ALTERNATIVE REPRESENTATIONS FOR SPINORS BY CLIFFORD ALGEBRAS

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ABSTRACT. Many different definitions and representations of spinors are given in the literature, but there are no single reference explaining how they are related, which may explain why considerable confusion on the subject persists in the literature. Here we deal with three different definitions for spinors, namely: (I) the *covariant definition* (E. Cartan), where a particular kind of a *covariant spinor* (c-spinor) is a set of complex variables defined by its transformations under a particular spin group; (II) the *ideal definition* (M. Riez) where a particular kind of an *algebraic spinor* (a-spinor) is defined as an element of a minimum lateral ideal in an appropriate Clifford algebra $\mathbb{R}_{p,q}$; (III) the *operator definition* (D. Hestenes), where a particular kind of an *operator spinor* (o-spinor) is a Clifford number (operator) in an appropriate Clifford algebra $\mathbb{R}_{p,q}$, determining a set of tensor fields by bilinear mappings. By introducing the concept of "spinorial metric" in the ideal space of algebraic spinors we prove for $p + q \leqslant 5$ that there exists an equivalence from the group theoretical point of view between covariant and algebraic spinors. We also study in which sense o-spinors are equivalent to covariant spinors. Our approach contain the following important physical cases: Pauli, Dirac, dotted and undotted two-component spinors. The advantage of the use of each kind of spinor is briefly discussed in relation with the Dirac equation in both flat and non-flat Lorentzian manifolds.

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1. INTRODUCTION

There appears in the literature three different definitions of spinors. These are:

(I) The *covariant definition*, (E. Cartan[1], R. Brauer and H. Weyl[2]), where a particular kind of a c-spinor is a set of complex variables defined by its transformations under a particular spin group.

(II) The *ideal definition* (M. Riez[3]), where a particular kind of an a-spinor is defined as an element of a minimum lateral ideal in an appropriate Clifford algebra $\mathbb{R}_{p,q}$ (*).

(III) The *operator definition* (D. Hestenes[4,5,6,7]), where a particular kind of an o-spinor is a Clifford number (operator) in an appropriate Clifford algebra $\mathbb{R}_{p,q}$, determining a set of tensor fields by bilinear mappings.

The usual presentation of a-spinors[3,4,5,8,9] as elements of minimal lateral ideals in Clifford algebras as well as the introduction in this context of the groups $Spin^+(p,q)$, does not leave clear the relation between these objects and the c-spinors and the universal covering groups of some groups $SO_+(p,q)$ used in theoretical physics. The same is true in relation with the o-spinors.

The main purpose of this paper is to clear up the situation and in the process we will obtain some very interesting results.

To formulate our problem we start by remembering that physicists use the following kinds of c-spinors,

(i) Pauli c-spinors - which are the vectors of a complex 2-dimensional space $\mathbb{C}(2)$ equiped with the spinorial metric

$$\beta_p : \mathbb{C}(2) \times \mathbb{C}(2) \to \mathbb{C}, \quad \beta_p(\psi,\varphi) = \psi^* \varphi$$

$$\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} ; \quad \varphi = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad z_i, y_i \in \mathbb{C}, \; i = 1,2 \text{ and } \psi^* = (\bar{z}_1, \bar{z}_2) \tag{1}$$

(*) four our notation see § 2

where in this text $\bar{z}$ always means the complex conjugate of $z \in \mathbb{C}$.

The spinorial metric is invariant under the action of the group SU(2), ie, if $u \in SU(2)$, then $\beta_p(u\psi, u\varphi) = \beta_p(\psi,\varphi)$. As it is well known, Pauli c-spinors carry the fundamental (irreducible) representation $D^{1/2}$ of SU(2)[10,11]

(ii) undotted and dotted two-component c-spinors (introduced by Van der Waerden[12]) - which are respectively the vectors of two complex 2-dimensional spaces $\mathbb{C}(2)$ and $\overset{\circ}{\mathbb{C}}(2)$. In both spaces there are defined as spinorial metrics β, $\overset{\circ}{\beta}$ such that

$$\beta : \mathbb{C}(2) \times \mathbb{C}(2) \to \mathbb{C} , \quad \beta(\psi,\varphi) = \psi^t C \varphi$$
$$\overset{\circ}{\beta} : \overset{\circ}{\mathbb{C}}(2) \times \overset{\circ}{\mathbb{C}}(2) \to \mathbb{C} , \quad \overset{\circ}{\beta}(\overset{\circ}{\psi},\overset{\circ}{\varphi}) = \overset{\circ}{\psi}{}^t C \overset{\circ}{\varphi} \tag{2}$$

where $\psi(\overset{\circ}{\psi})$ is of the type defined in eq (1) , $\psi^t(\overset{\circ}{\psi}{}^t)$ is the transpose of $\psi(\overset{\circ}{\psi})$ and

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is the representation of $\beta(\overset{\circ}{\beta})$ in the canonical basis of $\mathbb{C}(2)$ $(\overset{\circ}{\mathbb{C}}(2))$[13,14].

The spinorial metrics β, $\overset{\circ}{\beta}$ are invariant under the action of the group SL(2,$\mathbb{C}$), ie, if $u \in SL(2,\mathbb{C})$ then $\beta(u\psi,u\varphi) = \beta(\psi,\varphi)$ and $\overset{\circ}{\beta}((u^*)^{-1}\overset{\circ}{\psi}, (u^*)^{-1}\overset{\circ}{\varphi}) = \overset{\circ}{\beta}(\overset{\circ}{\psi},\overset{\circ}{\varphi})$. The matrices u and $(u^*)^{-1}$ are the (non-equivalent) representations $D^{(1/2,0)}$ and $D^{(0,1/2)}$ of the group SL(2,$\mathbb{C}$) and we say that the undotted (dotted) two-component c-spinors is the carrier of the representation $D^{(1/2,0)}$ $(D^{(0,1/2)})$.

(iii) Dirac c-spinors - these are the vectors of a complex 4-dimensional space $\mathbb{C}(4)$ equipped with the spinorial metric[13,14]

$$\beta_o : \mathbb{C}(4) \times \mathbb{C}(4) \to \mathbb{C} , \quad \beta_o(\psi_d,\varphi_d) = \psi_d^t B \varphi_d \tag{4}$$

where a Dirac c-spinor $\psi_d(\varphi_d)$ is defined as

$$\mathbb{C}(2) \oplus \overset{\circ}{\mathbb{C}}(2)^* = \mathbb{C}(4) \ni \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{5}$$

where $\xi \in \mathbb{C}(2)$ and $\eta = \overset{\circ}{\beta}(\overset{\circ}{\eta},) \in \mathbb{C}(2)^*$, the dual space of $\mathbb{C}(2)$. In the canonical basis of $\mathbb{C}(4)$ obtained through the canonical basis of $\mathbb{C}(2)$ and $\overset{\circ}{\mathbb{C}}(2)^*$ the matrix B is the representation of β_o and we have

$$\beta_o = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \tag{6}$$

where C is the matrix defined in eq(3).

Dirac c-spinors, as is well known, carry the $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ representation of $SL(2,\mathbb{C})$. Some authors like in ref (13) call bi-spinors the objects $\psi \in \mathbb{C}(4)$ of the form given by eq(3)

(iv) Standard Dirac c-spinors

In writting Dirac's equation it is more convenient to work with standard Dirac c-spinors. These are the objects ψ_s such that

$$\mathbb{C}(4) \ni \psi_s = \begin{pmatrix} \phi \\ \lambda \end{pmatrix} \tag{7}$$

where $\phi = \frac{1}{\sqrt{2}}(\xi+\eta)$, $\lambda = \frac{1}{\sqrt{2}}(\xi-\eta)$, where $\xi \in \mathbb{C}(2)$, $\eta = \overset{\circ}{\beta}(\eta,) \in$ $\in \overset{\circ}{\mathbb{C}}(2)^*$ and the sum in ϕ and λ are in the usual sense of sum of complex numbers for each component. It is well known that ψ_d and ψ_s are related by unitary transformations which leave unchanged the bilinear covariant forms constructed from ψ_d and ψ_d^*. Then, the ψ_s's also carry the $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ representation of $SL(2,\mathbb{C})$

We now ask the main question: to which minimal ideals, in which Clifford algebras are the c-spinors described in (i), (ii), (iii) and (iv) above to be associated ?

We are going to give an original answer to the above question by introducing a *unique natural scalar product* (see § 3) in certain appropriate lateral ideals of certain Clifford algebras that "mimic" what has been described in (i), (ii), (iii) and (iv) above. To this end in section 2 we give the main properties of Clifford algebras over the reals[16,17,18]. The material presented fixes our notation and is the minimum necessary to permit the formulation of our ideas in a rigorous way.

In section 3 we define the a-spinors as the elements of minimal lateral ideals in Clifford algebras. The a-spinors of each one of the Clifford algebras *studied in this paper* have a natural right F-linear space structure over one of the following fields $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively the real, complex and quaternion fields (§2)

We introduce for each a-spinor space $I \subset \mathbb{R}_{p,q}$, a *unique natural scalar product* (spinorial metric), ie, a non-degenerated bilinear application $\Gamma: I \times I \to F$, where F is the natural scalar field associated with the vector space structure of $I \subset \mathbb{R}_{p,q}$.

Our approach to the natural scalar product shows that for $p+q \leq 5$, the groups $\mathrm{Spin}^+(p,q)$ are the groups that leave invariant the spinorial metric. Thus our approach to the scalar product is different from the one discussed by Lounesto[8] and as we shall see offers a solution for the main question formulated above.

In §4 we analyse in detail the special cases $SU(2) \simeq \mathrm{Spin}(3,0)$ and $SL(2,\mathbb{C}) \simeq \mathrm{Spin}^+(1,3)$ and identify respectively the ideals that contain the objects corresponding to Pauli c-spinors $(I_p^+ = \mathbb{R}_{3,0}^+ e_{30})$ and undotted and dotted two components c-spinors $(I_u = R_{1,3}^+ e_{13}$, $I_d = I_u^*)$. Also in §4.3 we show that the minimal left ideals $I_D = \mathbb{R}_{1,3} f(e_{30})$ of $\mathbb{R}_{1,3}$, the space-time or Minkowski algebra[7] carry the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $SL(2,\mathbb{C})$, ie, the space-time a-spinors are a representation of the Dirac c-spinors.

In §4.4 we show that the original Dirac algebra $(\mathbb{C}(4))$ must be identified for physical reasons with the real Clifford algebra $\mathbb{R}_{4,1}$. We then show that the ideals $\bar{I}_D = \mathbb{R}_{4,1}^+ g(f(e_{30}))$ carry also a representation of the Dirac c-spinors.

In §5 we find the ideals in $\mathbb{R}_{1,3}$ representing the standard Dirac c-spinors ($I_{1D} = \mathbb{R}^{+}_{1,3}u_1$). Writing Dirac's equation in the standard representation in Minkowski space ($\mathbb{R}^{1,3}$) we show that the idempotent u_1 can be eliminated from the equation, thus resulting a new equation satisfied by an element of $\mathbb{R}^{+}_{1,3}$, the even sub-algebra of $\mathbb{R}_{1,3}$. This motivates the operator definition of spinors by Hestenes.

Finally in §6 we present our conclusions together with some comments concerning to the advantages of the use of each kind of spinors in the formulation and applications of the Dirac equation in flat and non flat Lorentzian manifolds.

2. SOME GENERAL FEATURES ABOUT CLIFFORD ALGEBRAS

Let V be a vector space of finite dimension n over the field F together with a non degenerate quadratic form Q. The Clifford algebra $C(V,Q) = T(V)/I_Q$ where $T(V)$ is the tensor algebra of V ($T(V) = \sum_{i=1}^{\infty} T^i(V)$; $T^{(0)}(V) = F$; $T^1(V) = V$; $T^r(V) = \otimes^r V$) and I_Q is the bilateral ideal generated by the elements of the form $x \otimes x - Q(x)1$, $x \in V$. The signature of Q is arbitrary. The Clifford algebra so constructed is an associative algebra with unit. The space V is naturally imbedded in $C(V,Q)$.

$$V \overset{i}{\hookrightarrow} T(V) \overset{j}{\rightarrow} T(V)/I_Q = C(V,Q)\,, \quad i_Q = j \circ i \quad \text{and} \quad V \equiv i_Q(V) \subset C(V,Q)$$

Let $C^+(V,Q)$ (respectively $C^-(V,Q)$) be the j-image of $\sum_{i=0}^{\infty} T^{2i}(V)$ (respectively $\sum_{i=0}^{\infty} T^{2i+1}(V)$) in $C(V,Q)$. The elements of $C^+(V,Q)$ form a subalgebra of $C(V,Q)$ called the even subalgebra of $C(V,Q)$.

$C(V,Q)$ has the following universal property: "If A is an associative F-algebra with unit then all linear mappings $\phi: V \to A$ such that $(\phi(x))^2 = Q(x)1$, $\forall x \in V$ can be extended in a unique way to a homomorphism $\phi : C(V,Q) \to A$".

In $C(V,Q)$ there exist three linear mappings which are quite natural. They are extensions of the mappings

(a) MAIN INVOLUTION - an automorphism $^{\blacksquare}$: $C(V,Q) \to C(V,Q)$ extension of $\alpha : V \to T(V)/I_Q$, $\alpha(x) = -i_Q(x) = -x$, $\forall x \in V$

(b) REVERSION - an antiautomorphism *: $C(V,Q) \to C(V,Q)$ extension of ^{t}: $T^r(V) \to T^r(V)$, $T^r(V) \ni x = x_{i_1} \otimes \ldots \otimes x_{i_r} \to x^t = x_{i_r} \otimes \ldots \otimes x_{i_1}$.

(c) CONJUGATION: $\tilde{}$: $C(V,Q) \to C(V,Q)$, defined by the composition of the automorphism $^{\blacksquare}$ with the anti automorphism *, ie, if $x \in C(V,Q)$, then $\tilde{x} = (x^{\star})^{\blacksquare}$

$C(V,Q)$ can be described through its generators, ie, if $\{e_i\}$, $i = 1,2,\ldots,n$ is a Q-orthonormal basis of V, then $C(V,Q)$ is generated by 1 and the e_i's subjected to the conditions $e_i e_i = Q(e_i)1$ and $e_i e_j + e_j e_i = 0$, $i \neq j$, $i,j = 1,2,\ldots,n$. If V is a n-dimensional real vector space then we can choose a basis $\{e_i\}$ for V such that $Q(e_i) = \pm 1$.

2.2. THE REAL CLIFFORD ALGEBRAS $\mathbb{R}_{p,q}$

Let $\mathbb{R}^{p,q}$ be a real vector space of dimension $p+q=n$ equipped with a metric $g: \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to R$. Let $\{e_i\}$ be the canonical basis of $\mathbb{R}^{p,q}$ such that

$$g(e_i,e_j) = g_{ij} = g(e_j,e_i) = g_{ji} = \begin{cases} +1 & i = j = 1,2,\ldots,p \\ -1 & i = j = p+1,\ldots p+q = n \\ 0 & i \neq j \end{cases}$$

The Clifford algebra $\mathbb{R}_{p,q} = C(\mathbb{R}^{p,q},Q)$, $p+q = n$, in the Clifford algebra over the real field $\mathbb{R}$, generated by 1 and the $\{e_i\}$,

$i = 1,\ldots,n$ such that $Q(e_i) = g(e_i,e_i)$. $\mathbb{R}_{p,q}$ is obviously of dimension 2^n and it is the direct sum of the vector spaces $\mathbb{R}^k_{p,q}$ of dimensions $\binom{n}{k}$, $0 \leq k \leq n$. The canonical basis for $\mathbb{R}^k_{p,q}$ are the elements $e_A = e_{\alpha_1} \ldots e_{\alpha_k}$ $1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n$. The element $e_J = e_1 \ldots e_n \in \mathbb{R}^n_{p,q}$ commutes (n-odd) or anti-commutes (n-even) with all vectors $e_1,\ldots,e_n$ in $\mathbb{R}^1_{p,q} = \mathbb{R}^{p,q}$. The center of $\mathbb{R}_{p,q}$ is $\mathbb{R}^0_{p,q} = \mathbb{R}$ if n is even and it is the direct sum $\mathbb{R}^0_{p,q} \oplus \mathbb{R}^n_{p,q}$ if n is odd.[8,18]

All Clifford algebras are semi-simple. If $p+q = n$ is even $\mathbb{R}_{p,q}$ is a simple algebra and if $p+q = n$ is odd we have the following possibilities:

(i) $\mathbb{R}_{p,q}$ is simple $\leftrightarrow e_J^2 = -1 \leftrightarrow p - q \neq 1 \pmod 4 \leftrightarrow$ center $\mathbb{R}_{p,q}$ is isomorphic to $\mathbb{C}$

(ii) $\mathbb{R}_{p,q}$ is not simple $\leftrightarrow e_J^2 = +1 \leftrightarrow p - q = 1 \pmod 4 \leftrightarrow$ center $\mathbb{R}_{p,q}$ is isomorphic to $\mathbb{R}^0_{p,q} \oplus \mathbb{R}^n_{p,q}$.

From the fact that all semi-simple algebras are the direct sum of two simple algebras[19] and from

WEDDENBURN'S THEOREM: "If A is a simple algebra then A is equivalent to $F(m)$, where F is a division algebra and m and F are unique (modulo isomorphisms)"

we obtain from the point of view of representation theory $\mathbb{R}_{p,q} \simeq F(m)$ or $\mathbb{R}_{p,q} \simeq F(m) \oplus F(m)$ where $F(m)$ is the matrix algebra of dimension $m\times m$ (for some m) with coefficients in $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Table I (where $[n/2]$ means the integral part of $(n/2)$ presents the representation of $\mathbb{R}_{p,q}$, as a matrix algebra[18]

p-q (mod8)	0	1	2	3	4	5	6	7
$\mathbb{R}_{p,q}$	$\mathbb{R}(2^{[n/2]})$	$\mathbb{R}(2^{[n/2]}) \oplus \mathbb{R}(2^{[n/2]})$	$\mathbb{R}(2^{[n/2]})$	$\mathbb{C}(2^{[n/2]})$	$\mathbb{H}(2^{[n/2-1]})$	$\mathbb{H}(2^{[n/2]-1}) \oplus \mathbb{H}(2^{[n/2]-1})$	$\mathbb{H}(2^{[n/2-1]})$	$\mathbb{C}(2^{[n/2]})$

Table I - Representation of the real Clifford algebra $\mathbb{R}_{p,q}$ as a matrix algebra

2.3. MINIMAL LATERAL IDEALS OF $\mathbb{R}_{p,q}$

The minimal left ideals of a semi-simple algebra A are of the type Ae, where $e(e^2 = e)$ is a primitive idempotent of A. A idempotent is primitive if it cannot be written as a sum of two non zero orthogonal idempotents, ie, $e \neq \hat{e} + \check{e}$, where $\hat{e}^2 = \hat{e}$, $\check{e}^2 = \check{e}$ and $\check{e}\hat{e} = \hat{e}\check{e} = 0$ [19]. Recall that when $p+q = n$ is even $\mathbb{R}_{p,q} \simeq F(m)$. (Table I). We also have the

THEOREM: The maximum number of pairwise orthogonal idempotents in $F(m)$ is m [21]

The decomposition of $\mathbb{R}_{p,q}$ into minimal ideals is characterized by a *spectral set* $\{e_{pq,i}\}$ of idempotent elements of $\mathbb{R}_{p,q}$ such that

(a) $\Sigma\, e_{pq,i} = 1$

(b) $e_{pq,i}\, e_{pq,j} = \delta_{ij}\, e_{pq,i}$

(c) rank of $e_{pq,i}$ is minimal $\neq 0$

where rank of $e_{pq,i}$ is defined as the rank of the $\oplus\Lambda^d(\mathbb{R}^{p,q})$-morphism $e_{pq,i} : \psi \to \psi\, e_{pq,i}$, where $\oplus\,\Lambda^d(\mathbb{R}^{p,q})$ is the exterior algebra of $\mathbb{R}^{p,q}$. Then $\mathbb{R}_{p,q} = \Sigma I^i_{p,q}$, $I^i_{p,q} = \mathbb{R}_{p,q} e_{pq,i}$ and $\psi \in I^i_{p,q} \subset \mathbb{R}_{p,q}$ is such that $\psi e_{pq,i} = \psi$. Conversely, any element $\psi \in I^i_{p,q}$ can be characterized by an idempotent $e_{pq,i}$ of minimal rank $\neq 0$ with $\psi e_{pq,i} = \psi$.

We have the

THEOREM: A minimal left ideal of $\mathbb{R}_{p,q}$ is of the type $I_{p,q} = \mathbb{R}_{p,q} e_{pq}$ where $e_{pq} = \tfrac{1}{2}(1 + e_{\alpha_1}) \ldots \tfrac{1}{2}(1 + e_{\alpha_k})$ is a primitive idempotent of $\mathbb{R}_{p,q}$ and where $e_{\alpha_1}, \ldots, e_{\alpha_k}$ is a set of commuting elements of the canonical basis of $\mathbb{R}_{p,q}$ such that $(e_{\alpha_i})^2 = 1$, $i = 1,\ldots,k$ that generates a group of order $k = q - r_{q-p}$ and r_i are the Radon-Hurwitz numbers, defined by the recurrence formula $r_{i+8} = r_i + 4$ (8) and

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

Table II - Radon-Hurwitz numbers

If we have a linear mapping $L_a : \mathbb{R}_{p,q} \to \mathbb{R}_{p,q} \ni L_a(x)$, $\forall x \in \mathbb{R}_{p,q}$ and where $a \in \mathbb{R}_{p,q}$, then since $I_{p,q}$ is invariant under left multiplication with arbitrary elements of $\mathbb{R}_{p,q}$, we can consider $L_a|I_{p,q} : I_{p,q} \to I_{p,q}$. We have the

THEOREM: If $p + q = n$ is even or odd with $p - q \neq 1 \pmod 4$ then

$$\mathbb{R}_{p,q} \simeq \mathcal{L}_F(I_{p,q}) \simeq F(m)$$

where $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $\mathcal{L}_F(I_{p,q})$ is the algebra of linear transformations in $I_{p,q}$ over the field F, $m = \dim_F(I_{p,q})$ and $F = eF(m)e$, e being the representation of e_{pq} in $F(m)$. If $p+q=n$ is odd, with $p-q=1 \pmod 4$, then $\mathbb{R}_{p,q} \simeq \mathcal{L}_F(I_{p,q}) \simeq F(m) \oplus F(m)$, $m = \dim_F(I_{p,q})$ and $e_{pq}\mathbb{R}_{p,q}e_{pq} \simeq \mathbb{R} \oplus \mathbb{R}$ or $\mathbb{H} \oplus \mathbb{H}$ (19)

With the above isomorphisms we can identify the minimal left ideals $\mathbb{R}_{p,q}$ with the column matrices of $F(m)$

Now, with the ideas introduced above it is a simple exercise to find a primitive idempotent of $\mathbb{R}_{p,q}$. We have the following algorithm. We first give a look in table I and find to which matrix algebra our particular $\mathbb{R}_{p,q}$ is isomorphic. Let $\mathbb{R}_{p,q} \simeq F(m)$ for a particular F and m(*). Next we take from the canonical basis $\{e_A\}$ of $\mathbb{R}_{p,q}$

$$e_A = e_{\beta_1} \cdots e_{\beta_k} \qquad 1 \leq \beta_1 \leq \cdots \leq \beta_k \leq n\ , \quad p+q=n$$

a commuting element $e_{\alpha_1} \in \{e_A\}$ such that $e_{\alpha_1}^2 = 1$(**). We then construct the idempotent $e_{pq} = \frac{1}{2}(1+e_{\alpha_1})$ and calculate $\dim_F(I_{p,q})$. If $\dim_F(I_{p,q}) = m$ then e_{pq} is primitive. If $\dim_F(I_{p,q}) \neq m$ then choose $\{e_A\} \ni e_{\alpha_2} \mid e_{\alpha_2}^2 = e_{\alpha_2}$ and construct the idempotent $e'_{pq} = \frac{1}{2}(1+e_{\alpha_1})\ \frac{1}{2}(1+e_{\alpha_2})$ and then calculate $\dim_F(I')$ where $I'_{p,q} = \mathbb{R}_{p,q}e'_{pq}$. If $\dim_F(I'_{p,q}) = m$, then e'_{pq} is primitive. Otherwise repit the procedure. According to the theorem above the process is finite.

Now, we must discuss the problem of the *equivalence of representations* when we take the minimal left ideals (instead of some vector space isomorphic to them) as representation modules of $\mathbb{R}_{p,q}$.

To this end, remembering that $\mathbb{R}_{p,q}$ is not just an algebra but an algebraic structure consisting of an algebra together with a distinguished subspace $\mathbb{R}^1_{p,q} \simeq \mathbb{R}^{p,q}$ and that our representation

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(*) We are supposing $\mathbb{R}_{p,q}$ is simple. The procedure is also straightforward when $\mathbb{R}_{p,q}$ is semi-simple

(**) All elements e_{α_i} are commuting elements as stated in the last theorem.

spaces $I_{p,q}$ are certain sub-algebras of $\mathbb{R}_{p,q}$ we have the following theorems [22]

THEOREM OF NOETHER-SKOLEN: When $\mathbb{R}_{p,q}$ is simple, its automorphisms are given by its inner automorphisms $\psi \to s(\psi) = s\,\psi\, s^{-1}$ $(\psi \in I_{p,q})$ such that $s\,\mathbb{R}^1_{p,q}\, s^{-1} \subset \mathbb{R}^1_{p,q}$

THEOREM: When $\mathbb{R}_{p,q}$ is simple all their finite-dimensional irreducible representations are equivalent under inner automorphisms.

In view of the above theorems we define that two representations $I_{p,q}$ and $I'_{p,q}$ of $\mathbb{R}_{p,q}$ are equivalent if there is an automorphism $s \in \mathbb{R}_{p,q}$ such that $I'_{p,q} = s(I_{p,q})$

3. ALGEBRAIC SPINORS, SPIN GROUP , SPINORIAL REPRESENTATION AND SPINORIAL METRIC.

3.1. ALGEBRAIC SPINORS. Given a real Clifford algebra $\mathbb{R}_{p,q}$ we call a-spinors the elements of the minimal left ideals $\mathbb{R}_{p,q}e_{pq}$ or $\mathbb{R}^+_{p,q}e'_{pq}$, where e_{pq} and e'_{pq} are primitive idempotents of $\mathbb{R}_{p,q}$.

3.2. THE SPIN GROUP - Spin(p,q)

The invertible elements $s \in \mathbb{R}_{p,q}$ such that $\forall x \in \mathbb{R}^1_{p,q} \equiv$ $\equiv \mathbb{R}^{p,q}$ we have $s^{\blacksquare} x\, s^{-1} \in \mathbb{R}^1_{p,q}$, form a multiplicative group called the Clifford group of $\mathbb{R}_{p,q}$ which we denote by Γ_x. This group is generated by the vectors $x \in \mathbb{R}^{p,q}$ such that $g(x,x) \neq 0$. Consider now the mapping $N: \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}$ defined by $N(s) = \tilde{s}\, s$. If $s \in \Gamma_x$, then N is a homomorphism of the group Γ_x into the multiplicative group of the non null multiples of $\mathbb{R}_{p,q}$.

We define the groups $Pin(p,q) = \{s \in \Gamma_x; N(s) = \pm 1\}$, $Spin(p,q) =$ $= Pin(p,q) \cap \mathbb{R}^+_{p,q}$ and $Spin^+(p,q) = \{s \in \Gamma_x,\ \tilde{s}\, s = +1\} \cap \mathbb{R}^+_{p,q}$ as the connected component of Spin(p,q) that contains the identity.

3.3. SPINORIAL REPRESENTATION.

Now, if we consider the definition of the group $Spin^+(p,q)$ we see that the ideals $I_{p,q}$ can be made into *spinorial representations* of $SO_+(p,q)$ (in the sense of group theory) by postulating $I_{p,q} \to s\,I_{p,q}$ under appropriated automorphisms ($s \in Spin^+(p,q)$) . This is exactly the idea behind the introduction of the spinorial metric (§3.4), which is necessary to "mimic" the results in (i), (ii),(iii) and (iv) of §1.

The transformation $\psi \to s\psi$ corresponds to the usual transformation of covariant spinors, but the use of this transformation in a formalism involving other Clifford numbers would contradict the fact that the $I_{p,q}$'s are substructures of $\mathbb{R}_{p,q}$

To clarify the ideas consider the case of $Spin^+(1,3)$ that is the universal covering group of the proper Lorentz group . We have that the (active) Lorentz transformation for an arbitrary Clifford number $m \in \mathbb{R}_{1,3}$ is the same as for a vector,

$$\mathbb{R}^1_{1,3} \ni v \to v' = s v s^{-1} \quad , \; s \in Spin^+(1,3) . \text{ Since}$$

$v' \in \mathbb{R}^1_{1,3} \subset \mathbb{R}_{1,3}$ and $(v')^2 = v^2 = g(v,v) \quad \forall v \in \mathbb{R}^1_{1,3}$ we can extend this mapping in an unique way to a homomorphism of $\mathbb{R}_{1,3}$. We get that for arbitrary $m \in \mathbb{R}_{1,3}$,

$$m \to s\, m\, s^{-1} \tag{12}$$

The bilinear (two-sided) transformation of eq(12) is the same for all multivectors and this is the distintive feature of the Clifford algebra formalism when applied to physics [24]. Observe that this feature is in contrast with the linear (one sided) transformations for the components in the usual tensor and covariant spinor formalism, where tensor and c-spinors of different ranks have different transformations laws under a Lorentz transformation.

In particular, consider $\phi \in \mathbb{R}^{+}_{1,3}$. It is a sum of a scalar, a pseudo-scalar and a bivector parts. From the point of view of the theory of the linear representations of the Lorentz group [25] the space of the ϕ's is the carrier space of the representation $D^{(0,0)} \oplus D^{(\frac{1}{2},0)} \otimes D^{(0,\frac{1}{2})} \oplus D^{(0,0)}$ of $SL(2,C)$

The fact that within the Clifford algebra formalism all Clifford numbers transform in the same way $(m \to s\, m\, s^{-1})$ suggests besides the covariant spinors and algebraic spinors a new definition of spinor, the operator definition by Hestenes [4,5,6,7] which we discuss in §5.

3.4. SCALAR PRODUCT OF SPINORS. THE SPINORIAL METRIC.

In §2.3 we saw that when $\mathbb{R}_{p,q}$ is simple, a minimal left ideal I of $\mathbb{R}_{p,q}$ is of the form $I = \mathbb{R}_{p,q} e_{pq}$ where e_{pq} is a primitive idempotent of $\mathbb{R}_{p,q}$ and $F = e_{pq} \mathbb{R}_{p,q} e_{pq}$ with $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ depending of $p - q = 0,1,2 \pmod 8$, $p - q = 3,7 \pmod 8$ or $p - q = 4,5,6 \pmod 8$ respectively (Table I). We can then define a right action F in I, $I \times F \to I$, by $I \times F \ni (\psi, \alpha) \to \psi\alpha \in I$. In this way I has a natural linear vector space structure over the field F, whose elements are the natural "scalars" of the vector space I.

These remarks suggest us to seach for a *natural "scalar product"* on I, ie, a non-degenerated bilinear mapping $\Gamma: I \times I \to F$. To this end we observe that if f and g are F-endomorphisms in $\mathbb{R}_{p,q}$ then we can define a bilinear mapping Γ in $\mathbb{R}_{p,q}$ using f and g. We simply take $\Gamma(\psi, \varphi) = f(\psi)\, g(\varphi)$, $\psi, \varphi \in \mathbb{R}_{p,q}$. Considering that $I = \mathbb{R}_{p,q} e_{pq}$ has a natural structure of vector space over F we can take the restriction of Γ to I, and ask the following question:

For $\psi, \varphi \in I$ when does $\Gamma(\psi, \varphi) \in F$?

As we saw in §2.1 we have three natural isomorphisms defined in $\mathbb{R}_{p,q}$, the main involution, the reversion and the conjugation , denoted respectively by $^{\blacksquare}, ^{\star}$ and $\tilde{}$. Combining these isomorphisms with the identity mapping we can define the following bilinear mappings

$$\Gamma_i : I\times I \to \mathbb{R}_{p,q} \, , \quad i = 1,2,3$$

$$\Gamma_1(\psi,\varphi) = \psi^{\blacksquare}\varphi \; ; \; \Gamma_2(\psi,\varphi) = \psi^{\star}\varphi \; ; \; \Gamma_3(\psi,\varphi) = \tilde{\psi}\varphi \, , \; \forall \psi,\varphi \in I$$

As already observed in §2.1 the main involution is an automorphism whereas the reversion and conjugation are antiautomorphisms. An automorphism (antiautomorphism) transforms an element of a minimal left ideal in an element of a minimal left ideal (minimal right ideal).

To see the validity of these statements it is enough to observe that the image of a primitive idempotent under a isomorphism is a primitive idempotent and that if $\psi \in I_{p,q} = \mathbb{R}_{p,q}e_{pq}$ then $\psi = xe_{pq}$ with $x \in \mathbb{R}_{p,q}$ and

$$\begin{aligned} \psi^{\blacksquare} &= (x\, e_{pq})^{\blacksquare} = x^{\blacksquare}e^{\blacksquare}_{pq} \Rightarrow \psi^{\blacksquare} \in I'_{p,q} = \mathbb{R}_{p,q}e^{\blacksquare}_{pq} \\ \psi^{\star} &= (x\, e_{pq})^{\star} = e^{\star}_{pq}\, x^{\star} \Rightarrow \psi^{\star} \in I^{\star}_{p,q} = e^{\star}_{pq}\,\mathbb{R}_{p,q} \\ \tilde{\psi} &= (x\, e_{pq})^{\sim} = \tilde{e}_{pq}\,\tilde{x} \Rightarrow \tilde{\psi} \in \tilde{I}_{p,q} = \tilde{e}_{pq}\,\mathbb{R}_{p,q} \end{aligned} \tag{8}$$

Using the isomorphisms $\mathbb{R}_{p,q} \simeq \mathcal{L}_F(I_{p,q}) \simeq F(m)$, $m = \dim_F(I_{p,q})$ (when $\mathbb{R}_{p,q}$ is simple, cf. §2.3) we identify the elements of the minimal left ideals of $\mathbb{R}_{p,q}$ with the column matrices of $F(m)$. Then, if $\psi \in I_{p,q}$ has a representation as a column matrix of $F(m)$ then $\psi^{\star}$ and $\tilde{\psi}$ have representations as row matrices of $F(m)$, and we get that $\psi^{\star}\varphi$ and $\tilde{\psi}\varphi$ are elements of F.

We identify the scalars of the vector structure of $I_{p,q}$ with multiples of

$$e_{pq} \equiv 1 = \begin{bmatrix} 1 & 0 & \cdots\cdots \\ 0 & 0 & \cdots\cdots \\ 0 & 0 & \cdots\cdots \\ \cdots & \cdots & \cdots\cdots \end{bmatrix} \tag{9}$$

ie, as matrices in $F(m)$ multiples of the matrix in eq(9). Through isomorphisms of $\mathbb{R}_{p,q}$ (multiplication by a convenient invertible element $u \in \mathbb{R}_{p,q}$) we can transport $\psi^{\star}\varphi$ or $\tilde{\psi}\varphi$ to the position (1,1) in the matrix representation of these operations. We then conclude that the *natural scalar products* in $I_{p,q}$ are

$$\beta_i : I_{p,q} \times I_{p,q} \to F \qquad i = 1,2 \tag{10}$$

$\beta_1(\psi,\varphi) = u'\psi^{\star}\varphi$ and $\beta_2(\psi,\varphi) = u\tilde{\psi}\varphi$, $\forall \psi,\varphi \in I_{p,q}$ and $u,u' \in \mathbb{R}_{p,q}$ are convenient invertible elements.

Lounesto[8] obtains the scalar products in eq(10) using similar arguments and immediately proceeds to the classification of the group of automorphisms of these scalar products, ie, the homomorphisms of right F-modules, $I_{p,q} \to I_{p,q}$, $\psi \to s\psi$, $s \in \mathbb{R}_{p,q}$ which preserve the products in eq(10). Observe that from $\beta_1(s\psi,s\varphi) = \beta_1(\psi,\varphi)$ we get $s^{\star}s = 1$ and from $\beta_2(s\psi,s\varphi) = \beta_2(\psi,\varphi)$ we get $\tilde{s}s = 1$ ($\psi,\varphi \in I_{p,q}$). Lounesto[8] calls $G_1 = \{s \in \mathbb{R}_{p,q}; s^{\star}s = 1\}$, $G_2 = \{s \in \mathbb{R}_{p,p}, \tilde{s}s = 1\}$

So in Lounesto's paper there does not appear in principle any relationship between the groups Spin(p,q) and the groups G_1 and G_2 with the consequence that we do not have a clear basis to mimic within the Clifford algebras $\mathbb{R}_{p,q}$ (for appropriate p and q) the results described in (i), (ii), (iii) and (iv) of §1. We can mimic these results within some Clifford algebras by introducing the concept of spinorial metric.

Observe that since $\mathrm{Spin}(p,q) \subset \mathbb{R}^{+}_{p,q}$ it seems interesting to define a scalar product in an ideal $I^{+}_{p,q} = \mathbb{R}^{+}_{p,q}e_{pq}$. The reason is that such a scalar product is now *unique*, since if $s \in \mathbb{R}^{+}_{p,q}$, then $s^{\star} = \tilde{s}$. This unique scalar product will be called in what follows the spinorial metric

$$\beta : I^+_{p,q} \times I^+_{p,q} \to F \tag{11}$$

define by $\beta(\psi,\varphi) = u\tilde{\psi}\varphi$. We see that $G = \{s \in \mathbb{R}^+_{p,q} \mid \tilde{s}s = 1\}$ is the group of automorphisms of the Spinorial metric just defined and $G \subset G_1$, $G \subset G_2$.

We now recall a result firstly obtained by Porteous[20]: Spin$^+(p,q)$ = $= \{s \in \mathbb{R}^+_{p,q} \mid \tilde{s}s = 1\}$ for $p+q \leq 5$ "[Proposition 13.58]

With this result we get a new interpretation of the groups Spin$^+(p,q)$ for $p+q \leq 5$, namely, these are the groups that leave the Spinorial metric of eq(11) invariant. But even more important is the fact that now we know the way to mimic within appropriate Clifford algebras (i), (ii), (iii) and (iv) of §1 and thus we can make a representation within Clifford algebras of the Pauli c-spinors, undotted and dotted bidimensional c-spinors and Dirac c-spinors. This is done in §4.

4. REPRESENTATION OF PAULI C-SPINORS , UNDOTTED AND DOTTED TWO-DIMENSIONAL C-SPINORS AND DIRAC C-SPINORS BY APPROPRIATED ALGEBRAIC SPINORS.

4.1. PAULI C-SPINORS AND THE GROUP SU(2)

The algebra $\mathbb{R}_{3,0}$ (Pauli algebra) is isomorphic to C(2) (see Table I), the algebra of complex two-dimensional matrices. $\mathbb{R}_{3,0}$ is generated by 1 and σ_i, $i = 1,2,3$ subject to the conditions $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$, $\delta_{ij} = +1$ or 0 depending if $i = j$ or $i \neq j$.

The element $e_{30} = \frac{1}{2}(1+\sigma_3)$ is a primitive idempotent of $\mathbb{R}_{3,0}$. We have that $\alpha = \{e_{30}, \sigma_1 e_{30}\}$ is a spinorial basis for $I_{3,0} \equiv I_p = \mathbb{R}_{3,0}e_{30}$. We shall see that the elements of $I^+_p = \mathbb{R}^+_{3,0}e_{30}$ (Pauli a-spinors) are the representatives of Pauli c-spinors ((1) of §1) within the Pauli algebra. The reason is as follows:

In the above basis we have the following matrix representation(*) for

$$x,\ x^{\blacksquare},x^{\star},\ \tilde{x} \in \mathbb{R}_{3,0}$$

$$\mathbb{C}(2) \ni x = \begin{bmatrix} z^1 & z^2 \\ z^3 & z^4 \end{bmatrix};\quad x^{\blacksquare} = \begin{bmatrix} \bar{z}^4 & -\bar{z}^3 \\ -\bar{z}^2 & \bar{z}^1 \end{bmatrix};\quad x^{\star} = \begin{bmatrix} \bar{z}^1 & \bar{z}^3 \\ \bar{z}^2 & \bar{z}^4 \end{bmatrix};$$

$$\tilde{x} = \begin{bmatrix} z^4 & -z^2 \\ -z^3 & z^1 \end{bmatrix} \tag{13}$$

Defining $\hat{\beta}: I_p \times I_p \to \mathbb{C}$, $\hat{\beta}(\psi,\varphi) = \psi^{\star}\varphi$, for $\psi = \begin{bmatrix} \psi^1 & 0 \\ \psi^2 & 0 \end{bmatrix}$ and $\varphi = \begin{bmatrix} \varphi^1 & 0 \\ \varphi^2 & 0 \end{bmatrix}$

$$\beta(\psi,\varphi) = \bar{\psi}^1\varphi^1 + \bar{\psi}^2\varphi^2 \tag{14}$$

We define now the spinorial metric $\beta = \hat{\beta}|I_p^+$. We have for $\psi,\varphi \in I_p^+$, $\beta(\psi,\varphi) = [\beta(\psi,\varphi)]^-$ (hermitian product). Since $\alpha = \{e_{30}, \sigma_1 e_{30}\}$ is an orthonormal basis for I_p^+ we have the following representation of β in the α-basis

$$[\beta]_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{1}_2 \tag{15}$$

$$\beta(\psi,\varphi) = \beta(s\psi, s\varphi) \Leftrightarrow s^{\star}s = \mathbb{1}_2 \Leftrightarrow s \in U(2)$$

(*) If $x \in \mathbb{R}_{3,0}$ we use the same letter for representing $x \in \mathbb{C}(2)$. This causes no confusion.

Now if $x \in \mathbb{R}^+_{3,0} \simeq \mathbb{R}_{0,2} \simeq \mathbb{H}$ we have the following representation for x in the α-basis

$$x = \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \quad \text{and} \quad \tilde{x} = x^\star = \begin{bmatrix} \bar{z} & \bar{w} \\ -w & z \end{bmatrix}$$

Observe now that $N(x) = \tilde{x}\, x = \det x\ \mathbb{1}_2$ and we get $N(x) = 1 \Leftrightarrow \det N = 1$. So the element $s \in \mathbb{R}^+_{3,0}$ such that $\beta(s\psi, s\varphi) = \beta(\psi,\varphi)$, $\psi,\varphi \in I^+_p$ satisfy $\tilde{s}\, s = \mathbb{1}_2$ and $\det s = +1$, which means that $s \in \in SU(2)$ and $SU(2) \simeq \mathrm{Spin}^+(3,0)$, and our assertion that Pauli c-spinors are represented by the elements of $I^+_p = \mathbb{R}^+_{3,0}\, e_{30}$ is proved.

4.2. UNDOTTED AND DOTTED TWO COMPONENTS A-SPINORS AND THE GROUP SL(2,C).

We have that $\mathbb{R}_{3,0} \overset{f}{\simeq} \mathbb{R}^+_{1,3}$ where f is the linear extension of $f(\sigma_i) = e_i e_0$, $i \neq 0$, $i = 1,2,3$, $\sigma_i \in \mathbb{R}^{3,0}$ as in § 4.1 and e_μ, $\mu = 0,1,2,3$, is an orthonormal basis of $\mathbb{R}^{1,3}$.

Since $e_{30} = \frac{1}{2}(1 + \sigma_3)$ is a primitive idempotent of $\mathbb{R}_{3,0}$, $f(e_{30}) = \frac{1}{2}(1 + e_3 e_0)$ is a primitive idempotent of $\mathbb{R}^+_{1,3}$. We have that $I_u = \mathbb{R}^+_{1,3} f(e_{30})$ is a minimal left ideal of $\mathbb{R}^+_{1,3}$ with basis $\alpha = \{f(e_{30}), e_1 e_0 f(e_{30})\}$. Using the isomorphism

$$\begin{aligned} \rho :\ & \mathbb{R}^+_{1,3} \to \mathcal{L}_{\mathbb{C}}(I_u) \\ & u \mapsto \rho(u) : I_u \to I_u \\ & \qquad\qquad \psi \mapsto u\psi \end{aligned} \tag{17}$$

we have the following matrix representation for $u \in \mathbb{R}^+_{1,3}$ in the α-basis

$$\mathbb{C}(2) \ni u = \begin{bmatrix} z^1 & z^2 \\ z^3 & z^4 \end{bmatrix} \; ; \quad u^\star = \tilde{u} = \begin{bmatrix} z^4 & -z^2 \\ -z^3 & z^1 \end{bmatrix} \tag{18}$$

Defining

$$\beta: I_u \times I_u \to C, \quad \beta(\psi,\varphi) = e_1 e_0 \tilde{\psi} \varphi \tag{19}$$

we get

$$\beta(\psi,\varphi) = \psi^1 \varphi^2 - \psi^2 \varphi^1 \tag{20}$$

and the representation of s in the α-basis is

$$[\beta]_\alpha = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{21}$$

Then $\beta(u\psi, u\varphi) = \beta(\psi,\varphi) \Leftrightarrow \tilde{u}\, u = \mathbb{1}_2 \Leftrightarrow \det u = 1 \Leftrightarrow u \in SL(2,\mathbb{C}) \simeq$ $\simeq \mathrm{Spin}^+(1,3)$.

We conclude that the elements of I_u (undotted a-spinors) can be said to give a representation of undotted two-component c-spinors within the space algebra $\mathbb{R}_{1,3}$. The vector space I_u carries the $D^{(½,0)}$ representation of $SL(2,\mathbb{C})$.

Now remenbering that $\star$ is an automorphism in the Clifford algebra $\star: \mathbb{R}^+_{1,3} \to \mathbb{R}^+_{1,3}$ that preserves the $\mathrm{Spin}^+(1,3)$ group we have: If $u \in \mathrm{Spin}^+(1,3) \Rightarrow u^\star \in \mathrm{Spin}^+(1,3) \Rightarrow (u^\star)^{-1} \in \mathrm{Spin}^+(1,3)$

Consider now the minimal right ideal $I_d = (\mathbb{R}^+_{1,3} f(e_{30}))^\star$ and the isomorphism

$$\begin{aligned} \overset{\circ}{\rho}: \mathbb{R}^+_{1,3} &\to \mathcal{L}_C(I_d) \\ u &\to \overset{\circ}{\rho}(u): I_d \to I_d \\ &\qquad \overset{\circ}{\psi} \to \overset{\circ}{\psi}\,(u^\star)^{-1} \end{aligned} \tag{22}$$

We conclude that the elements of I_d (dotted a-spinors) can be said to give a representation of the dotted two component c-spinors ((ii) of §1) within the space-time algebra $\mathbb{R}_{1,3}$. The vector space I_d carries the $D^{(0,½)}$ representation of $SL(2,\mathbb{C})$.

4.3. REPRESENTATION OF DIRAC C-SPINORS WITHIN THE SPACE-TIME ALGEBRA $\mathbb{R}_{1,3}$

We have that $\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$ and the idempotent $f(e_{30})$ is also primitive in $\mathbb{R}_{1,3}$. This means that $I_D = \mathbb{R}_{1,3}f(e_{30})$ is a minimal left ideal of $\mathbb{R}_{1,3}$. It is a bi-dimensional quaternion ideal in $\mathbb{R}_{1,3}$.

We can consider I_D as a 4-dimensional complex vector space and in this way we get a complex representation of $\mathbb{R}_{1,3} \simeq \mathbb{R}^+_{4,1} \subset \mathbb{R}_{4,1} \simeq$ $\simeq \mathbb{C}(4)$

Calling $f(e_{30}) = e_{13} \equiv \bar{e}$ we have

$$I_D = \mathbb{R}_{1,3}\bar{e} = a_1\bar{e} + a_2e_0\bar{e} + a_3e_1\bar{e} + a_4e_2\bar{e} + a_5e_0e_1\bar{e} + a_6e_0e_2\bar{e} + a_7e_1e_2\bar{e} + a_8e_0e_1e_2\bar{e}\ ;\ a_i \in \mathbb{R},\ i = 1,2,\ldots 8 \qquad (23)$$

Observing that

$$\bar{e}\mathbb{R}_{1,3}\bar{e} \simeq [\bar{e}, e_1\bar{e}, e_2\bar{e}, e_1e_2\bar{e}] \simeq \mathbb{H}\ ;\ \mathbb{C} \simeq [\bar{e}, e_1e_2\bar{e}] \subset \bar{e}\mathbb{R}_{1,3}\bar{e}$$

we can rewrite eq(23) as

$$\begin{aligned} I_D = \mathbb{R}_{1,3}\bar{e} &= \bar{e}(a_1\bar{e} + a_7e_1e_2\bar{e}) + e_0\bar{e}(a_2\bar{e} + a_8e_1e_2\bar{e}) + \\ &+ e_1\bar{e}(a_3\bar{e} - a_4e_1e_2\bar{e}) + e_0e_1\bar{e}(a_5\bar{e} - a_6e_1e_2\bar{e}) \end{aligned} \qquad (24)$$

A complex basis for I_D is then $\alpha_D = \{e_0\bar{e}, e_1\bar{e}, \bar{e}, e_0e_1\bar{e}\}$.

Consider now the injection

$$\begin{aligned} \gamma: \mathbb{R}_{1,3} &\to \mathcal{L}_{\mathbb{C}}(I_d) \\ u &\to \gamma(u): I_D \to I_D \\ &\qquad\qquad \psi \to u\psi \end{aligned} \qquad (25)$$

We get the following representation for e_μ , $\mu = 0,1,2,3$ in the α_D-basis

$$\gamma(e_0) \equiv \gamma_0 = \begin{bmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{bmatrix} ; \quad \gamma(e_i) = \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} , \quad i = 1,2,3 \tag{26}$$

where σ_i are the Pauli matrices.

In this basis we have the following representation for $x \in \mathbb{R}_{1,3}$

$$\gamma(x) = \left[\begin{array}{cc|cc} x_1 & x_2 & x_5 & x_6 \\ x_3 & x_4 & x_7 & x_8 \\ \hline x_8 & -x_4 & \bar{x}_4 & -\bar{x}_3 \\ -x_6 & x_5 & -\bar{x}_2 & \bar{x}_1 \end{array}\right] \tag{27}$$

Considering the restriction $\gamma|_{\mathbb{R}^+_{1,3}}$ we get for $z \in \mathbb{R}^+_{1,3}$ the following representation in the α_D-basis

$$\gamma(z) = \left[\begin{array}{cc|cc} z_1 & z_2 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ z_3 & z_4 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \bar{z}_4 & -\bar{z}_3 \\ & & -\bar{z}_2 & \bar{z}_1 \end{array}\right] \tag{28}$$

Now, since $\mathbb{R}_{3,0} \overset{f}{\simeq} \mathbb{R}^+_{1,3}$ there exists an unique $y \in \mathbb{R}_{3,0} \simeq \mathbb{C}(2)$ such that $z = f(y)$ and we have

$$\gamma(f(y)) = \left|\begin{array}{c|c} y & 0 \\ \hline 0 & (y^*)^{-1} \end{array}\right| , \quad y \in \mathbb{C}(2) \tag{29}$$

and

$$\gamma o f: \mathbb{R}_{3,0} \to \mathcal{L}_{\mathbb{C}}(I_D)$$

$$y \to \left[\begin{array}{c|c} y & 0 \\ \hline 0 & (y^{\star})^{-1} \end{array}\right] \; ; \quad y \in \mathbb{R}_{3,0} \tag{30}$$

We see that the restriction $\gamma|\mathbb{R}^{+}_{1,3}$ gives a complex 4-dimensional representation of $Spin^{+}(1,3) \simeq SL(2,\mathbb{C})$ - namely the representation $D^{(½,0)} \oplus D^{(0,½)}$ of $SL(2,\mathbb{C})$.

We call the elements $\psi \in I_D$ space-time a-spinors. From the above discussion it is quite clear that space-time a-spinors represent in $\mathbb{R}_{1,3}$ the Dirac c-spinors introduced in (iii) of §1.

We also mimic the spinorial metric in $\mathbb{C}(4)$ [(iii) of §1] defining

$$\beta_D: I_D \times I_D \to C \quad , \quad \beta(\psi,\varphi) = b\,\tilde{\psi}\,\varphi \tag{31}$$

for an appropriate $b \in \mathbb{R}_{1,3}$.

4.4. REPRESENTATIONS OF DIRAC C-SPINORS WITHIN THE $\mathbb{R}_{4,1}$ ALGEBRA

From Table I we see that $\mathbb{R}_{4,1}$, $\mathbb{R}_{2,3}$ and $\mathbb{R}_{0,5}$ are isomorphic to the algebra $\mathbb{C}(4)$ which is the usual Dirac algebra of physicists. In order to identity the algebra that carries the physical interpretation associated with space-time $(\mathbb{R}^{1,3})$ we procced as follows. Let E_A, $A = 0,1,2,3,4$ be an orthonormal basis for $\mathbb{R}^{p,q}$ with $p + q = 5$. The volume element is $E_J = E_0 E_1 E_2 E_3 E_4$ and we get $E_J^2 = -1$ for $q = 1,3,5$. Now define

$$e_\mu = E_\mu E_4 \tag{32}$$

and impose that e_μ is an orthonormal basis for $\mathbb{R}^{1,3}$, ie

$$e_0^2 = -E_0^2 E_4^2 = +1 \ , \quad e_k^2 = -E_k^2 E_4^2 = -1 \ , \ k = 1,2,3 \qquad (33)$$

Eq.(33) is satisfied when $p = 4$, $q = 1$,ie , $E_4^2 = E_k^2 = -E_0^2 = 1$ and we conclude that the real Clifford algebra associated with space-time ($\mathbb{R}^{1,3}$) and isomorphic to $\mathbb{C}(4)$ is $\mathbb{R}_{4,1}$.

Eq.(32) shows that $\mathbb{R}_{1,3} \overset{g}{\simeq} \mathbb{R}^+_{4,1}$ where g is the linear extension of $g(e_\mu) = E_\mu E_4$, $\mu = 0,1,2,3$. We already saw in §4.2 and §4.3 that $f(e_{30})$ is a primitive idempotent of $\mathbb{R}_{1,3}$ and we have that $g(f(e_{30}))$ is a primitive idempotent of $\mathbb{R}^+_{4,1}$. Then $\bar{I}_D = \mathbb{R}^+_{4,1} g(f(e_{30}))$ is a minimal ideal of $\mathbb{R}^+_{4,1}$ which is a 4-dimensional vector-space over the complex field and its elements , the Dirac a-spinors, are representations in $\mathbb{R}_{4,1}$ of Dirac c-spinors.

5. THE DIRAC EQUATION AND OPERATOR SPINORS (O-SPINORS)

In Ref.(4) Hestenes searchs a representation, using the Clifford bundle[22] [over Minkowski space-time (M)] of the usual Dirac equation

$$i\hbar\gamma^\mu(\partial_\mu - qA_\mu(x))\psi_s(x) = m\,\psi_s(x) \qquad (34)$$

where $M \ni x \to \psi_s(x)$ is a section of the spinor bundle[10] (over M), q is the electric charge, A_μ, $\mu = 0,1,2,3$ are the components at x (in an orthonormal frame in T_xM) of the eletromagnetic potential, and γ^μ are the Dirac matrices satisfying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} = 2\,\mathrm{diag}(+1,-1,-1,-1)$. ψ_s is the standard Dirac c-spinors introduced in (iv) of §1 . In the standard representation the γ-matrices are

$$\gamma_0 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{bmatrix} \ ; \ \gamma_k = \begin{bmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{bmatrix} \ . \qquad (35)$$

Introducing for each $x \in M$ a spinor basis at the fiber over $x(\mathbb{C}(4))$ in the spinor bundle where

$$u_{s_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} ; \quad u_{s_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} ; \quad u_{s_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} ; \quad u_{s_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (36)$$

we can show that ψ_s has the following representation in $\mathbb{R}_{1,3}$

$$\mathbb{C}(4) \ni \psi_s \to \psi = \psi u_1 \ , \quad \psi \in \mathbb{R}^+_{1,3} \ ; \ u_{s_1} \to u_1 = \tfrac{1}{2}(1 + e_0); \ u_1^2 = u_1 \qquad (37)$$

where e_μ, $\mu = 0,1,2,3$, is an orthonormal basis of $\mathbb{R}^+_{1,3}$ (*) and u_1 is the representation of u_{s_1} in $\mathbb{R}_{1,3}$. Interpreting the γ^μ as representations of the vectors $e^\mu = g^{\mu\nu} e_\nu$, we get the following representation of eq(34) in the Clifford bundle over M

$$(\hbar \,\Box \psi e_2 e_1 - qA\psi) e_0 u_1 = m\psi u_1 \ ; \quad \Box = e^\mu \partial_\mu \ ; \quad A = e^\mu A_\mu \qquad (38)$$

We observe that in eq(38) the $i = \sqrt{-1}$ has been eliminated! Now, although u_1 has no inverse, the coefficients of u_1 can be equated and we have

$$\hbar \,\Box \psi e_2 e_1 - qA\psi = m\psi e_0 \qquad (39)$$

Finally considering

$$\phi = \psi u \ ; \quad u = \tfrac{1}{4}(1 + e_0)(1 + e_3 e_0) \qquad (40)$$

eq(39) can be written as

$$\hbar \Box \phi e_5 - qA\phi = m\phi \ ; \quad e_5 = e_0 e_1 e_2 e_3 \qquad (41)$$

(*) Observe that the Minkowski space M is the affine space constructed with $\mathbb{R}^{1,3}$

which appears originaly in Ref.(25). It is quite clear that ϕ is an element of a minimal ideal in $\mathbb{R}_{1,3}$ since u is a primitive idempotent of $\mathbb{R}_{1,3}$.

In resume, the standard Dirac c-spinors are represented in $\mathbb{R}_{1,3}$ by the elements of the minimal ideal $I_{1D} = \mathbb{R}^{+}_{1,3}u_1$ of $\mathbb{R}_{1,3}$ and a simple Dirac equation can be written using the elements of the minimal ideal $I_{2D} = \mathbb{R}^{+}_{1,3}u$ of $\mathbb{R}_{1,3}$.

Obviously I_{1D} and I_{2D} are isomorphic. We can show without dificulty that I_{1D}(or I_{2D}) is the carrier space (in the group theoretical sense) of the representation $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ of $SL(2,\mathbb{C})$.

In Refs.(4,5,6,7) Hestenes gives a new definition of the Dirac-spinor which we shall call Dirac operator spinor (O-spinor)

Dirac Operator-Spinor: A Dirac σ-spinor ψ is an element of $\mathbb{R}^{+}_{1,3}$, $\psi^{\star}\psi = 1$, such that $\psi\colon \mathbb{R}^{1,3} \ni \gamma^{\mu} \to V^{\mu} = \psi\gamma^{\mu}\psi^{\star} \in \mathbb{R}^{1,3}$

The consistence of Hestenes definition follows from the fact that $\psi \in \mathbb{R}^{+}_{1,3}$ produces "ideal representations" (ie, a-spinors) by *operating* on idempotents and it produces observables (in the Dirac theory) by operating on vectors, ie $\psi\gamma^{\mu}\psi^{\star}$ is a vector. This is quite obvious since from our previous discussion in §4 we know that $\psi \in$ $\in \mathrm{Spin}^{+}(1,3)$.

Also we must recall at this point our discussion of § 3.4. Indeed, $\psi \in \mathbb{R}^{+}_{1,3}$ is a sum of scalar, pseudo-scalar and bivector parts and from the point of view of the theory of the linear representations of the Lorentz group, the space of the ψ's is the carrier space of the representation $D^{(0,0)} \oplus D^{(\frac{1}{2},0)} \otimes D^{(0,\frac{1}{2})} \oplus D^{(0,0)}$ of $SL(2,\mathbb{C})$ [11]. Nevertheless from the point of view of the representation theory of Clifford algebras $\psi \in \mathbb{R}^{+}_{1,3}$ trasforms as $\psi \to s\psi s^{-1}$, $s \in \mathrm{Spin}^{+}(1,3) \equiv \mathbb{R}^{+}_{1,3}$ $\tilde{s}s = 1$ and $\mathrm{Spin}^{+}(1,3)$ is the $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ representation of $SL(2,\mathbb{C})$ in $\mathbb{R}_{1,3}$. (§4.3)

In Ref. (6) Hestenes also shows that there exists operators representations of Pauli c-spinors. The operator definition of spinors has been generalized to arbitrary Clifford algebras by Dimakis (26)

6. CONCLUSIONS

Hestenes[7] said about the theory of spinors: "I have not met anyone who was not dissatisfied with his first reading on the subject!"

Well, the reasons for such statement are in our view due to two main facts

(A) the usual presentation of c-spinors such as introduced in (i), (ii), (iii), (iv) of §1 does not emphasize the geometrical meaning of these objects

(B) There are not clear connection between the abstract concepts of c-spinors of §1 and the more abstract concepts of a-spinors or o-spinors as elements of particular Cifford algebras.

As to (A) we think that the situation has been partially clarified with the presentation by Hestenes of the geometrical meaning of Pauli-spinors[6] and of Dirac spinors [4,5] and also by Penrose and Rindler[27] of the geometrical meaning of the undotted and dotted two component spinors.

As to (B) we think that our paper shows in a clear way how to obtain relations between all the different representations of spinors.

It is important also to realize that the detailed relations presented in §4 between a-spinors and appropriated sums of multivectors show that the usual claim[27] that spinors are more fundamental then tensors is non sequitur.

We now briefly comment on some distintive features of the different representations of Dirac spinors. A more detailed discussion will be presented elsewhere.

It is well known that due to topological obstructions[27] the c-spinor bundle over a non-flat manifold does not exists in general. Only Lorentzian manifolds where the second Stiefel-Whitney class vanish possess a c-spinor bundle structure.

Now, the global generalization of the Dirac algebraic spinor to a Dirac algebraic spinor field depends on the existence on the Clifford bundle over space-time (M,g,D)* of an idempotent e of minimal global rank where the global rank of e is defined by

$$\text{rank } e = \max_{x\in M} (\text{rank } e_x)$$

Indeed, we say that an elementary a-spin structure exists(22) if the rank of e is locally minimal for almost all $x \in M$. In general we say that a *generalized a-spin* structure is given if and only if a non zero idempotent e is given. In particular $e = 1$ is the trivial spin structure which always exists for any given manifold. The existence of an elementary a-spin structure imposes global topological conditions on the manifold M.

Nevertheless, it is easy to see that any Lorentzian manifold of signature (+1,-1,-1,-1) admits an elementary spin structure. Indeed, in such a manifold there always exists a time-like vector field $e_0(g_x(e_{0x}) = 1)$ and then $e = (1 + e_0)/2$ has global minimal rank 8 which is the local minimal rank.

An e-algebraic spinor field corresponding to a generalized a-spin structure e is defined as a cross section ψ of the Clifford bundle such that $\psi e = \psi$

Now, it is important to emphasize that although an elementary a-spinor bundle can be defined for all Lorentzian manifolds, the Dirac equation cannot in general be written using an elementary a-spinor field in a general non-flat Lorentzian manifold. This time we have

(*) Space-time is a triple (M,g,D) where M is a paracompact 4-dimensional manifold, g is a metric with signature (+,-,-,-) and and D is the Levi-Civita connection of g in M.

"metrical obstructions" [27]. This is true for the usual Kähler-Dirac equation as well as for the generalization of eq(38) for non-flat Lorentzian manifolds. When this is the case we need to use a generalized e-algebraic spinor field. Here is where the Dirac o-spinors of Hestenes become really important.

At last we must comment that as $\mathbb{R}_{1,3}$ has only two primitive pairwise orthogonal idempotents and we can have in the Dirac theory using the $\mathbb{R}_{1,3}$ formalism only two projection operators using the elements of the algebra. Remember that $\mathbb{C}(4)$ has four primitive pairwise orthogonal idempotents (and this is also the case for $\mathbb{R}_{3,1}$-the Majorana algebra). To overcome this difficulty Hestenes [7] introduced operators in the Dirac theory, that belongs to the dual space of $\mathbb{R}_{1,3}$ (here considered as a vector space over $\mathbb{R}$). To end, we resume that the present paper gives the relation between c-spinors and a-spinors (and o-spinors). Our method of identification of c-spinors with a-spinors is based on the concept of spinorial metric and on the observation that for $p + q \leq 5$, $Spin^+(p,q)$ is the invariance group of the spinorial metrics.

Among the important results obtained we emphasize that here for the first time there appears the representation of undotted and dotted two-component and Dirac c-spinors in $\mathbb{R}_{1,3}$. In particular we gave a rigorous proof that the space of Dirac a-spinors, ie, the elements of $I_D = \mathbb{R}_{1,3} f(e_{30})$ carry the representation $D^{(\frac{1}{2},0)} \oplus$ $\oplus D^{(0,\frac{1}{2})}$ of $SL(2,\mathbb{C})$ and thus can be said to give a representation of Dirac c-spinors as introduced in (iii) of §1

Also we clear in what sense the Dirac o-spinors can be said to be a representation in $\mathbb{R}_{1,3}$ of Dirac c-spinors.

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