $\nabla^{ji}_{\frac{\partial}{\partial \bar{z}}}(A^{ij}_z) = 0$.

Now by making use of the Theorem of Koszul and Malgrange we can put on $\pi^*_j \otimes \pi_i$ a complex structure whose $\bar{\partial}$-operator is given $\nabla^{ji}_{\frac{\partial}{\partial \bar{z}}}$. Hence with respect to such complex structure we have that A^{ij}_z is a holomorphic section of $\mathrm{Hom}(\pi_j, \pi_i)$.

In holomorphic coordinates (U,z) $\overset{o}{A}{}^{ij} \circ \overset{o}{A}{}^{ji}$ has the form $dz \otimes dz\, A^{ij}_z \circ A^{ji}_z$. Let s_1 be a elementary symmetric function of $A^{ij}_z \circ A^{ji}_z$. It is clearly holomorphic on U. Moreover $S_1 = s_1 dz \otimes \otimes dz$ is globally defined on M^2. By above S_1 is clearly holomorphic. Therefore:

4.6. COROLLARY. If $\phi = (\pi_1, \ldots, \pi_n) : \mathbb{C}P^1 \longrightarrow F(n)$ is a totally isotropic map then $S_1 = 0$. Therefore $A^{ij}_z A^{ji}_z = 0$, $\forall 1 \leq i,j \leq n$ that is, ϕ is holomorphic with respect to some almost complex structure on $F(n)$.

PROOF. S_1 is a holomorphic section of $\overset{2}{\otimes} T(S^2)^*$. Hence by

Riemann-Rock's theorem $S_1 = 0$. Therefore $tr(A_z^{ij} A_z^{ji}) = 0$, $\forall i,j$. But each A_μ^{ij} is analytic and rank of of $A_z^{ij} A_z^{ji}$ is less or equal to 1 hence $A_z^{ij} A_z^{ji} = 0$, $\forall i,j$, i.e., ϕ is holomorphic.

Now suppose $h : M^2 \longrightarrow \mathbb{C}P^{n-1}$ is a full (non-degenerate) holomorphic map. See [13] for more details. Let h be given locally by $\mu(z) = [\mu_o(z),\ldots,\mu_{n-1}(z)]$ then we define the k-th associated curve of h called σ_k by: $\sigma_k : M^2 \longrightarrow G_{k+1}(\mathbb{C}^n)$ where $\sigma_k(z) = \mathrm{span}[\mu(z) \wedge \mu'(z) \wedge \ldots \wedge \mu^{(k)}(z)]$. We can see that σ_k is a well-defined. Then we consider $h_k : M^2 \longrightarrow \mathbb{C}P^{n-1}$ given by:

$$h_k(z) = \sigma_k(z) \cap \sigma_{k+1}^{\perp}(z) .$$

We have the following important theorem due to Burns, Din, Eells, Glaser, Stora, Wood and Zakarewski.

4.7. THEOREM. For each $0 \le k \le n-1$, we have that $h_k : M^2 \longrightarrow \mathbb{C}P^{n-1}$ is harmonic. Furthermore, given $\phi : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^{n-1}$ harmonic, then there exists a unique $0 \le k \le n-1$ and $h : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^{n-1}$ full and holomorphic such that $\phi = h_k$ (Note that $h_o = h$ is holomorphic and h_{n-1} as antiholomorphic).

PROOF. See [10].

Therefore we have canonical maps $\psi : M^2 \longrightarrow F(n)$ called Eells-Wood maps given by: $\psi(x) = (h_o(x),\ldots,h_{n-1}(x))$. We can see that the Eells-Wood maps are holomorphic. Moreover ψ is harmonic and $A_{\bar{z}}^{ij} = 0$ unless i and j are consecutive integers between 1 and n.

Reciprocally, we can prove the interesting result:

4.8. THEOREM. If $\psi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow F(n)$ is harmonic and holomorphic with respect to the almost complex structure on $F(n)$ determined by S^+. Then ψ is a Eells-Wood map.

PROOF. If X is a nxn-matrix we denote the by X^{S^+} the matrix $\sum_{(i,j)\in S^+} \pi_i X \pi_j$.

By using 4.8 Corollary we see that $(\frac{\partial}{\partial \bar z}(A_z))^{S^+} - (\frac{\partial}{\partial z}(A_{\bar z}))^{S^+} =$ $= [A_{\bar z}, A_z]$.

But ϕ is harmonic if and only if $\frac{\partial}{\partial \bar z}(A_z) + \frac{\partial}{\partial z}(A_{\bar z}) = 0$. Therefore

$$(\frac{\partial}{\partial \bar z}(A_z))^{S^+} = \frac{1}{2}[A_{\bar z}, A_z]^{S^+} \qquad (*)$$

But 4.3 Lemma tells us that:

$$(\frac{\partial}{\partial z}(A_{\bar z}))^{S^+} = \frac{\partial}{\partial z}(A_{\bar z}^{S^+}) + [A_{\bar z}^{S^+}, A_z] + [A_z, A_{\bar z}]^{S^+} \quad (**) .$$

But by hypothesis ϕ is holomorphic with respect to S^+ then $A_{\bar z}^{S^+} = 0$.

Therefore by using $(**)$ we see that $(\frac{\partial}{\partial z}(A_{\bar z}))^{S^+} = [A_z, A_{\bar z}]^{S^+}$.
Now if we use $(*)$ we conclude that $[A_z, A_{\bar z}]^{S^+} = 0$. But $[A_z, A_{\bar z}]^* =$ $= [A_z, A_{\bar z}]$ then $[A_z, A_{\bar z}]^{S^+} = 0$. Hence $[A_z, A_{\bar z}]_p = 0$; i.e., ϕ must be a Eells-Wood map.

§5. STABILITY OF HARMONIC MAPS FROM COMPACT RIEMANN SURFACES TO FLAG MANIFOLDS

We start this paragraph by proving

5.1. LEMMA. $\delta(A^{ij}_\mu)(q) = [A^{ij}_\mu, q] - \pi_i \frac{\partial q}{\partial \mu} \pi_j$, for any $q: M^2 \longrightarrow \mu(n)$ and $\mu = z$ or $\bar z$.

PROOF. $\delta(\pi_i \frac{\partial \pi_j}{\partial \mu})(q) = \delta(\pi_i)(q) \frac{\partial \pi_j}{\partial \mu} + \pi_i \frac{\partial}{\partial \mu}([\pi_j, q]) =$

$$[\pi_i,q]\frac{\partial\pi_j}{\partial\mu} + \pi_i\left[\frac{\partial\pi_j}{\partial\mu},q\right] + \pi_i\left[\pi_j,\frac{\partial q}{\partial\mu}\right] = \pi_i q\frac{\partial\pi_j}{\partial\mu} - qA^{ij}_\mu +$$

$$+ A^{ij}_\mu q - \pi_i q\frac{\partial\pi_j}{\partial\mu} - \pi_i\frac{\partial q}{\partial\mu}\pi_j = [A^{ij}_\mu,q] - \pi_i\frac{\partial q}{\partial\mu}\pi_j\ .$$

Let g_a be the left-invariant metric on $F(n)$ induced by: $\langle A,B\rangle_a = tr(a^{ij}E_iAE_jB^*)$ where $a^{ij} = a^{ji} > 0$. Also consider ${}^aA^{ij}_\mu = a^{ij}A^{ij}_\mu$ where $\mu = z$ or $\bar z$ and $A^a_\mu = ({}^aA^{ij}_\mu)$.

5.2. PROPOSITION. Let $\phi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow (F(n),g_a)$ be any smooth map. Then $(\delta E)(\delta q) = -2Re\{a^{ij}\int_{M^2}\langle A^{ij}_\mu,\frac{\partial q}{\partial\mu}\rangle\} =$ $= -2Re\{\langle\langle {}^aA_\mu,\frac{\partial q}{\partial\mu}\rangle\rangle\}$ where $\mu = z$ or $\bar z$ and $\langle\langle\ ,\ \rangle\rangle$ denotes the L^2-Hilbert inner product.

PROOF. $E = \int_{M^2} a^{ij}\langle\pi_i\frac{\partial\pi_j}{\partial\mu},\pi_i\frac{\partial\pi_j}{\partial\mu}\rangle v_g$. From now on we will omit the symbol $\int_{M^2}$. Then: $\delta E = -2Re\{a^{ij}\langle A^{ij},\delta(A^{ij})\rangle\} =$

$$\underset{\text{position}}{\overset{\text{by 5.1 Pro-}}{=\!=\!=}} -2Re\{a^{ij}\langle A^{ij}_\mu,[A^{ij}_\mu,q] - \pi_i\frac{\partial q}{\partial\mu}\pi_j\rangle\} =$$

$$= -2Re\{a^{ij}\langle[A^{ji}_{\bar\mu},A^{ij}_\mu],q\rangle\} - 2Re\{a^{ij}\langle A^{ij},\frac{\partial q}{\partial\mu}\rangle\} =$$

$$= 2Re\{\langle {}^aA^{ij}_\mu,\frac{\partial q}{\partial\mu}\rangle\}\quad\text{since}\quad Re\{a^{ij}\langle[A^{ij}_{\bar\mu},A^{ji}_\mu],q\rangle\} = 0\ .$$

Then by using 5.2 Proposition we can prove that the Eells-Wood maps are harmonic with respect to any left-invariant metric g_a on $F(n)$ (See [17] for a proof of this fact and [14] for a different proof of the same fact).

Now let us compute the second variation of the energy.

5.3. PROPOSITION. Let $\phi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow (F(n),g_a)$ be a harmonic map. Then

$$(\delta^2 E_a)(\delta\phi(q)) = I_a^{\phi}(q) = 2\mathrm{Re}\{\int_{M^2} \langle a^{ij}\pi_i \frac{\partial q}{\partial \bar z}\pi_j +$$

$$+[q,{}^aA_{\bar z}], \frac{\partial q}{\partial \bar z}\rangle v_g \quad .$$

PROOF. We will omit $\int_{M^2} \sum_{i,j}$. We have: $\delta E_a = -2\mathrm{Re}\{a^{ij}\langle A^{ij}, \frac{\partial q}{\partial \bar z}\rangle\}$.

Hence:

$$\delta^2 E_a = -2\mathrm{Re}\{a^{ij}\langle [A^{ij}_{\bar z}, q] - \pi_i \frac{\partial q}{\partial \bar z}\pi_j , \frac{\partial q}{\partial \bar z}\rangle\} =$$

$$= -2\mathrm{Re}\,\{\langle [{}^aA_{\bar z}, q] - a^{ij}\pi_i \frac{\partial q}{\partial \bar z}\pi_j , \frac{\partial q}{\partial \bar z}\rangle\} \quad \text{hence}$$

$$(\delta^2 E_a)(\delta\phi(q)) = 2\mathrm{Re}\{\int_{M^2} \langle a^{ij}\pi_i \frac{\partial q}{\partial \bar z}\pi_j + [q,{}^aA_{\bar z}] , \frac{\partial q}{\partial \bar z}\rangle v_g\} \ .$$

We now prove a very useful lemma, namely

5.4. LEMMA. Let $\psi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow (F(n),g_a)$ be a Eells--Wood map. Then:

$$I^{\psi}_{(a_1,\ldots,a_{n-1},a_1+a_2+\varepsilon_1,\ldots,a_1+\ldots+a_{n-1}+\varepsilon_\ell)} =$$

$$= I^{\psi}_{(a_1,\ldots,a_{n-1},a_1+a_2,\ldots,a_1+\ldots+a_{n-1})} + 4\varepsilon_1|\delta(A^{13}_z)|^2 +$$

$$+\ldots+4\varepsilon_\ell|\delta(A^{1n}_z)|^2$$ for any $\varepsilon_1,\ldots,\varepsilon_\ell$ such that $a_1+a_2+\varepsilon_1 > 0, \ldots, a_1+a_2+\varepsilon_\ell > 0$.

PROOF. We recall that since ψ is a Eells-Wood map $A^{ij}_z = 0$ if i and j are not consecutive integers. Therefore:

$$I^{\psi}_{(a_1,\ldots,a_{n-1},a_1+a_2+\varepsilon_1,\ldots,a_1+\ldots+a_{n-1}+\varepsilon_\ell)} =$$

$$\underset{\text{position}}{\overset{\text{by 5.3 Pro-}}{=\!=\!=}} 2\mathrm{Re}\{\int_{M^2} \langle [q, a_1(|A_z^{12}|^2 + |A_z^{21}|^2) + \ldots +$$

$$+\ldots+a_{n-1}(|A_z^{(n-1)n}|^2 + |A_z^{n(n-1)}|^2) + (a_1+a_2+ {}_1)0 + \ldots +$$

$$+\ldots+(a_1+\ldots+a_{n-1}+\varepsilon_\ell).0] + a_1\pi_1 \frac{\partial q}{\partial z}\pi_2 + \ldots +$$

$$+\ldots+a_{n-1}\pi_{n-1}\frac{\partial q}{\partial z}\pi_1 + \ldots + (a_1+a_2+\varepsilon_1)(\pi_1\frac{\partial q}{\partial z}\pi_3 + \pi_3\frac{\partial q}{\partial z}) +$$

$$+\ldots+(a_1+\ldots a_{n-1}+\varepsilon_\ell)(\pi_1\frac{\partial q}{\partial z}\pi_n + \pi_n\frac{\partial q}{\partial z}\pi_1), \frac{\partial q}{\partial z}\rangle v_g =$$

$$= I^{\psi}_{(a_1,\ldots,a_{n-1},a_1+a_2,\ldots,a_1+\ldots+a_{n-1})} + 2\varepsilon_1(|\pi_1\frac{\partial q}{\partial z}\pi_3|^2 +$$

$$+ |\pi_3\frac{\partial q}{\partial z}\pi_1|^2) + \ldots + 2\varepsilon_\ell(|\pi_1\frac{\partial q}{\partial z}\pi_n|^2 + |\pi_n\frac{\partial q}{\partial z}\pi_1|^2) =$$

$$= I^{\psi}_{(a_1,\ldots,a_{n-1},a_1+a_2,\ldots,a_1+\ldots+a_{n-1})} + 4\varepsilon_1|\delta(A_z^{13})\ ^2 +$$

$$+\ldots+4\varepsilon_\ell|\delta(A_z^{1n})|^2.$$

Now by using the lemma above we can analyse the effect on the index form if we perturb a Kähler metric on $F(n)$. We recall that $\phi : M^2 \longrightarrow (F(n), g_a)$ is called stable if $I_a^{\phi}(q) \geq 0$ for any $q : M^2 \longrightarrow \mu(n)$. Then we will prove that the Eells - Wood maps are stable with respect to a precise set of left - invariant metrics which by Lichenerowicz's remark of course include the Kähler metrics, and are not stable with respect to another precise set of left-invariant metrics which include the Killing form metric.

Lemma we have

$$I^{\psi}_{(a_1,\ldots,a_{n-1},a_1+a_2-\varepsilon_1,\ldots,a_1+\ldots+a_{n-1}-\varepsilon_\ell)} =$$

$$= I^{(q)}_{(a_1,\ldots,a_{n-1},a_1+a_2,\ldots,a_1+\ldots+a_{n-1})} - 4\varepsilon_1 |\delta(A_z^{13})(q)|^2 -$$

$-\ldots-4\varepsilon_\ell |\delta(A_z^{1n})(q)|^2$. Therefore since ψ is full if we choose a holomorphic variation q such that $\delta(A_z^{13})(q) \neq 0$ or ... $\ldots\delta(A_z^{1n})(q) \neq 0$ and we have that:

$I^{\psi\ (q)}_{(a_1,\ldots,a_{n-1},a_1+a_2-\varepsilon_1,\ldots,a_1+\ldots+a_{n-1}-\varepsilon_\ell)} < 0$. Therefore ψ is not stable. We will show later that such holomorphic variation always exists.

5.7. COROLLARY. Let $\psi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow F(n)$ be a full Eells-Wood map, where $F(n)$ is equipped with the Killing form metric. Then ψ is not stable.

PROOF. Just apply 5.6 Theorem for $a_1=\ldots=a_{n-1}=1$, $\varepsilon_1=1,\ldots,\varepsilon_\ell=n-2$.

Now we will compute the index of the Eells-Wood map with respect to those metrics that such map are not stable.

Let S^+ be a positive system. We call $q : M^2 \longrightarrow u(n)$ a S^+-holomorphic variation if $\delta(A_z^{ij})(q) = \pi_i \frac{q}{z} \pi_j = 0$ when $(i,j) \in S^+$.

A S^+-holomorphic variation q is called a Eells-Wood S^+-variation if $\pi_i \frac{\partial q}{\partial z} \pi_j = 0$, $\forall (i,j) \in S^-$ and i and j are not consecutive integers.

Let V_{S^+} denotes the space of S^+-holomorphic variations minus the space of Eells-Wood S^+-variations. Then we can prove:

5.8. COROLLARY. Let $\psi = (\pi_1,\ldots,\pi_n) : M^2 \longrightarrow (F(n), g_{(a_1,\ldots,a_{n-1},a_1+a_2-\varepsilon_1,\ldots,a_1+\ldots+a_{n-1}-\varepsilon_\ell)})$ be a full Eells-Wood map holomorphic with respect to the almost complex structure determined by S^+ and $\varepsilon_1,\ldots,\varepsilon_\ell > 0$ such that $a_1+a_2-\varepsilon_1 > 0,$ $,\ldots,a_1+\ldots+a_{n-1}-\varepsilon_\ell > 0$. Then index $\psi \geq \dim V_{S^+}$.

PROOF. Let $q \in V_{S^+}$. According to 5.4 Lemma we have:

$$I^{(q)}_{(a_1,\ldots,a_{n-1},a_1+a_2-\varepsilon_1,\ldots,a_1+a_2+\ldots+a_{n-1}-\varepsilon_\ell)} =$$

$$= I_{(a_1,\ldots,a_{n-1},a_1+a_2,\ldots,a_1+\ldots+a_{n-1})} - 4\varepsilon_1|\delta(A_z^{13})(q)|^2 -$$

$$-\ldots-4\varepsilon_\ell|\delta(A_z^{1n})(q)|^2 < 0$$

becausa $I^{\psi\,(q)}_{(a_1,\ldots,a_{n-1},a_1+a_2,\ldots,a_1+\ldots+a_{n-1})} = 0$

according to Lichnerowicz's remark and $\delta(A_z^{ij})(q) \neq 0$ for some $(i,j) \in S^-$ where i,j are non-consecutive integers because q is not a Eells-Wood S^+-variation.

Now we will compute precisely the index of the Eells-Wood maps when $F(3)$ is equipped with the Killing form metric.

5.9. COROLLARY. Let $\psi = (\pi_1,\pi_2,\pi_3) : M^2 \longrightarrow F(3)$ be a full Eells-Wood map holomorphic with respect to S^+, where $F(3)$ is equipped with the Killing form metric. Then index $\psi = \dim V_+$.

PROOF. According to 5.8 Corollary, index $\psi \geq \dim V_{S^+}$. Therefore to prove our Corollary we only have to show that index $\psi \leq \dim V_{S^+}$.

Then, let $q : M^2 \longrightarrow u(3)$ be such that $I^{(q)}_{(1,1,2-1)} =$ $= I^{(q)}_{(1,1,2)} - 4|\delta(A^{13}_z)(q)|^2 < 0$. But then $I^{(q)}_{(1,1,2)} = 0$ and $((A^{13}_z)(q) \neq 0$; i.e., $q \in V_{S^+}$. Suppose not say, $I^{(q)}_{(1,1,2)} \neq 0$. But $g_{(1,1,2)}$ is a Kähler metric and ψ is holomorphic with respect to S^+ then:

$$I^{(q)}_{(1,1,2)} = 2|\delta(A^{12}_z)(q)|^2 + 2|\delta(A^{23}_z(q)|^2 + 4|\delta(A^{13}_z)(q)|^2 > 0.$$

Therefore

$$I^{(q)}_{(1,1,1)} = I^{(q)}_{(1,1,2)} - 4|\delta(A^{13}_z)(q)|^2 =$$

$$= 2(|\delta(A^{12}_z)(q)|^2 + |\delta(A^{23}_z)(q)|^2 \geq 0$$

impossible according to the choice of q. Therefore $q \in V_+$ and index $\psi \leq \dim V_{S^+}$.

Now let us compute $\dim V_{S^+}$. See [13] fore more details.

We can see that the dimension of the space of holomorphic maps from M^2 to $\mathbb{C}P^{n-1}$ of degree d is 2nd-1 .

Therefore the dimension of the space of S^+-holomorphic variations is $n(n-1)(2d - \frac{1}{2})$ and the dimension of the space of Eells -Wood S^+-holomorphic variations is 2nd-1 hence
$\dim V_{S^+} = n(n-1)(2d - \frac{1}{2}) + 1 - 2nd.$

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5.5. THEOREM. Let $\psi = (\pi_1,\dots,\pi_n) : M^2 \longrightarrow (F(n), g_{(a_1,\dots,a_{n-1},a_1+a_2+\varepsilon_1,\dots,a_1+a_2+\dots+a_{n-1}+\varepsilon_\ell)})$ be a Eells-Wood map and $\varepsilon_1,\dots,\varepsilon_\ell$ are non-negative. Then ψ is stable.

PROOF. Since ψ is holomorphic with respect to some almost complex structure on $F(n)$, according to Lichnerowiz's remark (see [9] or [17] fore more details) we see that $I^\psi_{(a_1,\dots,a_{n-1},a_1+a_2,\dots,a_1+\dots+a_{n-1})}$ is positive semi-definite. Then if we apply 5.4 Lemma we have:

$$I^\psi_{(a_1,\dots,a_{n-1},a_1+a_2+\varepsilon_1,\dots,a_1+\dots+a_{n-1}+\varepsilon_\ell)}(q) =$$

$$= I^\psi_{(a_1,\dots,a_{n-1},a_1+a_2,\dots,a_1+\dots+a_{n-1})}(q) + 4\varepsilon_1|\delta(A^{13}_z)(q)|^2 +$$

$$+\dots+ 4\varepsilon_\ell|\delta(A^{1n}_z)(q)|^2 \geq 0$$ since $\varepsilon_1,\dots,\varepsilon_\ell$ are non-negative. Therefore ψ is stable.

Now let $\phi = (\pi_1,\dots,\pi_n) : M^2 \longrightarrow F(n)$. We say that ϕ is full or non-degenerate if for all $i < n$ there exists j such that $A^{ij}_\mu \neq 0$ where $\mu = z$ or $\bar{z}$.

Then we can prove:

5.6. THEOREM. Let $\psi = (\pi_1,\dots,\pi_n) : M^2 \longrightarrow (F_n, g_{(a_1,\dots,a_{n-1},a_1+a_2-\varepsilon_1,\dots,a_1+\dots+a_{n-1}-\varepsilon_\ell)})$ be a full Eells-Wood map where $\varepsilon_1,\dots,\varepsilon_\ell$ are positive and $a_1+a_2-\varepsilon_1 > 0,\dots,\dots,a_1+a_2+\dots+a_{n-1}-\varepsilon_\ell > 0$. Then ψ is not stable.

PROOF. According to Lichnerowicz's remark we know that the index form of the energy functional is the same that the index form for the $\bar{\partial}$-energy when the metric is Kähler. But $g_{(a_1,\dots,a_{n-1},a_1+a_2,\dots,a_1+\dots+a_{n-1})}$ is Kähler therefore, if q is a holomorphic variation in the same almost complex structure that ψ is holomorphic, we must have: $I^\psi_{(a_1,\dots,a_{n-1},a_1+a_2,\dots,a_1+\dots+a_{n-1})}(q) = 0$. Hence, if we aplly 5.4