# GRASSMAN'S FIELDS AND GENERALIZED MAGNETIC MONOPOLES

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**ABSTRACT.** We present a theory of dual charges with the introduction of a generalized potential and a generalized field which are respectively elements of the odd and even parts of the Grassman algebra of space-time, with values in the Lie algebra of a "gauge group" G.
Defining a generalized Dirac operator and its dual, we get the field equations of the theory. When $G = U(1)$ we obtain a theory of electrodynamics with magnetic monopoles without string. We show that the generalized field is invariant under harmonic gauge transformations and we obtain Dirac's quantization condition for the dual charges.

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## GENERAL THEORY

Let $P(M,G,\pi)$ be a principal fiber-bundle over space-time $M$ here considered as a Lorentzian manifold where the metric is taken with signature $(+1,-1,-1,-1)$. Let $\alpha$ and $\alpha'$ be two connections defined in $P(M,G,\pi)$ with values in the Lie-algebra $\hat{G}$ of $G$, and such that the pull-backs to $M$ are respectively the gauge potentials A and B.

DEFINITION 1. The Generalized Potential is

$$\omega = A + \star B \in \Lambda^1(M,\hat{G}) \oplus \Lambda^3(M,G) \tag{1}$$

where as usual $\star$ is the Hodge-star operator.

In Electrodynamics the Dirac operator $d + \delta = (\gamma^\mu \nabla_\mu)$ corresponds to the "Cliffordization" of the covariant derivative(12). Here we must have

DEFINITION 2. The Generalized Dirac operator associated to $\omega$ is

$$\mathbb{D}^\omega = D^A + \delta^B \tag{2}$$

where $D^A$ and $\delta^B$ are the usual covariant derivatives and co-derivatives of the usual gauge theories with gauge group $G$(3). Next, we need

DEFINITION 3: The generalized field is given by

$$\Omega \triangleq \mathbb{D}^\omega \omega \triangleq (\mathbb{D}^A + \delta^{\mathbb{B}})(A + \star \mathbb{B}) \tag{3}$$

$$= \underbrace{\Omega^A + \star\,\Omega^B}_{\text{2-form}} + \underbrace{\delta^B A}_{\text{0-form}} + \underbrace{D^A(\star B)}_{\text{4-form}} .$$

where $\Omega^A = D^A A$ and $\Omega^B = D^B B$ as usual

Eqs (1) and (3) show that in the general theory the potential is an element of the odd part of the Grassman algebra of space-time $\Lambda(M,\hat{G})$ and the generalized field is an element of the even part of $\Lambda(M,\hat{G})$

The first important remark is that contrary to the usual gauge theories(3) here we do not have the validity of Bianchi's identity. Instead we have

$$D^{\omega}\Omega = (D^A\delta^B + \delta^B D^A)(\Lambda + * B) \tag{4}$$

which is in general different from zero . The tripotential $*B$ allows degree of freedom to describe a generalized magnetic monopole as we will see below it is them responsible for the non-integrability of the generalized field $\Omega$.

In orden to present the field equations of the general theory we need

DEFINITION 4: The dual operator to $D^{\omega}$ is

$$\Delta^{\omega} = *D^{\omega}* = *(D^A + \delta^B)* = D^B + \delta^A \tag{5}$$

we have,

$$\begin{aligned}\Delta^{\omega}\Omega = D^B\Omega^A + D^B(*\Omega^B) + D^B\delta^B\Lambda + D^B D^A(*B) + \delta^A\Omega^A + \\ + \delta^A(*\Omega^B) + \delta^A\delta^B\Lambda + \delta^A D^A(*B)\end{aligned} \tag{6}$$

Eq(6) can be simplified since as $\dim M = 4$ and $A,B \in \Lambda^1(M,\hat{G})$ we have identically

$$D^B D^A(*B) = 0\ , \quad \delta^A\delta^B(\Lambda) = 0 \tag{7}$$

We now introduce the field equations through

POSTULATE 1: The field equation of the general theory is

$$\Delta^{\omega}\Omega = J + \star G \tag{8}$$

where J and $\star G$ describe the sources of the generalized field. In what follows we call *dual charges* the charges associated to the current $\star G$.

Eq(8) can be written using (6) and (7)

$$\delta^{A}\Omega^{A} + \star D^{A}\Omega^{B} + D^{B}\delta^{B}{}_{A} = J \tag{9.a}$$

$$\delta^{B}\Omega^{B} + \star D^{B}\Omega^{A} + D^{A}\delta^{A}{}_{B} = G \tag{9.b}$$

In the usual gauge theory based on a principal fiber bundle we have associated to the connections $\alpha$ and $\alpha'$ the field equations

$$\delta^{A}\Omega^{A} = J \quad \text{and} \quad \delta^{B}\Omega^{B} = G \tag{10}$$

The additional terms in the first member of eqs(9) show that the general theory contains a non-trivial interaction between the potentials A and B which are represented by interaction currents

DEFINITION 5. The interaction currents are

$$J^{(A,B)}_{int} = \star D^{A}\Omega^{B} + D^{B}\delta^{B}{}_{A}$$
$$J^{(B,A)}_{int} = \star D^{B}\Omega^{A} + D^{A}\delta^{A}{}_{B} \tag{11}$$

Using Cartan's structural equation $\Omega^{\alpha} = d\alpha + \frac{1}{2}[\alpha,\alpha]$ for $\alpha = A,B$ we can obtain an expression in components for the generalized field. We get

$$\Omega^{i}_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \varepsilon_{\mu\nu\rho\sigma}\, \partial^\rho (B^i)^\sigma + \tag{12}$$

$$+ \frac{1}{2} A^k_\mu A^j_\nu C^i_{kj} - \varepsilon_{\mu\nu\rho\sigma} \frac{1}{2} (A^k)^\rho (A^j)^\sigma C^i_{kj}$$

where $[E_k, E_j] = C^i_{k_j} E_i \;,\quad E_i \in \hat{G}$.

Eq(12) is a generalization of the Cabibbo-Ferrari relation[4] for a non-abelian group G

GRASSMAN ELECTRODYNAMICS

When $G = U(1)$ we have $[A,A] = [B,B] = 0$ and $D^\omega = \Delta^\omega = d + \delta$ and eq(3) gives

$$\Omega = (dA + {*dB}) + \delta A + d(*B) \tag{13}$$

The field equations (Postulate 1) then are

$$(d+\delta)\Omega = J + *\,G \Rightarrow \Box A = J \;;\; \Box B = G \;,\; \Box = (d+\delta)^2 \tag{14}$$

The equations $\Box A = J$; $\Box B = G$ are always true in our formalism independently of the "gauge" since in the Grassman electrodynamics $\Omega$ is the sum of scalar, pseudo-scalar and two-form terms. This is in contrast with the approach of ref [4] where eqs(14) are valid only in the Lorentz gauge. In order to have $\Omega$ a two-form its is necessary to fix the Lorentz gauge for the potentials. Indeed, when $\delta A = \delta B = 0$ eq(13) yields

$$\Omega = dA + {*dB} \tag{13'}$$

Here we observe that recently Teitelboim and Hennaux[5] presented an electrodynamics including magnetic monopoles where the potential is described by a p-form, $p \neq 1$. In[5] electric and magnetic charges are extended objects. In our approach charges and monopoles (dual charges) continue to be point like-objects. This is done through the introduction of the generalized potential as element of the odd part of $\Lambda(M,\hat{G})$ and the generalized field as element of the even part of $\Lambda(M,\hat{G})$. Here the Dirac operator $D = d + \delta$ is the natural operator in the sense that

$$D: \underbrace{\Lambda^1(M) \oplus \Lambda^3(M)}_{\text{potentials and currents}} \longrightarrow \underbrace{\Lambda^0(M) \oplus \Lambda^2(M) \oplus \Lambda^4(M)}_{\text{fields}}$$

This generalization of the concept of field and potential can suggest a new concept for matter fields if we take also into account that the Dirac spinor field can be represented as an element of $\Lambda^0(M) \oplus \Lambda^2(M) \oplus \Lambda^4(M)$ (5,6). The theory in the case of electrodynamics can also be formulated in a Clifford bundle (2,7,8). Incidentally in ref [9] a similar formalism in the context of tachyons monopoles is developed.

## QUASI GAUGE INVARIANCE OF THE GENERALIZED POTENTIAL

Remember that in usual electrodynamics the field $F = dA$ is gauge invariant under the gauge transformation

$$A \mapsto A + dg \,, \quad g: \mathbb{R}^4 \to R \quad \text{a differentiable function} \tag{14}$$

However, the generalized field given by eq(13) is not invariant under arbitrary transformations given by eq(14). The generalized field is invariant under a more restrict class of transformations, where $g$ is harmonic. Indeed, let

$$A \mapsto A + dg \,, \quad B \mapsto B + dh \tag{14'}$$

We have that

$$\Omega \rightarrow \Omega + \delta dg + \delta dh \tag{15}$$

and impossing invariance of the field we get

$$\delta dg = (\delta d + d\delta)g = \Box g = 0 \; ; \quad \delta dh = (\delta d + d\delta)h = \Box h = 0 \tag{16}$$

QUANTIZATION CONDITION FOR THE GENERALIZED ELECTRODYNAMICS

Here we show that our theory satisfy Dirac's quantization condition under a reasonable condition.

Let $\phi(x,\Gamma)$ be Mandelstam's path dependent wave function[9] for a charged particle in an usual electromagnetic field $F = dA$. If $\phi(x)$ is the usual wave function of the particle we have

$$\phi(x,\Gamma) = \varphi(x) \exp\int_{\Gamma}^{x} -ieA \tag{17}$$

where $\Gamma$ is an arbitrary path from $\infty \rightarrow x$. If we choose two paths $\Gamma$ and $\Gamma'$ differing only for a finite part we get using Stokes theorem

$$\phi(x,\Gamma') = \phi(x,\Gamma) \exp\int_{S} -ie\,dA \tag{18}$$

where $S$ is an arbitrary surface such that $\partial S = \Gamma' - \Gamma$

We want now to know how to generalize eq(18) for the case where the charge e interacts with the generalized potential given by eq(13). To start we observe that as $S$ is a bidimensional surface we must use the Lorentz gauge, ie, we put $\delta A = \delta B = 0$ and then $\Omega = dA + \star\, dB'$. We next have

POSTULATE 2. The interaction of an eletric charge e with the generalized field is described by the path-dependent wave function $\phi(x,\Gamma)$ which satisfies

$$\phi(x,\Gamma') = \phi(x,\Gamma) \exp \int_S -ie\Omega \tag{19}$$

The independence of eq(19) of the surface S implies

$$\exp \oint_{S_o} -ie(dA + \star d B) = 1 \tag{20}$$

where $S_o$ is a closed surface. By Stokes theorem we can write eq(20) as

$$\exp \int_V -ie \star \delta d B = 1 \; ; \; S_o = \partial V \tag{21}$$

Supposing now, without loss of generality that the origim is inside V and that we have a static monopole at the origim we have $G = (g\delta(\vec{r}),0,0,0)$ and

$$\Box B = (d\delta + \delta d) G = dB = G \tag{22}$$

Using (22) in (21) we have

$$\exp \int_V -ie \star G = \exp(-ieg) = 1 \Longrightarrow \frac{eg}{4\pi} = n/2 \; , \; n \in Z \tag{23}$$

We observe here that the new point in order to obtain Dirac's quantization condition $eg/4\pi = n/2$, $n \in Z$ is postulate 2. This point is not clear in ref [4]. We end this paper with the remark that postulate 2 is consistent with a quantization scheme of the monopole-charge system which give the right equations of motion and from which the Dirac quantization condition can be obtained in a very elegant manner.[8]

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