THE MEANING OF TIME IN THE THEORY OF RELATIVITY AND "EINSTEIN'S LATER VIEW OF THE TWIN PARADOX"

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ABSTRACT. The purpose of the present paper is to reply to a very misleading paper by M. Sachs entitled "Einstein's later view of the Twin Paradox (TP) (Found, Phys. 15, 977 (1985))". There, by selecting some passages from Einstein's papers he tried to convince the reader that Einstein changed his mind regarding the asymmetric aging of the twins on different motions. Also Sachs insinuates that he presented several years ago "convincent mathematical arguments" proving that the theory of Relativity does not predict asymmetrical aging in the TP. Here we give a definitive treatment to the clocks problem showing that Sachs' "convincent mathematical arguments" are non-sequitur. Also by properly quoting Einstein we show that his later view of the TP coincides with the one derived from the rigorous theory of time developed in this paper.

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1. INTRODUCTION

Mendel Sachs paper[1] with the title 'Einstein's later view of the twin paradox' is misleading in (at least) two aspects:

(i) It uses certain quotations from Einstein's papers[2,3] to suggest to the reader that Einstein abandoned his earlier view that if two identical standard clocks meet in a point x_1 in space-time and are synchronized and then follows different world-lines that meet again in a second point $x_2 \neq x_1$, then they will not in general show identical times at x_2.

(ii) It gives to reader the impression that the arguments presented in an old Sachs paper[4] are correct within General Relativity and that in reference[5] he contested the "majority view" in a legitimate manner.

In what follows we show that both (i) and (ii) are non sequitur. To this end we introduce in §2 and §3 several concepts necessary in order to understand the meaning of time in the Theory of Relativity.

In §2 we present the fundaments of the Theory and the Standard Clock Postulate (SCP).

We introduce in §3 the notion of reference frames in a Lorentzian manifold and the notion of the coordinate system naturally adapted to a given reference frame. We classify the reference frames according to their synchronizability. This classification is extremely important for it shows in which conditions the time-like coordinate x^o has the meaning of time as measured by standard clocks at rest in a given reference frame.

In §4 we present a rigourous mathematical treatment of the clocks problem (no paradox, of course) which is independent of the introduction of charts in the space-time manifold. All observers in all reference frames must then agree with the result.

In §5 we discuss the old Sachs paper [4] and show explicitly that Sachs calculation are not in accord with the Theory of Relativity.

In §6 we present a small selection of passages from Einstein's Autobiographical Notes [1] which endorse the theory of time presented in this paper and show very clearly that Sachs paper [1] is ill conceived. The Appendix contains the derivation of the fundamental anti-Minkowski inequality used in §4 as well as several important results related to the Linear Algebra of Minkowski space which should be very well-known.

2. THE SPACE-TIME OF THE THEORY OF RELATIVITY AND THE CHRONOMETRIC HYPOTHESIS

The most important feature of the Theory of Relativity is the hypothesis that the collection of all possible happenings , i.e., all possible events constituting space-time, i.e., $V_4 = (M,g,D)$ is a connected 4-dimensional oriented , and time oriented Lorentzian manifold (M,g) together with the Levi-Civita connection D of g on M[6,7]. The events in $U \subset M$ in a particular chart of a given atlas have coordinates (x^0,x^1,x^2,x^3), x^0 is called time-like coordinate and the x^i , $i = 1,2,3$ are called space-like coordinates. These labels *do not necessarily have* a metrical meaning.

The metric of the manifold (in a coordinate basis) is

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu$$

$$g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = g_{\nu\mu} \tag{1}$$

$g(\frac{\partial}{\partial x^{\mu}}, \frac{\partial}{\partial x^{\nu}})$ being calculated, of course, for each $x \in M$ in M_x, the tangent space to M at x(*). Now, tangent space magnitudes defined by the metric are related to magnitudes on the manifold by the following definition.

Let $I \subset \mathbb{R}$ be an interval on the real line and $\Gamma : I \to M$ a map. We suppose that Γ is a C^{0}, piecewise C^{1} curve in M. We denote the inclusion function $I \to \mathbb{R}$ by u, and the distinguished vector field on I by d/du. For each $u \in I$, $\Gamma_{*}u$ denotes the tangent vectors at $\Gamma u \in M$; thus

$$\Gamma_{*}u = [\Gamma_{*}(\frac{d}{du})] \ (u) \in M_{\Gamma_u} .$$

Finally, the path-length between points $x_1 = \Gamma(a)$, $x_2 = \Gamma(b)$, $a,b \in I$, $x_1, x_2 \in M$ along the curve(**) $\Gamma : I \to M$ and such that $g(\Gamma_{*}u, \Gamma_{*}u)$ has the same sign at all points along Γu, is the quantity

$$\int_a^b du[|g(\Gamma_{*}u, \Gamma_{*}u|]^{1/2} \tag{2}$$

Observe now that taking the point $\Gamma(a)$ as a reference point we can use eq(2) to define the function

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(*) The properties of the vectors at M_x (Minkowski space) are studied in the Appendix.

(**) curves are classified as timelike, lightlike and spacelike when (for all $u \in I$) $g(\Gamma_{*}u, \Gamma_{*}u) > 0$, $g(\Gamma_{*}u ; \Gamma_{*}u) = 0$ $g(\Gamma_{*}u, \Gamma_{*}u) < 0$ respectively.

$$s : \Gamma(I) \to R \quad \text{by} \quad s(u) = \int_a^u [\,|g(\Gamma_* u', \Gamma_* u')|\,]^{1/2} du' \tag{3}$$

With eq(3) we can calculate the derivative ds/du . We have

$$\frac{ds}{du} = [\,|g(\Gamma_* u\,,\, \Gamma_* u)|\,]^{1/2} = [\,|g_{\mu\nu} \frac{dx^\mu}{du} \frac{dx^\nu}{du}|\,]^{1/2} \tag{4}$$

From eq(4) old text-books on differential geometry and general relativity infers the equation.

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{5}$$

supposed to represent the square of the lenght of the "infinitesimal" arc determined by the coordinate displacement

$$x^\mu(a) \to x^\mu(a) + \frac{dx^\mu}{du}(a)\varepsilon,$$

where ε is an "infinitesimal" and $a \in I$.

The abusive and non careful use of eq(5) has produced many incorrect interpretations in the Theory of Relativity, as we will see in what follows.

Now, given a time-like curve $\gamma : R \supset I \to M$, any event $e \in \gamma(I)$ separates all other events in two disjoint classes, the past and the future (see Appendix). The theory models an observer as

DEFINITION 1: An observer in V_4 is a future-pointing time-like curve $\gamma : R \supset I \to M$ by $I \ni u \to e \in \gamma(I) \subset M$, and such that $g(\gamma_*, \gamma_*) = 1$

We now introduce,

POSTULATE I (Standard Clock Postulate)(SCP) - Let γ be an observer. Then there exist *standard clocks* that "can be carried by γ" and such that they register (in γ) *proper-time* , i.e., the inclusion parameter u of the definition of observer.[8]

It seems that atomic-clocks are standard clocks[10], but see also[11].

DEFINITION 2. A reference frame Q in V_4 is a time-like vector field such that each of its integral lines is an observer

This definition is due to R. Sachs and H. Wu[6]. B. O'Neill[18] calls observer fields the reference frames fields.

Given $U \subset M$ where Q is defined, there are an infinity of charts (coordinate systems) $\langle x^\mu \rangle : U \to \mathbb{R}^4$ of the maximal oriented atlas of M. We have

DEFINITION 3. A chart in $U \subset M$ is said to be a naturally adapted coordinate system to a reference frame Q (nacs/Q) if in the natural coordinate base of $T_x U$ ($x \in U$) associated with the chart the space-like components of Q are null.

3. THE MEANING OF THE TIME-LIKE COORDINATE x^o

Old treatments of the clocks problem involve at least two reference frames Q and Q' each one containing a standard clock at rest. A (nacs/Q), $\langle x^\mu \rangle$ and a (nacs/Q'), $\langle x'^\mu \rangle$ are also used For $U \subset M$ where both Q and Q' are defined we have the coordinate transformation $\langle x^\mu \rangle \to \langle x'^\mu \rangle$. In particular we have $x'^o = f^o(x^o, x^1, x^2, x^3)$ relating the time-like coordinate of an event $e \in U \subset M$ in Q' with the time-like and space-like coordinates of the same $e \in U \subset M$ in Q.

Now - *and this point is crucial* - we must find an answer to the following question: Given an arbitrary reference frame Q when does there exist a (nacs/Q) such that (for $U \subset M$ where Q is defined) x^o has the meaning of proper-time as determined by standard clocks at rest in Q and synchronized by Einstein's method(*)?

Let be $\alpha = g(Q, \ )$. We have the

DEFINITION 3. [6]

(i) Q is locally synchronizable if and only if $d\alpha \wedge \alpha = 0$

(ii) Q is locally proper-time synchronizable if and only if $d\alpha = 0$

(iii) Q is synchronizable if there are mappings $f: M \to R$ and $x^o: M \to \mathbb{R}$, such that $f > 0$ and $\alpha = f dx^o$

(iv) Q is proper-time synchronizable if and only if $\alpha = dx^o$

It is clear that (ii) $\Rightarrow$ (i), and (iv) $\Rightarrow$ (ii) and the reciprocals are valid only locally.

DEFINITION 4. When Q is synchronizable (proper-time syncronizable) whatever function x^o like in Definition 3 is called a time function (proper-time function)

When there exists a time function, it obviously is not unique. If there exists a proper-time function we have $du = \gamma^*(dx^o)$, $\forall \gamma \in Q$ and $\gamma: \mathbb{R} \supset I \to M$, u being the inclusion function of the curve γ.

When Q is synchronizable, all hypersurfaces of the time function x^o are orthogonal to Q, being then orthogonal to all observers in Q. These hypersurfaces are space-like. In this case we say that the observers in Q can separate M in time $\times$ space. When M is contractible (ie, $\pi_1(M) = 0$) we have, using the reciprocal

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(*) Einstein's synchronization method is given by Definition 6.

of Poincarè's lemma [19]

PROPOSITION 1. If $V_4 = (M,g,D)$ is contractible and $d\alpha = 0$, $\alpha = g(Q,\ )$, $g(Q,Q) = +1$, then observers in Q can separated M in time x space.

When Proposition 1 holds true there exists $x^o: M \to \mathbb{R}$, such that $\alpha = dx^o$, and it is possible to give to the time-like co-ordinate x^o the meaning of proper-time as measured by standard clocks at rest in Q and synchronized à l'Einstein. This statement will be proved below.

Suppose now that M is not contractible. Let α be a 1-form field ($\alpha = g(Q,\ )$) such that $d\alpha \neq 0 \ \forall x \in M$. We have the following question:

When does it exist a function $f: M \to \mathbb{R}$ such that $df \neq 0$ $\forall x \in M$ and such that the hypersurfaces of the type

$$N = \{x \in U \mid f(x) = \text{constant},\ df(x) = 0\}$$

are integral manifolds of α?

A necessary condition for the existence of this f is the Frobenius condition [19]

$$d\alpha \wedge \alpha = 0 \tag{6}$$

We then have

PROPOSITION 2. Let (M,g,D) be a Lorentzian manifold. If there exist in M a reference-frame Q such that $d\alpha\wedge\alpha = 0$, $\alpha = g(Q,\ )$ then Q can separate M in time × space

The proof is a direct application of the Frobenius condition.

We now introduce the concept of synchronization of clocks necessary in order to physically justify the definitions of this section. We need to introduce

POSTULATE II - (Light Axiom). Let (λ, m_λ) be a photon, ie, $m_\lambda = 0$ and $\lambda : \mathbb{R} \supseteq I \to M$ a null curve. Then in any Lorentzian manifold $V_4 \equiv (M,g,D)$ the world-line of any photon λ is a null geodesic.

We have the

PROPOSITION 3. Let $\gamma : \mathbb{R} \supset I \to M$ be an observer in (M,g,D). Suppose that $u_e \in I$ is given. Then, there exists an open interval $E \subset I$, $u_e \in E$ and an open neighborhood V of $e = \gamma u_e$ such that $\forall\, e' \in V - \gamma E$, there exist u_e, and u_{e_2} and a light signal λ from e' to $e_2 = \gamma u_{e_2}$ and a light signal λ' from $e_1 = \gamma u_{e_1}$ to e; u_{e_1}, u_{e_2}, λ, λ' are unique

The proof can be found in ref [6]

Let Z be a reference frame in $V_4 = (M,g,D)$, μ_s its flux and $\gamma : \mathbb{R} \supset I \to M$ an integral curve of Z.

DEFINITION 5. An infinitesimally nearby observer is a vector field $W: I \to TM$ which is Lie parallel with respect to Z, and such that for $u \in I$, there is a neighborhood ε on u, a neighborhood μ of γu and a vector field V on μ such that $\mathcal{L}_Z V = 0$ and $W = V o \gamma$ on ε.

The reason for calling W an infinitesimally nearby observer is the following: Let $\langle x^\mu \rangle : u \to \mathbb{R}^4$ and $W|_\varepsilon = a^\mu (\partial_\mu o\, \gamma|_\varepsilon)$. We may write $U = \{(x^o, x^1, x^2, x^3) \mid |x^\mu| < \varepsilon\ \forall\, \mu\}$ and assume that $\gamma_u = (u,o,o,o)$, since in U we can always choose $Z|_U = \partial/\partial x^o$ 21. There is a congruence of integral curves of Z determined by

$$(u,t) \longmapsto (u + a^o x^o,\ a^1 x^o,\ a^2 x^o,\ a^3 x^o)$$

and $x^o = 0$ gives $\gamma|_\varepsilon$ and x^o times an appropriate constant gives another curve of the congruence in U where the parametrization given by $\langle x^\mu \rangle$ holds. Now when $W|_\varepsilon$ and $Z\circ\gamma$ are linearly independent, different curves have distinct images. This family uniquely determines $W|_\varepsilon$ as its *transversal vector field*, ie,

$$(Wf)(u) = [(\frac{\partial}{\partial x^o}) \{f(u + a^o x^o, a^1 x^o, a^2 x^o, a^3 x^o)\}]_{x^o = 0}$$

for each $f: U \to R$ and $u \in \varepsilon$. Conversely, once $W|_\varepsilon$ is given, the family is determined up to first order in x^o, in the sense of a Taylor expansion in x^o.

Now, let Q be a reference frame in (M,g,D) and let γ and γ' be two 'infinitesimal nearby' observers of Q. Suppose that γ' contains e' of Proposition 3. [Fig.1].

According to Postulate I, the observers in γ and γ' can order all the events in their respective world lines. We write $e_1 < e < e_2$ to indicate that according to γ the event e is later than e_1 and e_2 is later than e.

The problem of the synchronization of clocks is as follows: which event e in the world-line γ is simultaneous to the event e' in γ'?

The answer to this question depends on a *definition*. Intuitively we consider that the event e' simultaneous to e must not be causaly related to e, ie, there must be no causal curve(*) connecting e' to e. Now, Proposition 3 and the definition of a nearby observer imply that there is no causal curve connecting e' to e if $e_1 < e < e_2$. Let $\langle x^\mu \rangle : M \supset U \to R^4$ be a local chart of the maximal oriented atlas of M naturally adapted to Q and let

(*) A causal curve is a mapping $\gamma: I \to M$ $(I \subseteq R)$ such that $g(\gamma_*, \gamma_*) \geq 0 \;\forall\; x = \gamma u \in \gamma(I)$

$$e_1 = (x^o_{e_1},0,0,0)\ ;\ e_2 = (x^o_{e_2},0,0,0);\ \ e = (x^o_e,0,0,0)\ ,\ e = (x^o_e,\Delta x^1,\Delta x^2,\Delta x^3) \quad (7)$$

be the space-time coordinates of the respective events. We have the

DEFINITION 6. The event e in γ simultaneous to the event e' in γ' is the one such that its time-like coordinate is given by

$$x^o_e = x^o_{e_1} + \frac{1}{2}[(x^o_{e'} - x^o_{e_1}) + (x^o_{e_2} - x^o_{e'})] \quad (8)$$

Physically, the synchronization procedure given by eq(8) [Einstein's method] means that the observer at γ proceeds as follows: (i) at e_1 he sends a light signal λ' to e' in γ' , (ii) the signal is immediatly reflected back through the path λ and arrives at γ at the event e_2.

Calling $\Delta x^o_1 = x^o_{e'} - x^o_{e_1}$, $\Delta x^o_2 = x^o_{e_2} - x^o_{e'}$, and taking into account that the lengths of the arcs e_1e' in λ' and $e'e_2$ in λ can be represented by the vectors $W_{\lambda'} = (-\Delta x^o_1,\Delta x^1,\Delta x^2,\Delta x^3)$ and $W_{\lambda'} = (\Delta x^o_2,\Delta x^1,\Delta x^2,\Delta x^3)$ and since $g(W_{\lambda'}W_{\lambda}) = g(W_{\lambda'},W_{\lambda'}) = 0$, we get

$$x^o_e = x^o_{e'} - \frac{g_{oi}}{g_{oo}}\Delta x^i \neq x^o_{e'} \quad (9)$$

This equation permits the synchronization of two infinitesimal nearby clocks in γ and γ' at rest in the refernce frame Q and in the local chart $\langle x^\mu\rangle : M \supset U \to \mathbb{R}^4$

We also observe that an unique synchronization of all clocks at rest in Q in the region $U \subset M$ is possible only if there exists a (nacs/Q) such that in this coordinate system $g_{io} = 0$, $\forall x \in U$. When Q is proper-time synchronizable, ie, $\alpha = dx^o$, $\alpha = g(Q,\ )$ and $x^o: M \to R$ then there exists a local chart where $Q = \partial/\partial x^o$ and $g_{oo} = 1\ \forall x \in M$. As all level surfaces of the function x^o are orthogonal to Ω (and then orthogonal to all observers in Ω) the spatial coordinates x^i are such that $g(\partial/\partial x^o, \partial/\partial x^i) = g_{ij} = 0 \ \forall x \in M$.

Given an arbitrary reference frame Z in $U \subset M$ in general there does not exist a (nacs/Z) such that $g_{oo} = 1$ and $g_{io} = 0 \ \forall x \in U$. This is the reason why we classified Z according to Definition 3. Now, suppose that Z is an arbitrary reference frame in $U \subseteq M$ and $\langle x^\mu \rangle$ a (nacs/Z). Let Ω be another frame defined also in $U \subseteq M$ which is proper-time synchronizable and let $\langle \tilde{x}^\mu \rangle$ be a (nacs/Ω). The coordinate transformation $\langle x^\mu \rangle \to \langle \tilde{x}^\mu \rangle$ must then satisfy

$$g^{\mu\nu}(x) \frac{\partial \tilde{x}^o}{\partial x^\mu} \frac{\partial \tilde{x}^o}{\partial x^\nu} = \tilde{g}^{oo}(\tilde{x}) = 1$$

$$g^{\mu\nu}(x) \frac{\partial \tilde{x}^i}{\partial x^\mu} \frac{\partial \tilde{x}^o}{\partial x^\nu} = g^{io}(\tilde{x}) = 0 \qquad (10)$$

$$g^{\mu\nu}(x) \frac{\partial \tilde{x}^i}{\partial x^\mu} \frac{\partial \tilde{x}^j}{\partial x^\nu} = \tilde{g}^{ij}(\tilde{x})$$

where $g^{\mu\nu}(x) g_{\nu\alpha}(x) = \delta^\mu_\alpha \ \forall x \in U \subset M$

Eqs (10) have the form of the relativistic Hamilton-Jacobi equation for a free-particle. It can be very hard to find solutions to these equations and the interested reader can consult ref. [20].

To finish we must comment on a basic point [9]. The reference frames Ω introduced above are mathematical instruments. This means that a given frame does not need to have a material support in all points of the world manifold. An example will ilustrate this point. Let $V_4 = (M,g,D)$ be a flat Lorentzian manifold, namely

Minkowski space-time. Let $i = \partial/\partial t$ be an inertial frame (*), defined of course for all $x \in M$. Let now (t,r,ϕ,z) be the cilindrical coordinates naturally adapted to i. Then g is

$$g = dt \otimes dt - dr \otimes dr - r^2\, d\phi \otimes d\phi - dz \otimes dz \tag{11}$$

Let

$$Q = (1 - \omega^2 r^2)^{-1/2} \frac{\partial}{\partial t} + \omega (1 - \omega^2 r^2)^{-1/2} \frac{\partial}{\partial \phi} \tag{12}$$

be a reference frame defined in $U \equiv (-\infty < t < \infty;\ 0 < r < \frac{1}{\omega};\ 0 \leq \phi < 2\pi,\ -\infty < z < \infty)$ $(U \subset M)$.

Then

$$\alpha = g(Q,\ ) = (1 - \omega^2 r^2)^{-1/2} dt - \omega r^2 (1 - \omega^2 r^2)\, d\phi \tag{13}$$

and

$$d\alpha \wedge \alpha = -2\omega r\, dt \wedge dr \wedge d\phi \neq 0 \tag{14}$$

The rotation vector [7] associated to Q is $\Omega = g(*(d\alpha \wedge \alpha),\ ) = \omega\, \partial/\partial z$. This means that Q is rotating with constant angular velocity ω relative to the z axis of i. Now Q can be materialized in $U \subset M$ by a solid rotating disc, but it is obvious that in U, i cannot have material support.

The reference frame Q defined by eq(12) is also an example where it does not exist a (nacs/Q) such that the time-like coordinate of the system can have the meaning of proper-time registred by standard clocks at rest in Q, for all $x \in U$.

(*) Inertial frames, which exist only in an flat manifold are such that $Di = 0$. Then $d\alpha \wedge \alpha = 0$, $\alpha = g(i,\ )$.

4. SOLUTION OF THE CLOCKS PROBLEM

We are now prepared to discuss the 'clocks problem', unfortunately known as the clock paradox. As the problem first arised in the Special Theory of Relativity we will first discuss the clocks problem in the case where D is a flat connection. In this case the manifold M is an affine vector manifold, known as Minkowski space-time (see Appendix). Due to this fact it is possible to present a coordinate free treatment of the clocks problem.

Let there Γ_1, $\vec{\Gamma}_2$ and $\overleftarrow{\Gamma}_2$ be three timelike and straight lines in M, as in Fig. 2. Γ_1 and $\vec{\Gamma}_2$ has x_i as a common point and Γ_1, $\overleftarrow{\Gamma}_2$ has x_f as a common point and $\vec{\Gamma}_2$ and $\overleftarrow{\Gamma}_2$ has x_m as common point. Γ_1 represents the path of a standard clock called ① and $\Gamma_2 = \vec{\Gamma}_2 + \overleftarrow{\Gamma}_2$ represents the path of a standard clock called ②. Now, according to eq(2) the proper-time registred by clock ① between the events x_j and x_f is given by $T_{①} = \|x_f - x_i\|$, i.e. the norm of the vector $x_f - x_i \in M$. The proper-time registred by clock ② is given by $T_{②} = \|x_m - x_i\| + \|x_f - x_m\|$. Now, according to the fundamental *anti-Minkowski inequality* valid for timelike vectors in the same class (Appendix, Prop. 9) we have

$$\|x_f - x_i\| \geq \|x_m - x_i\| + \|x_f - x_m\| \quad (15)$$

and thus $T_{①} \geq T_{②}$.

This result is an intrinsic consequence of the mathematical model of the Theory of Relativity. All observers in all reference frames in M (inertial or not) must agree with the validity of the result $T_{①} \geq T_{②}$.

We observe that path Γ_1 is a geodesic path between x_f and x_i as can be trivially proved. We also can prove the following theorem which is valid in a general Lorentzian manifold (i.e., D does not need to be flat).

THEOREM: Among all timelike curves in $V_4 = (M,g,D)$ passing through the points $x_i = \Gamma(a)$ $x_f = \Gamma(b)$ the integral in eq(2) is a maximum when Γ is a timelike geodesic.

If the reader did not succeed to build up the proof of the Theorem he can consult reference [12].

To end this section we mention that in the case of the Special Theory of Relativity we can give a proof of the non-existence of a Lorentz-invariant clock, i.e., a clock such that, when in motion relative to an inertial frame S, does not lag behind relative to a series of clocks synchronized ã la Einstein in S. Indeed, in reference [13] it is proved that the existence of one such clock implies the breakdown of Lorentz invariance.

5. THE "SQUARE-ROOT" OF g

In this section we give the promised proof that the "solution of the clock paradox" offered in the Mendel Sachs' paper [4] is not in accord with the Theory of Relativity.

To start, let $V_4 = (M,g,D)$ like in Section 2 be a model of the space-time manifold. In a coordinate basis, $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$ [eq(2)]. We now ask if g can be "factored" as the tensor product of two 1-forms ω_1 and ω_2, i.e., if we can write

$$g = \omega_1 \otimes \omega_2 \tag{16}$$

The answer to the above question is yes, and we can exhibit easily two solutions, where ω_1 and ω_2 are Clifford-valued 1-forms

(i) $\omega_1 = \omega_2 = \gamma_\mu(x)dx^\mu$ and the $\gamma_\mu(x)$ satisfy

$$\gamma_\mu(x)\gamma_\nu(x) + \gamma_\nu(x)\gamma_\mu(x) = 2g_{\mu\nu}(x) \tag{17}$$

γ_μ are then the generators of the local Clifford algebra $\mathbb{R}_{1,3}$ of space-time [14,15].

(ii) $\omega_1 = \tilde{\omega}_2$; where $\omega_1 = q_\alpha(x)dx^\alpha$; $\omega_2 = q_\beta(x)dx^\beta$, and q_α are the generators of the quaternion field, $\tilde{q}_\beta$ is the quaternion conjugate field and we have

$$q_\alpha(x)\tilde{q}_\beta(x) + q_\beta(x)\tilde{q}_\alpha(x) = 2g_{\alpha\beta}(x) \tag{18}$$

Choosing the solution given by eq(9) we ask now what conditions the q_α-fields must satisfy in order for $d\omega = 0$ (*). The solution is that q_α must obey the Cauchy-Riemann's like identity

$$\frac{\partial q_\alpha}{\partial x^\beta} - \frac{\partial q_\beta}{\partial x^\alpha} = 0 . \tag{19}$$

So, if eq(10) is satisfied we can write

$$\omega = d\mathbf{s} \tag{20}$$

where $\mathbf{s} : M \to H$ is a quaternion valued function defined in the space-time manifold. We have

$$\mathbf{s} = \int \omega \tag{21}$$

independent of the path.

(*) This is the condition for no "clock paradox" if "time" where to be associated with the quaternion valued function of eq(21).

We can now understand what happened in Mendel Sachs papers[4,16,17]. Using the Neanderthal notation of eq(5) , namely, $(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ he concluded that it is possible to write ds = **ds** . This is obvious impossible since **s** : M → H is a quaternion valued function, whereas eq(5) defines only the real-valued function s:Γ(I) → R given by eq(3). It is impossible to extend s as defined in eq(3) to a function defined on all the space-time manifold.

6. EINSTEIN'S TRUE VIEW CONCERNING THE CLOCK'S PROBLEM.

In this section we present a small selection of passages (A,B,C) from Einstein's Autobiographical Notes together with some comments that show very clearly that Sachs' paper[1] is ill conceived. Indeed Einstein said

A - "A clock at rest relative to the system of inertia defines a local time. The local times of all space points taken together are the "time" which belongs to the selected system, if a means is given to "set" these clocks relative to each other."

B - "...The presuposition of the existence (in principle) of (ideal, viz, perfect) measuring rods and clocks is not independent of each other: since a light signal, which is reflected back and forth between the ends of a rigid rod, constitutes an ideal clock, provided that the postulate of the constancy of the light velocity in vacuum does not lead to contradictions.

The above paradox may then be formulated as follows. According to the rules of connection, used in classical physics , of the spatial co-ordinates and of the time of events in the transition from one inertial system to another the two assumptions of

(1) the constancy of the light velocity

(2) the independence of the laws (thus specially also of the law of the constancy of the light velocity) of the choice of the inertial system (principle of special relativity) are mutually incompatible (despite the fact that both taken separately are based on experience)

The insight which is fundamental for the special theory of relativity is this: The assumptions (1) and (2) are compatible if relations of a new type ("Lorentz-transformation") are postulated for the conversion of co-ordinates and the times of events. With the given physical interpretation of co-ordinates and time, this is by no means merely a conventional step, *but implies certain hypothesis concerning the actual behaviour of measuring-rods and clocks, which can be experimentally validated or disproved*" (our italics)

Our comments concerning passages A and B above are as follows: In an inertial system i_1 standard clocks at rest read directly the time-like coordinate x^o, (see §3) if the clocks are synchronized à l'Einstein in order to obtain a one way velocity of light which is isotropic [21]. The Lorentz-transformations between two inertial frames are not physical cause-effect relations but implies certain hypothesis concerning the actual behaviour of moving measuring-rods and clocks, which can be experimentally validated or disproved.

The hypothesis concerning the behaviour of clocks is the one introduced in §2, namely that there exist standard (or ideal) clocks which measure proper-time, ie, the integral given by eq(2) when Γ is time-like. (Postulate 1)

If real clocks do not satisfy the standard clock postulate then the Lorentz transformations could not give the relation between the times x^o in i_1 and $x^{o'}$ in i_2 (i_1, i = 1,2 being inertial frames) where x^o and $x^{o'}$ are measured by standard clocks at rest respectively in i_1 and i_2 and synchronized à l'Einstein.

C - "One is struck [by the fact] that the theory (except for the four dimensional space) introduces two kinds of physical things, i.e., (1) measuring rods and clocks, (2) all other things, e.g., the electro-magnetic field, the material point, etc. This in a certain sense is inconsistent, strictly speaking measuring rods and clocks would have to be represented as solutions of the basic equations (objects consisting of moving atomic configurations), not, as it were, as theoretically self sufficient entities. However, the procedure justifies itself because it was clear from the very beginning that the postulates of the theory are not strong enough to deduce from them sufficiently complete equations for physical events sufficiently free from arbitrariness , in order to base upon such a foundation a theory of measuring rods and clocks. If one did not wish to forego a physical interpretation of the co-ordinates in general (*something which, in itself, could be possible*) it was better to permit such inconsistency- with the obligation, however, of eliminating it at a later stage of the theory"

In passage C Einstein remind us that it cannot be an accident that standard clocks register the time defined by eq(3). This basic fact must be explained as an adjustement to the field in which the clocks are embeded as test bodies. This needs, of course, a detailed theory of matter, which unfortunately does not yet exist. In C Einstein also remind us that in an arbitrary reference frame Q in a Lorentzian manifold the coordinate labels (x^o, x^1, x^2, x^3) of a particular chart valid for $U \subset M$ of the (nacs/Q) (as defined in §3) does not have a metrical meaning in general. This means in particular that in general x^o is not the time registred by standard clocks at rest in Q. Indeed, this is the case since the standard clocks register the time given by eq(3)! Passage C is in complete agreement with the theory of time developed in this paper.

7. CONCLUSIONS

In this paper we presented a rigorous theory of the meaning of time in the Theory of Relativity. The material is not completely original since it can be found scattered in the literature [6,8,9]. However we think that our work can be of utility for all readers that have yet some doubt concerning the twin paradox. The paper also demonstrates that Sachs' treatment of the twin paradox [1,16,17] is non sequitur and that Einstein never wrote a single line which endorses Sachs' misleading point of view.

APPENDIX

The objective of this appendix is to prove the anti-Minkowski triangle inequality, used in section 4. We take the opportunity to present some results related to the Linear Algebra of Minkowski Space, which should be very well-known. The tangent space to any point of the space time manifold is the Minkowski space , and its precise definition is

DEFINITION 1: Minkowski space M is a 4-dimensional vector space over the real field with a Lorentzian inner product, that is, we can associate the metric tensor to the matrix diag(1,-1,-1,-1) in one orthonormal basis.

DEFINITION 2: Let be $v \in M$, we say that v is spacelike if $v^2 < 0$ or $v = 0$, that v is lightlike if $v^2 = 0$ and $v \neq 0$, that v is timelike if $v^2 > 0$.

DEFINITION 3: Let $S \subset M$ be a subspace. We say that S is spacelike if all its vectors are spacelike , that S is lightlike if it contains a lightlike vector but no timelike vector , that S is timelike if it contains a timelike vector.

We note that the definitions stablish that a subspace $S \subset M$ is spacelike or lightlike or timelike . We are going to prove some propositions that will permit us to understand the Linear Algebra of M.

PROPOSITION 1: A subspace S is timelike if and only if its orthogonal complement $S^{\perp}$ is spacelike.

PROOF: Let $S \subset M$ be a timelike subspace, there exists a timelike vector $v_o \in S$ and we define $e_o = \|v_o\|^{-1} v_o (\|v_o\| = (|v_o^2|)^{1/2})$ and add to it other three vectors $e_i\, (i = 1,2,3)$ in such a way to construct an orthonormal basis $\{e_\mu\}$ ($\mu = 0,1,2,3$) with $e_\mu . e_\nu = \eta_{\mu\nu}$, $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$. Then $S^\perp \subset \text{span}\,[e_1,e_2,e_3]$ (the space generated by e_1,e_2,e_3) and $S^\perp$ is spacelike (we define $u.v \equiv g(u,v)$).

Conversely (we note that $(S^\perp)^\perp = S$) if S is spacelike we have $M = S \oplus S^\perp$ (direct sum), and for any timelike vector $v \in M$ we have $v = v' + v''$ with $v' \in S$ and $v'' \in S^\perp$. Then we have $v''.v' = = v.v - v''.v'' < 0$ and therefore v'' is timelike and $S^\perp$ is timelike.

PROPOSITION 2: Let $S \subset M$ be a lightlike subspace. Then its orthogonal complement $S^\perp$ is lightlike and $S \cap S^\perp \neq \{0\}$.

PROOF: If S is lightlike, $S^\perp$ cannot be timelike or spacelike by prop. 1, then $S^\perp$ is lightlike. There is a light vector $n \in S$, but there isn't any timelike vector belonging to S. Then $\forall a \in \mathbb{R}$, $\forall s \in S$, $(s + an)(s + an) = s.s + 2as.n \leq 0 \quad \forall a \in \mathbb{R}$, therefore $s.n = 0 \quad \forall s \in S$ and $n \in S^\perp \therefore S^\perp \cap S \neq \{0\}$.

PROPOSITION 3: Two lightlike vectors $n_1, n_2 \in M$ are orthonormal if and only if they are proportional.

PROOF. Let $v \in M$ be a timelike vector. By prop. 2 we have $v.n_1 \neq 0$, $v.n_2 \neq 0$, then there exists $\alpha \in \mathbb{R}$, $\alpha \neq 0$, such that $v.(n_1 + \alpha n_2) = 0$ and by prop. 1, $n_1 + \alpha n_2$ is spacelike. But $(n_1 + \alpha n_2)^2 = 2\alpha n_1 . n_2$, then if n_1, n_2 are orthogonal $n_1 . n_2 = 0 \therefore n_1 + \alpha n_2 = 0$ and n_1, n_2 are proportional. Conversely if n_1 and n_2 are proportional we have $n_1 = \beta n_2$ and $n_1 . n_2 = \beta (n_2)^2 = 0$.

PROPOSITION 4: There are only two orthogonal spacelike vectors which also are orthogonal to a lightlike vector.

PROOF: Let $n \in M$ lightlike, $s_1, s_2 \in M$ spacelike such that

$s_1 . s_2 = 0$. We define a basis $\{e_\mu\}$ with $-e_o^2 = e_i^2 = -1$ in such a way that $s_1 = (0, s_1^1, 0, 0)$ and $s_2 = (0,0,s_2^2,0)$ in this basis. If $s_1 . n = s_2 . n = 0$ we'll have that $n = (n^o, 0, 0, n^3)$ with $(n^o)^2 = (n^3)^2 \neq 0$. Another spacelike vector orthogonal to s_1 and s_2 must have the form $s_3 = (s_3^1, 0, 0, s_3^3)$, with $(s_3^3)^2 > (s_3^1)^2$, then $s_3 . n = n^3 s_3^3 \neq 0$.

PROPOSITION 5: The unique way to construct an orthonormal basis for M is with one timelike vector and three spacelike vectors.

PROOF: If we have a lightlike vector in our basis , we have no timelike vector by prop. 1 . Then we must have only lightlike vectors and spacelike vectors in our basis. By prop. 3, orthogonal light vectors are proportional, and we must have only one lightlike vector in our basis, the other three must be spacelike, what is an absurd by prop. 4 .

Therefore we only have timelike and spacelike vectors in the basis in the proportion 1:3 in order for the signature to be -2.

Now we are going to show that we can divide the set $\tau \subset M$ of all timelike vectors into two disjoint subsets τ^+ and τ^-, which we identify, by convention which future and past.

PROPOSITION 6: The relation $u \uparrow v$, defined by $u \uparrow v$ if and only if $u.v > 0$ is an equivalence relation for timelike vectors and this relation divides τ into two disjoint equivalence classes τ^+ and τ^-.

PROOF: First we take $e_o, e_o^2 = 1$, as the future timelike direction, we add to it three spacelike vectors $e_i, e_i^2 = -1$ in such a way to construct an orthonormal basis $\{e_\mu\}$. If $u, v \in \tau$ we have $u = (u^o, u^1, u^2, u^3)$ and $v = (v^o, v^1, v^2, v^3)$ in this basis with $(u^o)^2 > \Sigma_i (u^i)^2$, $(v^o)^2 > \Sigma_i (v^i)^2$, $\therefore u.v = u^o v^o - \Sigma_i u^i v^i$. Then we observe that if u^o, v^o have the same signal, $u.v > 0$, if not, $u.v < 0$, since by Schwarz inequality in $\mathbb{R}^3$ we have

$$(\Sigma_i (u_i)^2)^{1/2} (\Sigma_i (v^i)^2)^{1/2} \geq \Sigma_i u^i v^i \therefore |u^o| |v^o| > \Sigma_i u^i v^i .$$

Therefore $\uparrow$ is obviously an equivalence relation that divides τ into two equivalence classes, τ^+ such that if $u \in \tau^+$, $u.e_o > 0$ and τ^- such that $u \in \tau^- \Rightarrow u.e_o < 0$. We call τ^+ the future component of τ and τ^- the past component and we note that this definition depends of the choice of a future timelike direction.

PROPOSITION 7: τ^+(and τ^-) are convex sets, that is, given $u,v \in \tau^+$, $a \in (0,\infty)$, $b \in [0,\infty)$ then $w = au + bv \in \tau^+$.

PROOF: $u,v \in \tau^+ \Longleftrightarrow u.e_o > 0$, $v.e_o > 0$, where e_o as fixed future timelike direction, therefore, $w.e_o = au.e_o + bv.e_o > 0$ and $w \in \tau^+$.

At this point we are prepared to show the anti-Minkowski inequality. First we are going to show the anti-Schwarz inequality.

PROPOSITION 8; Let $u,v \in M$ be timelike vectors. Then we have for them the anti-Schwarz inequality, that is, $|u.v| \geq \|u\| \|v\|$ and the equality only occurs if u,v are proportional.

PROOF: We choose an orthonormal basis $\{\vec{e}_\alpha\}$ with $e_o = \|u\|^{-1} u$ such that $u = (u^o,0,0,0)$ in this basis and $v = (v^o,v^1,v^2,v^3)$ with $(v^o)^2 > \Sigma_i (v^i)^2$. Then we have

$$\|u\| = |u^o|, \ \|v\| = ((v^o)^2 - \Sigma_i (v^i)^2)^{1/2} \leq |v^o|, \ |u.v| = |u^o| \ |v^o| \geq \|u\| \ \|v\| .$$

If the equality is satisfied we have

$$\|v\| = |v^o| \therefore v = (v^o,0,0,0)$$

being proportional to u.

PROPOSITION 9: Let $u,v \in \tau^+$, then we have for u,v the anti-Minkowski inequality, that is,

$$\|u + v\| \geq \|u\| + \|v\| .$$

PROOF: We note that by prop. 6 $u.v > 0$, and using prop. 8 we have

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2u.v \geq \|u\|^2 + \|v\|^2 + 2\|u\| \, \|v\|$$

$$= (\|u\| + \|v\|)^2 \Rightarrow \|u + v\| \geq \|u\| + \|v\| .$$

To complete this appendix we are going to study the possibility of any Schwarz like or Minkowski like inequality for spacelike vectors $u,v \in M$.

PROPOSITION 10: Let $u,v \in M$ be spacelike vectors such that span $[u,v]$ is spacelike, then it holds the usual Schwarz inequality, $|u.v| \leq \|u\| \, \|v\|$. It also holds the usual Minkowski inequality, $\|u + v\| \leq \|u\| + \|v\|$. If the equalities are satisfied, then u and v are proportional.

PROOF: If span $[u,v]$ is spacelike, it has an orthonormal basis $\{e_1,e_2\}$ with $e_1^2 = e_2^2 = -1$ such that $u = (u^1,0)$ and $v = (v^1,v^2)$ in this basis. Then

$$\|u\| = |u^1|, \|v\| = ((v^1)^2 + (v^2)^2)^{1/2} \geq |v^1|, \; |u.v| = |u^1| \, |v^1| \leq \|u\| \, \|v\| .$$

If the equality holds then $v^1 = 0$ and u,v are proportional.

We therefore have

$$\|u + v\|^2 = -(u + v)^2 = -u^2 - v^2 - 2u.v = \|u\|^2 + \|v\|^2 - 2u.v \leq$$

$$\leq \|u\|^2 + \|v\|^2 + 2\|u\| \, \|v\| = (\|u\| + \|v\|)^2 \therefore \|u + v\| \leq \|u\| + \|v\| .$$

The equality only holds if $-u.v = |u.v| = \|u\| \ \|v\|$, then u,v are proportional.

PROPOSITION 11: Let $u,v \in M$ be spacelike vectors such that span $[u,v]$ is timelike, then it holds the anti-Schwarz inequality, $|u.v| \geq \|u\| \ \|v\|$, if the equality holds, then u,v are proportional. If $u.v \leq 0$ that is u,v have different "time direction", and more, if $u + v$ is spacelike; we then have the anti-Minkowski inequality, $\|u + v\| \geq \|u\| + \|v\|$, and the equality only holds if u,v are proportional.

PROOF: If span $[u,v]$ is timelike we choose an orthonormal basis $\{e_o, e_1\}$ for it with $e_o^2 = e_1^2 = 1$ and such that in this basis $u = (0, u^1)$, $v = (v^o, v^1)$ with $|v^o| < |v^1|$. Then we have $\|u\| = |u^1|$, $\|v\| = [(v^1)^2 - (v^o)^2]^{½} \leq |v^1|$ and $|u.v| = |u^1| \ |v^1| \geq \|u\| \|v\|$. If the equality holds, $v^o = 0$ and u,v are proportional.

If $u.v \leq 0$, we have $|u.v| = -u.v$, if also $u + v$ is spacelike we have

$$\|u + v\|^2 = -(u + v)^2 = -u^2 - v^2 - 2u.v = \|u\|^2 + \|v\|^2 + 2|u.v| \geq$$

$$\geq \|u\|^2 + \|v\|^2 + 2\|u\| \ \|v\| = (\|u\| + \|v\|)^2 \therefore$$

$$\therefore \|u + v\| \geq \|u\| + \|v\| .$$

The equality only holds if $-u.v = |u.v| = \|u\| \|v\|$ then u,v are proportional.

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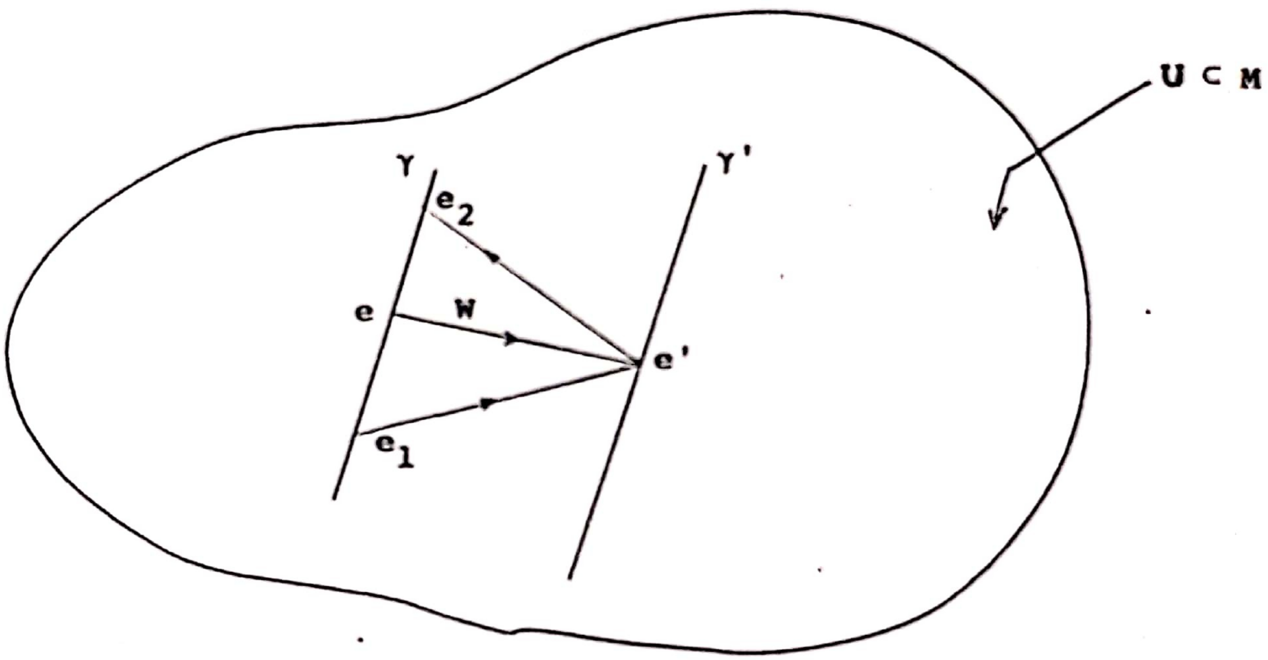

FIG.1 NEARBY INTEGRAL CURVES OF Q

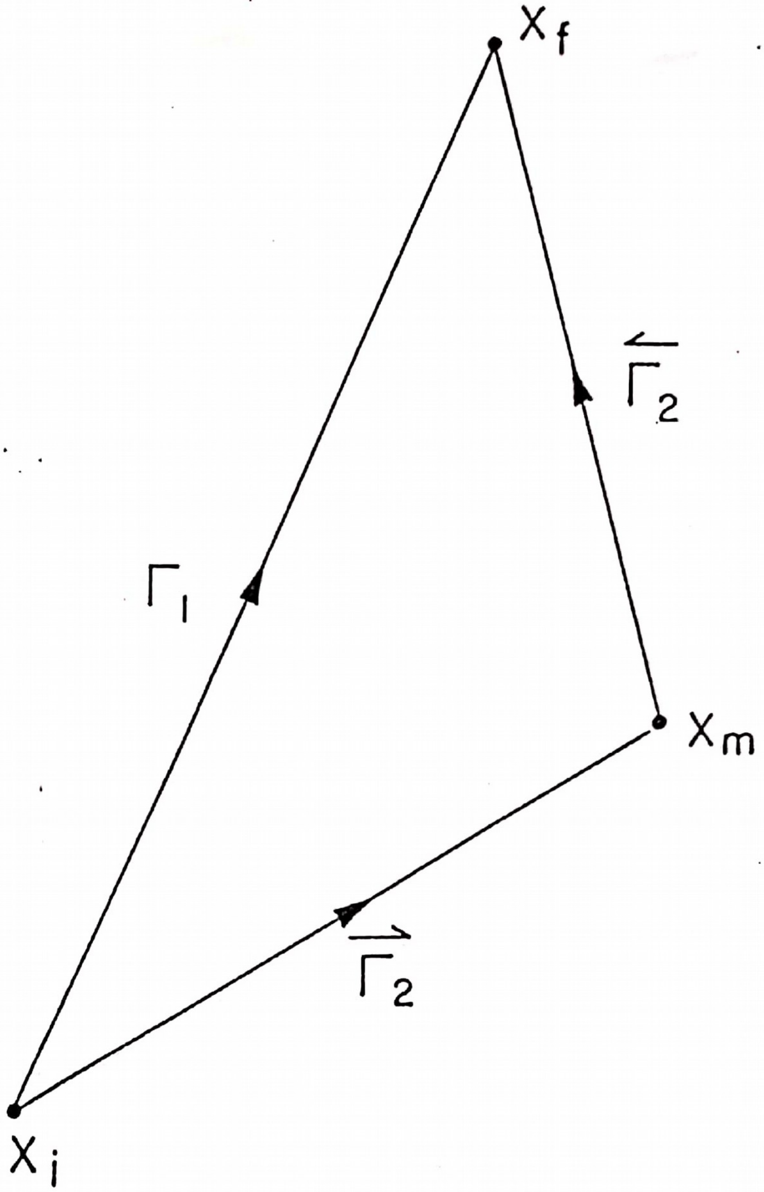

Fig. 2 Paths of two clocks ① and ② synchronized at x_i and that meet again in x_f in M