VECTOR FIELDS NEAR THE BOUNDARY OF A 3–MANIFOLD

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ABSTRACT. The simplest patterns of qualitative changes – bifurcations – located around a compact two dimensional submanifold, that occur on smooth one-parameter families of vector fields on a three dimensional manifold, are studied here.

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1. INTRODUCTION

Let M be a compact C^∞ 3-dimensional manifold. Denote by X^r the space of C^r vector fields tangent to M, endowed with the C^r topology where is big-enough and finite. Let S be a compact 2-dimensional submanifold of M which in general will be the boundary of a compact region in M.

1.1 DEFINITION. A vector field X in X^r is said to be S-stable if for a neighborhood $T(S)$ of S there are neighborhoods V of X in X^r and V of S in M such that V of S in M such that V is contained in $T(S)$ and for every Y in V there is a homeomorphism $h(Y)$ mapping V onto itself, preserving S, sending orbits of $X|_V$ onto orbits of $Y|_V$. A homeomorphism such as $h(Y)$ will be refered to as an S-equivalence between X and Y.

The first theorem of this paper characterizes the set $\Sigma_0 = \Sigma_0(S)$ of S-stable vector fields in X^r.

THEOREM 1. i) X belongs to Σ_0 if and only if:

a) $X(p) \neq 0$ for every p in S;

b) for every local implicit definition, f, of S at p, either b_1) $Xf(p) \neq 0$, b_2) $Xf(p) = 0$ and $X^2f(p) \neq 0$ or b_3) $Xf(p) = X^2f(p) = 0$ and $\{df(p), dXf(p), dX^2f(p)\}$ are linearly independent.

ii) Σ_0 is open and dense in X^r.

The points in S at which $Xf \neq 0$ (resp. $Xf = 0$) are called S-regular (resp. S-singular) points of X. The points of S where b_2 is satisfied are called *fold singularities*; they form smooth curves in S, along which X has quadratic contact with S. The set where b_3 is satisfied is the union of isolated points of cubic contact between X and S, located at the extremes of the curves of fold singuralarities, called *cusp singularities*.

In fact, by projecting S along the orbits of X into a surface N transverse to the orbit through a singularity, we get a Witney singularity of fold (case b_2) or cuspidal types (case b_3) [W].

Call $\mathfrak{X}_1^r = \mathfrak{X}_1^r(S)$ the complement of $\Sigma_0(S)$ in $\mathfrak{X}^r$, which is the union of the set of vector fields $\mathfrak{X}_1^r(a)$ and $\mathfrak{X}_1^r(b)$ which violate respectively conditions a) and b) in Theorem 1. We can assume that vector fields in $\mathfrak{X}_1^r(b)$ have no critical point on S and at least one point p at which either c_1) $X^2f(p) = 0$ and $X^3f(p) = 0$ or c_2) $X^2f(p) = 0$, $X^3f(p) \neq 0$ but $df(p), d(Xf)(p)$, $d(X^2f)(p)$ are linearly dependent.

1.2 DEFINITION. An S-hyperbolic critical point of X is a critical point p in S such that i) it is a hyperbolic critical point of X (the eigenvalues of $DX(p)$ have non vanishing real parts); ii) the eigenvalues of $DX(p)$ are pairwise distinct and the correspondent eigenspaces are transversal to T_pS and iii) each pair of non complex conjugate eigenvalues of $DX(p)$ have distinct real parts. We may also refer to a S-hyperbolic critical point as a singularity of type Σ.

Call $\Sigma_1(a)$ to be the set of vector fields X in $\mathfrak{X}_1^r(a)$ such that X has a unique critical point p in S which is S-hyperbolic, while all the other points in S, conditions b) of Theorem 1 are satisfied.

We may also refer to a S-hyperbolic critical point as a singularity of type $\Sigma_1(a)$.

1.3 DEFINITION. Call $\Sigma_1(b)$ the set of vector fields X in $\mathfrak{X}_1^r(b)$ which have a unique point p in S such that $X(p) \neq 0$, $Xf(p) = 0$, $X^2f(p) = 0$ and one of the following conditions hold:

Q_1: $X^3f(p) \neq 0$, $\operatorname{rank}\{df(p), d(Xf)(p), d(X^2f)(p)\} = 2$ and the function $Xf|_S$ has at p a non-degenerate (Morse) critical point.

Q_2: $X^3f(p) = 0$, $X^4f(p) \neq 0$ and p is a regular point of $Xf|_S$.

A point $p \in S$ which satisfies Definition 1.3 is called singularity of X of type $\Sigma_1(b)$.

1.4 DEFINITION. A vector field X in X_1^r is said to be S-stable with respect to X_1^r if for some neighborhood $T(S)$ of S, there are neighborhoods $\mathcal{V}$ of X in X^r and V of S in M, contained in $T(S)$, such that for every Y in $\mathcal{V} \cap X_1^r$ there is an S-equivalence between $X|_V$ and $Y|_V$.

We have the following result:

THEOREM 2.

i) X in X_1^r is S-stable relative to X_1^r if and only if it belongs to $\Sigma_1 = \Sigma_1(a) \cup \Sigma_1(b)$.

ii) Σ_1 is a submanifold of codimension one of X^r.

iii) Σ_1 is open in X_1, endowed with the topology induced from X^r.

iv) For a residual set of smooth curves $\gamma: R \to X^r$, γ meets transversally Σ_1 and $\gamma^{-1}(X_2) = \phi$.

The background on the study of stability and bifurcation properties of vector fields on manifolds with boundary can be found in [A, P-P, Pe, S_2, T, V].

This paper is organized as follows. In sections 2 and 3 will be studied the singularities of vector fields, of critical and tangential types. Section 4 is devoted to the study of local structural stability of families of vector fields. In section 5 some considerations concerning Transient Vector Fields. are made. These sections prepare the way to the proof of Theorems 1 and 2 given in section 6.

§2. SINGULARITIES OF TANGENCIAL TYPE.

Let p be a singularity of $X \in \mathcal{X}$ of type $\Sigma_1(b)$.

Consider coordinates $x = (x_1, x_2, x_3)$ around $p \in S$ such that $X = \frac{\partial}{\partial x_1}$ and of course we may solve $f(x_1, x_2, x_3) = 0$ by $x_3 = g(x_1, x_2)$ with $g(0,0) = 0$.

Fix $N = \{x_1 = 0\}$ as being the section transverse X at p.

Define

$$\sigma_X : S,p \to N,p \qquad \text{by}$$

$$\sigma_X(x_1, x_2, g(x_1,x_2)) = (0, x_2, g(x_1,x_2))$$

Moreover $Xf(x) = \frac{\partial g}{\partial x_1}(x) = g_1(x)$, $X^2 f(x) = g_{11}(x)$, $X^3 f(x) = g_{111}(x)$ and $X^4 f(x) = g_{1111}(x)$.

The mapping σ_X is of the same class of differentiability as X and it is called the projection of S along the orbits of X onto a transverse surface N. See more details in $[S_1]$.

All results obtained in the sequel of this section follow directly from the Theory of Singularity of Mappings (the work of Chincaro [C] plays a fundamental role in the development of this work) by using standard known techniques.

2.1. LEMMA: The point $p \in S$ is a singularity of X of type $\Sigma'(b)$ if and only if σ_X is C^r equivalent (in the usual sense of the theory of singularity of mappings) to one of the following mappings:

$$Q_{11}: \sigma_1(x,y) = (x^2 + xy^2, y) \qquad \text{(Lips)}$$

$$Q_{12}: \sigma_2(x,y) = (x^3 - xy^2, y) \qquad \text{(Bec to Bec)}$$

$$Q_2 : \sigma_3(x,y) = (x^4 + xy, y) \qquad \text{(Dove's Tail)}$$

2.2. LEMMA: There exist neighborhoods B of X in X^r, V of p in M and a C^{r-3}-function $G : B \to \mathbb{R}$ such that:

a) $G(Y) = 0$ if and only if Y has a unique point $p_Y \in L_Y \cap V$ which is a singularity of Y of the same type as p; if $G(Y) \neq 0$ then L_Y contains only generic singularities of Y;

b) $dG(X) \neq 0$.

2.3. PROPOSITION: There exist neighborhoods B of X in X^r, V of p in M such that:

i) $X|_V$ is C^r-equivalent to $Y|_V$ if and only if $G(Y) = 0$;

ii) If $Y_1, Y_2 \in B$ and $(G(Y_1)G(Y_2)) > 0$ then $Y_1|_V$ is C^r-equivalent to $Y_2|_V$.

2.4. REMARK: In the class C^∞ the equivalence obtained follow directly from the Chincaro's work [C] and it is clear that they have to respect the stratification as given in §2.

2.5. NORMAL FORMS FOR A SINGULARITY OF TYPE $\Sigma_1(b)$: For some of simplicity we give our normal forms in terms of a "straight" vector field and a "twisted" boundary.

$$Q_1 : X(x_1,x_2,x_3) = \frac{\partial}{\partial x_1} \quad \text{and} \quad f(x_1,x_2,x_3) = x_3 - (x_1^3 + x_1x_2^2)$$

$$Q_2 : X(x_1,x_2,x_3) = \frac{\partial}{\partial x_1} \quad \text{and} \quad f(x_1,x_2,x_3) = x_3 - (x_1^3 - x_1x_2^2)$$

$$Q_3 : X(x_1,x_2,x_3) = \frac{\partial}{\partial x_1} \quad \text{and} \quad f(x_1,x_2,x_3) = x_3 - (x_1^4 + x_1x_2).$$

2.6. NORMAL FORMS FOR THE UNFOLDING OF A SINGULARITY OF TYPE $\Sigma_1(b)$:

i: $X(x_1,x_2,x_3) = \frac{\partial}{\partial x_1}$ and $f_\lambda(x_1,x_2,x_3) = x_3 - (x_1 \pm x_1x_2 + \lambda x_1)$

ii) $X(x_1,x_2,x_3) = \frac{\partial}{\partial x_1}$ and $f_\lambda(x_1,x_2,x_3) = x_3 - (x_1^4 + x_1x_2 + \lambda x_1^2)$.

§3. S-HYPERBOLIC CRITICAL POINTS

Let p in S be an S-hyperbolic critical point of X. In what follows we will study the local behavior of trajectories near such a point and their relation with S.

We choose coordinates $x = (x_1, x_2, x_3)$ around p such that $f(x) = x_1$, $X = (X^1, X^2, X^3)$, $X^1(x) = x_2$. Then we have: i) $Xf(x) = x_2$, $X^2f(x) = X^2(x)$ and $L_X = \{x_1 = x_2 = 0\}$; ii) If $\alpha \neq 0$ then the order of tangency between the trajectory of X through $q = (0,0,\alpha)$ and L_X (at q) is two (this means that q is a fold singularity of X); $\frac{\partial X^2}{\partial x_3}(0) \neq 0$, provided that the eigenspaces of DX(p) are transverse to S at p.

3.1. LEMMA: If $q \in L_X - \{p\}$, then in a neighborhood of p, $\varphi_t(q)$ does not meet L_X for $t \neq 0$.

PROOF: We have to consider the following cases:

a) All eigenvalues of DX(p) are real.

In this case there is, at least, a pair of invariant two dimensional manifolds of X in general position which are transverse to L_X at p. Moreover all trajectories through any point of $L_X - \{p\}$ tend asymptotically to directions transverse to L_X. These facts permit us to conclude the proof in this case.

b) The eigenvalues of dX(p) are $\lambda \in \mathbb{R}$ and $\lambda' = a \pm ib$ with $a \neq 0$ and $b \neq 0$.

Assume first that the vector field is linear. We may choose canonical coordinates $y = (y_1, y_2, y_3)$ around p such that the orbits of X are given by $\varphi_t(y) = (u(t), z(t))$ where

$$u(t) = e^{at}\begin{pmatrix}\cos bt & \sin bt\\ -\sin bt & \cos bt\end{pmatrix}u_o \qquad \text{and}$$

$$z(t) = e^{\lambda t}z_o \quad \text{with} \quad u_o = (y_1, y_2) \text{ and } z_o = y_3.$$

Assume $z_o \neq 0$.

So each trajectory of the vector field lies in the set π given by the equation

$$|u| = K|z|^{\alpha} \quad \text{with} \quad K = |u_o|^{\lambda}|z_o|^{-a} \quad \text{and} \quad \alpha = a/\lambda.$$

Observe that:

i) if $\alpha > 0$ then the trajectories of X lie in branches of "hyperbole" (see Figure 4.1);

ii) if $0 < \alpha < 1$ then the trajectories tend (for $t \to -\infty$) to p, asymptotically tangent to the (y_1, y_2)-plane (Figure 4.2);

iii) If $\alpha > 1$ then the trajectories tend to p asymptotically tangent to the y_3-axis (Figure 3.3).

iv) Moreover L_X and π are in general position. We check this by proving that L_X is projected regularly on the y_3-axis and observing that the points of $L_X - \{p\}$ are fold singularities of X. Next we are going to prove the first assertion.

Let $G : M,p \to \mathbb{R}^2,0$ given by $G = (f, Xf)$ with $S = f^{-1}(0)$. So $L_X = G^{-1}(0)$ and in the above coordinates we have

$$\frac{\partial G}{\partial y_1 \partial y_2}(0) = \begin{pmatrix} \frac{\partial f}{\partial y_1}(0) & \frac{\partial f}{\partial y_2}(0) \\ a\frac{\partial f}{\partial y_1} - b\frac{\partial f}{\partial y_2}(0) & b\frac{\partial f}{\partial y_1}(0) + a\frac{\partial f}{\partial y_2}(0) \end{pmatrix}$$

Observe that $\mathrm{Det}(\frac{\partial G}{\partial y_1 \partial y_2}(0))$ is no zero provided that the eigenspaces of dX_p are transverse to S at p. This implies that L_X projects regularly on the y_3-axis.

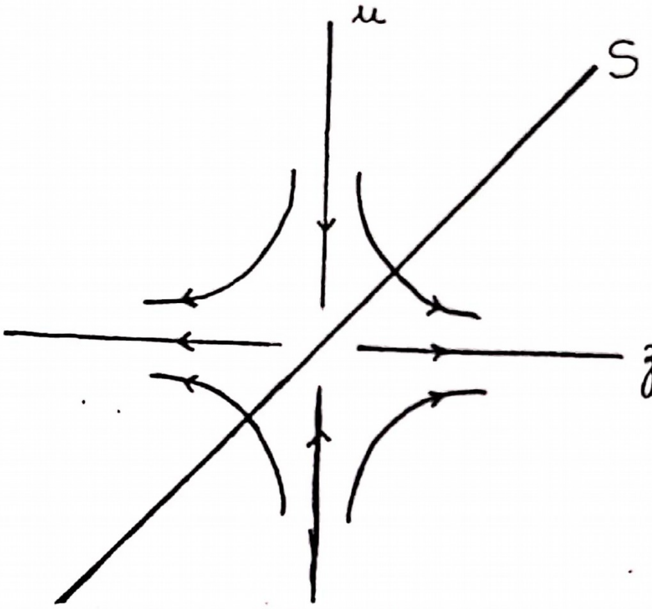

Figure 3.1. $\alpha < 0$

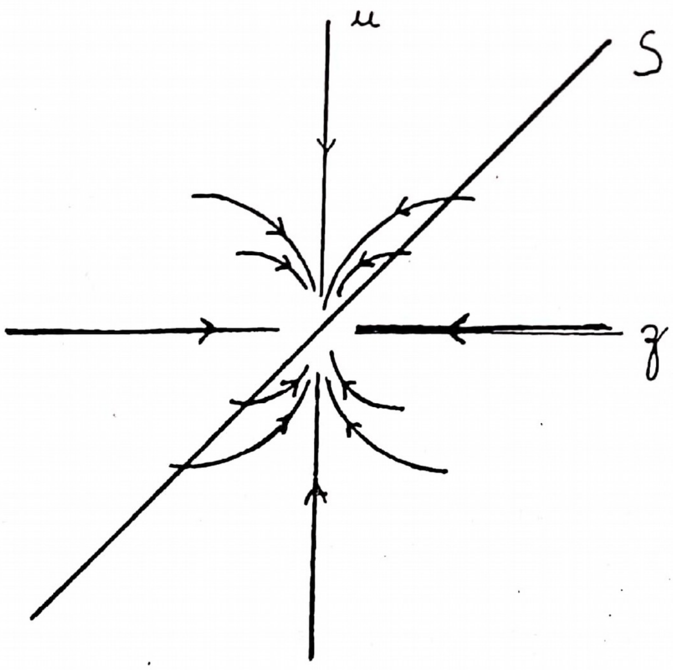

Figure 3.2. $0 < \alpha < 1$

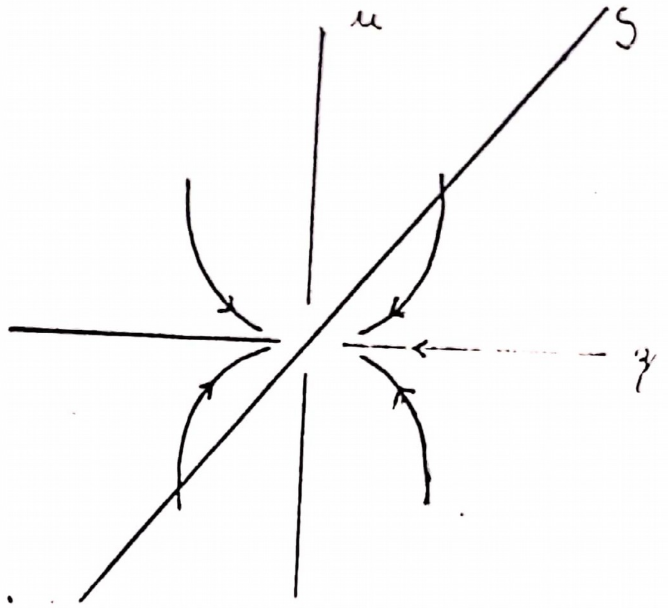

Figure 3.2. $\alpha > 1$.

The conclusion of the lemma in this particular case is now immediate.

We are going to prove that this situation persists in the general nonlinear case.

We can introduce a C^1 change of coordinates in a full neighborhood of p in such a way that X is represented by the equations:

$$(3.2.)\qquad \begin{aligned} \dot{u} &= (a + ib)u + uh_1(u,z) \\ \dot{z} &= \lambda z + zh_2(u,z) \end{aligned} \qquad \text{where } u \in \mathbb{C} \text{ and such that}$$

$$\lim_{(u,z)\to(0,0)} h_j(u,z) = 0 , \quad j = 1,2.$$

Assume for instance that $a < \lambda < 0$ (the other cases are treated in a similar way).

The solutions of (3.2.1) through (u_o, z_o) are given by

$$u(t) = u_o \exp[(a + ib)t + \int_0^t h_1(u(s), z(s))ds]$$

$$z(t) = z_o \exp[\lambda t + \int_0^t h_2(u(s), z(s))ds].$$

So, for $z_o = 0$,

$$\zeta(t) = \left|\frac{u(t)}{z(t)}\right| = \left|\frac{u_o}{z_o}\right| (\exp(a - \lambda)t)\exp(\int_0^t h_1 - h_2)ds)$$

tends to 0, when $t \to \infty$.

Now similar comments, as in the linear case, can be applied to finish the proof. □

3.3. REMARK: a) A proof of above lemma can also be obtained by performing the spherical blowing up of the critical point; b) using the above notation, we observe that if $\alpha = 1$ then there exists a trajectory tending to p (in positive or negative time) tangent to S.

3.4. LEMMA: There exist neighborhoods V of p in M, B of X in $\mathcal{X}^r$ such that: i) each $Y \in B$ has one unique hyperbolic critical point $P_Y \in V$. Moreover, the correspondence $Y \to P_Y$ is C^r; ii) there exists a C^r function $G : B \to \mathbb{R}$ such that $G(Y) = 0$ if and only if $P_Y \in S$. Moreover $DG(X) \neq 0$.

PROOF: Part i) is immediate.

Let N be a one dimensional section transverse to S at p and $\rho : M,p \to N,p$ be a projection "paralell" to S. The required function is defined by $G(Y) = \rho(P_Y)$. □

3.5. FUNDAMENTAL LEMMA: There exist neighborhoods V of p in M, B of X in $\mathcal{X}^r$ and a C^{r-2} function $\tau : B \to V$ such that $\tau(X) = p$

and $f(\tau(Y)) = 0$. Moreover: i) $Y^2f(q) = 0$ if and only if $q=\tau(Y)$; ii) if $P_Y \notin S$ then df, $d(Yf)$, dY^2f, at $\tau(Y)$, are linearly independent (this means that $\tau(Y)$ is a cusp stable singularity of Y); iii) if $P_Y \in S$ then $p_Y = \tau(Y)$ and it is a S-hyperbolic critical point of Y.

PROOF: Call $C(I,S) = \{h : I = (-\varepsilon,\varepsilon) \to S$ of class $C^{r-1}\}$.

Let $h_X \in C(I,S)$ be a C^{r-1} parametrization of L_X with $h_X(0) = p$. We can pick a neighborhood B of X in $\mathcal{X}^r$ and a C^{r-1} - mapping $h : B,X \to C(I,S),h_X$ such that $h(Y) = h_Y$ is a C^{r-1} parametrization of L_Y.

We have $Yf(h_Y(\alpha)) = 0$ for every $\alpha \in I$.

Define now a C^{r-2} mapping $F : B\times I \to \mathbb{R}$ by

$$F(Y,\alpha) = Y^2f(h_Y(\alpha)).$$

It satisfies $F(X,0) = X^2f(p) = 0$ and $\frac{\partial F}{\alpha\alpha}(X,0) = \frac{d}{d\alpha}[X^2f(h_X(\alpha))]_{\alpha=0} \neq 0$; we applied here the assertions contained in Remark 4.1.

So we can find a C^{r-2} mapping $\alpha = \alpha(Y)$ which is a solution of $F(Y,\alpha) = 0$.

The required mapping is defined by $\tau(Y) = h_Y(\tau(Y))$.

Part i) follows directly from the definition of τ.

We mention that in the coordinates given at the begining of this section we have

$$\text{Det}\,[df(p),\ d(Xf)(p),\ d(X^2f)(p)] = \frac{\partial X^2}{\partial x_3}(0) \neq 0.$$

The conclusion follows by continuity. $\square$

4.6. PROPOSITION: There exist neighborhoods V of p in M and B of X in $\mathfrak{X}^r$ such that: i) X,p is C^0 S-equivalent to $Y,p_Y \in B \times V$ if and only if $G(Y) = 0$; ii) if $Y_1,Y_2 \in B$ and $(G(Y_1).G(Y_2)) > 0$ then $Y_1|_V$ is C^0 S-equivalent to $Y_2|_V$ where G is defined in Lemma 3.4.

In what follow we define a stratification of a neighborhood of a critical point $p \in S$ which is essential for the proof of this proposition.

3.7. DISTINGUISHED SETS.

Following Lemma 3.5 let $Y \in B$.

First of all, we shall distinguish some subsets in V. We have to elaborate the following list:

1) S, L_Y, $I_Y = \bigcup_{q \in L_Y} (\varphi_t(q))$, $I_Y \cap S$

2) $\tau(Y)$ and p_Y must be distinguished as well as the trajectories of Y passing through them.

Consider the following possibilities:

3) p_Y is a node; this means that the respective eigenvalues λ_i are real; $i = 1,2,3$ and $\lambda_1 < \lambda_2 < \lambda_3 < 0$ (resp. $0 < \lambda_3 < \lambda_2 < \lambda_1$). We list:

3a) W_1^s : the invariant one dimensional manifold of Y tangent to T_1 (resp. W_1^u).

3b) W_{12}^s: the invariant two dimensional manifold of Y tangent to the linear space generated by T_1 and T_2 (resp. W_{12}^u).

4) p_Y is a nodal-focus: this means that λ_1 is real, $\lambda_2 = a \pm ib$ with $\lambda_1 < a < 0$ (resp. $0 < a < \lambda_1$). We list

W_1^s: the invariant one dimensional manifold tangent to T_1 (resp. W_1^u).

5) p_Y is a focal node: this means that λ_1 is real, $\lambda_2 = a \pm ib$ with $a < \lambda_1 < 0$ (resp. $0 < \lambda_1 < 2$). We list

W_2^s: the invariant two dimensional manifold tangent to T_2 (resp. W_2^u).

6) p_Y is a nodal-saddle: this means that λ_i are real, $i=1, 2,3$ and $\lambda_3 < 0 < \lambda_2 < \lambda_1$ (resp. $\lambda_1 < \lambda_2 < 0 < \lambda_3$). Using the above notations the following sets are distinguished: W_3^s, W_{12}^u and W_1^u (resp. W_1^s, W_{12}^s and W_3^u).

7) p_Y is a focal saddle: this means that λ_1 is real and $\lambda_2 = a \pm ib$ with $\lambda_1 < 0 < a$ (resp. $a < 0 < \lambda_1$). We list W_1^s and W_2^u (resp. W_1^u and W_2^s).

8) The boundary of V.

9) We include in our list the intersections between each two distinguished sets listed above.

3.8. REMARK: If $\tau(Y) \neq p_Y$ then: a) the trajectory of Y passing through $\tau(Y)$ never meets the distinguished sets given in 3, 4, 5, 6 and 7 (see above). This can be seen immediately by considering the restriction of X to some suitable two dimensional invariant set and using the results of [T]; b) in a similar may we can check that no trajectory of Y contained in I_Y is a distinguished one dimensional invariant manifold of Y listed in 3, 4, 5, 6 and 7 of (3.7).

4.9. ESTRATIFICATION OF A NEIGHBORHOOD OF p_Y.

Denote by $E_o(Y)$ the union of all 0-dimensional distinguished sets defined above. If $p_Y = \tau_Y$ we observe that $E_o \subset S$. If $p_Y \neq \tau_Y$ then $p_Y \in E_o(Y)$ and $p_Y \notin S$. In this way we define $E_1(Y)$, $E_2(Y)$ and $E_3(Y) = V$. We have to consider the following cases:

a) p_Y is not a saddle. In this case we may consider transverse to ∂V.

b) p_Y is a saddle. In this case there is a one dimensional submanifold of ∂V which is the set of external tangencies between Y and ∂V. This submanifold must be included in the list of distinguished sets and of course it is far away from L_Y.

We see that $E_o \subset E_1 \subset E_2 \subset E_3$ define a stratification on V in such a way that the Whitney's Conditions are naturally satisfied [Th].

Let us now indicate how Proposition 4.6 can be obtained from the above stratification.

3.10. PROOF OF PROPOSITION 3.6.

Let $p \in S$ be a singularity of X of type $\Sigma_1(a)$.

This proof will be done in a geometrical way and we are going to use results and techniques contained in [S.1] and [T] without explicit references to them.

PROOF OF PART i). To prove the sufficient condition we proceed as follows: a) we have to study each of the cases listed in 3, 4, 5, 6 and 7 of (4.7); b) we analyse the behavior of $I_X \cap \partial V$ with respect to the intersection between ∂V and the distinguished invariant manifolds of X; c) because of the definition of a singularity of type $\Sigma_1(a)$, this behavior persists for small perturbations of X in X_1^r ; d) the equivalence is first defined on ∂V and then extended to the full neighborhodd as a stratified mapping. From Lemma 3.2 follows that any trajectory passing through a point $q \in I_X \cap \partial V$ encounters L_X just once.

Consider the following cases:

I) The eigenvalues of $DX(p)$ are real with $\lambda_1 < \lambda_2 < \lambda_3 < 0$ (Node).

Assuming the above notations, call $V_1 = W_1^s \cap \partial V = \{q_1\} \cup \{q_2\}$ $V_2 = W_{12}^s \cap \partial V$ and $U = I_X \cap \partial V = U_1 \cup U_2$ (see Figure 4.4).

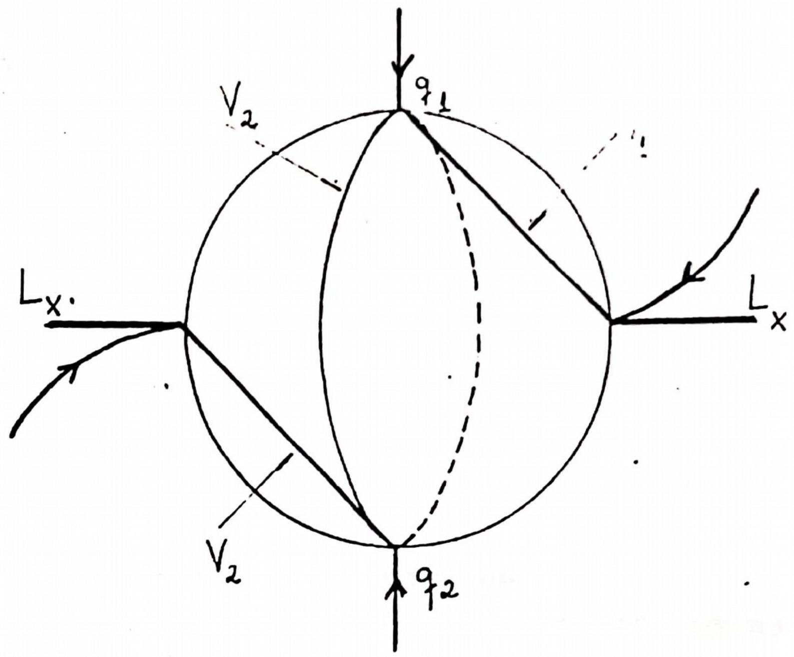

Figure 3.4, The boundary of neighborhood of a Node

We see that, both V_1 and U have two connected components composed by isolated points and semi-intervals respectively. Moreover, $C\ell(U)$ and V_2 meet transversally.

For $Y \in B$ we have the analogous objects $\tilde{V}_1 = \{\tilde{q}_1\} \cup \{\tilde{q}_2\}$, V_2 and $\tilde{U} = \tilde{U}_1 \cup \tilde{U}_2$.

We define the equivalence $h : \partial V \to \partial V$ by imposing $q_i \to \tilde{q}_i$, $U_i \to \tilde{U}_i$, $V_2 \to \tilde{V}_2$ and then we extend some how to V (for example by preserving the rate of arc lenght) respecting the stratifications.

II) The eigenvalues of $DX(p)$ are real with $\lambda_3 < 0 < \lambda_1 < \lambda_2$ (Nodal-Saddle).

Consider the sets $V_1 = W_1^u \cap \partial V = \{p_1\} \cup \{q_1\}$, $V_2 = W_{12}^u \cap \partial V$ and $V_3 = W_3^s \cap \partial V = \{p_3\} \cup \{q_3\}$.

We give an orientation to W_3^s and consider that W_{12}^u separates V in two regions $V = V^+ \cup V^-$. Observe that the set of external tangencies between X and ∂V has two connected components $T = T^+ \cup T^-$ with $T^+ \subset V^+$ and $T^- \subset V^-$. Moreover $U = I_X \cap \partial V$ has four connected components $U = U_1^+ \cup U_2^+ \cup U_1^- \cup U_2^-$ where $U_1^+ \cup U_2^+ \subset V^+$, $U_1^- \cup U_2^- \subset V^-$ and such that X enters V on $U_1^+ \cup U_1^-$ and leaves V on $U_2^+ \cup U_2^-$ (see Figure 3.5)

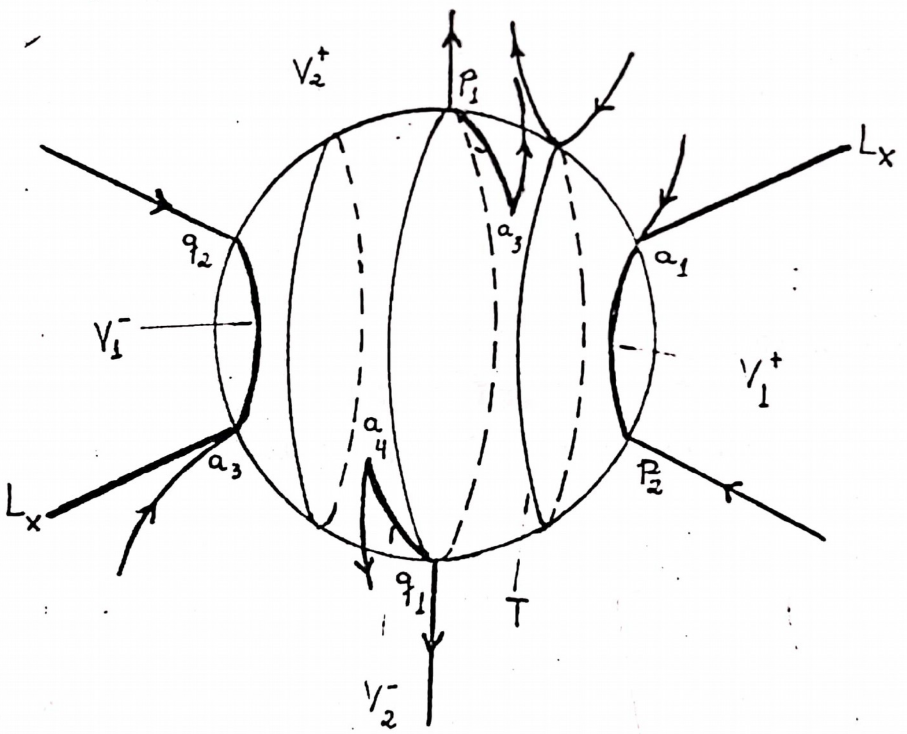

Figure 3.5. The boundary of a neighborhood of a Nodal-Saddle.

We add that each component of U is a semi-interval $[a,b)$ where b is one of the following points p_1, q_1, p_2 and q_2 and a is a point beloging to the trajectories of X passing through $L_X \cap \partial V$.

Call $U_1^+ = [a_1,p_2)$, $U_2^+ = [a_2,p_1)$, $U_1^- = [a_3,q_2)$ and $U_2^- = [a_4,q_1)$.

If Y is a small C^r perturbation of X in X^r, we have similar objects $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{q}_1$, $\tilde{q}_2$, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{a}_3$, $\tilde{a}_4$, $\tilde{U}_1^+$, $\tilde{U}_2^+$, $\tilde{U}_1^-$, $\tilde{U}_2^-$, $\tilde{V}_1$, $\tilde{V}_2$, $\tilde{V}_3$ and $\tilde{T}$.

The required homeomorphism on ∂V must preserve the above distinguished sets.

Now the conclusion of the proof in this case is straighforward.

III) The eigenvalues of $DX(p)$ are $\lambda_1 \in \mathbb{R}$ and $\lambda_2 = a \pm ib$ with $a < \lambda_1 < 0$ (Focal-Node).

Let $V_1 = W_2^s \cap \partial V$ and $U = I_X \cap \partial V$. As before, W_2^s determines on V two connected components $V = V^+ \cup V^-$ and $U = U^+ \cup U^-$ with $U^+ \subset V^+$ and $U^- \subset V^-$.

Observe now that U^+ (resp. U^-) is a semi-interval with an end point beloging to $L_X \cap \partial V$ and $V_1 \subset C\ell(U^+)$ (resp.$V_1 \subset C\ell(U^-)$). This means that U accumulates on V_1 spiraling as illustraded in Figure 4.6.

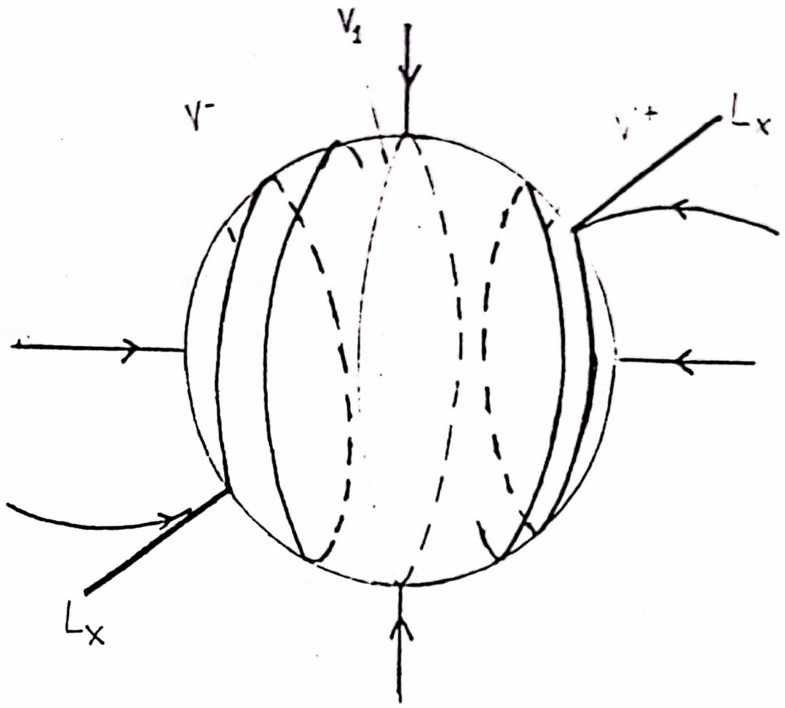

Figure 3.6. The boundary of neighborhood of a Focal-Saddle.

We choose Γ a small one dimensional section (in ∂V) transverse V_1 at some point $p \in V_1$.

For a small perturbation Y of X in X_1^r there exist similar objects $\tilde{V}_1$, $\tilde{U}$ and $\tilde{\Gamma}$.

We have to start the construction of the equivalence by imposing that Γ and U must be sent to $\tilde{\Gamma}$ and $\tilde{U}$ respectively. Notice that: a) U (resp. $\tilde{U}$) is transverse to $S \cap \partial V$; b) any trajectory of X (resp. Y) meets ∂V exactly once; c) we can use the rate of arc lenght with respect to S and ∂V to extend the homeomorphism.

PROOF OF PART ii). Let $Y_1, Y_2 \in B$ such that $G(Y_1).G(Y_2) > 0$. We will proceed to construct the equivalence between Y_1 and Y_2 as in Part i).

Notice that any $Y \in B$ with $G(Y) \neq 0$ has also the following sets to be distinguished: $\tau(Y)$, $\alpha_Y = L_Y \cap W$ where W is either some distinguished two dimensional invariant manifold of Y or ∂V, as well as the trajectories of X passing through them. Moreover: a) if p_Y is not a saddle then $U = I_Y \cap \partial V$ is a closed interval in ∂V with and points on L_Y, transverse to the strong 2-dimensional invariant manifold of X (see Figure 4.6); b) if p_Y is a saddle then $U = I_Y \cap \partial V$ is composed by two semi intervals $U_1 = [a_1, b_1)$, $U_3 = [a_3, b_3)$ and by one closed interval $U_2 = [a_2, b_2]$ where b_1, b_3 are in the 1-dimensional invariant manifold of Y and a_1, a_2, a_3, b_2 are in the trajectories of Y passing through $L_Y \cap \partial V$ (see Figure 3.7). It should be mentioned that X enters (or leaves) V in U_1 and U_3 and leaves (or enters) V in U_2. In addition the trajectories of Y passing through $\tau(Y)$ and α_Y meet ∂V in the interior of U_2. We may classify the trajectories of Y in the following way:

I) The trajectories of Y which enter or leave V in a finite time, and do not meet L_Y.

II) The trajectories which has a unique external tangency with ∂V.

III) The trajectories which either enter or leave V passing through points distinct from α_Y.

IV) The trajectories passing through U_1, U_2, U_3.

V) This class is composed by p_Y.

Now we follow the same lines as in the case of Part i) to finish the proof. □

The following result is a immediate consequence of the Fundamental Lemma and Proposition 3.6:

3.11. COROLLARY: If $G(Y) \neq 0$ then $Y|_V$ is S-structurally stable for perturbations of Y in X^r.

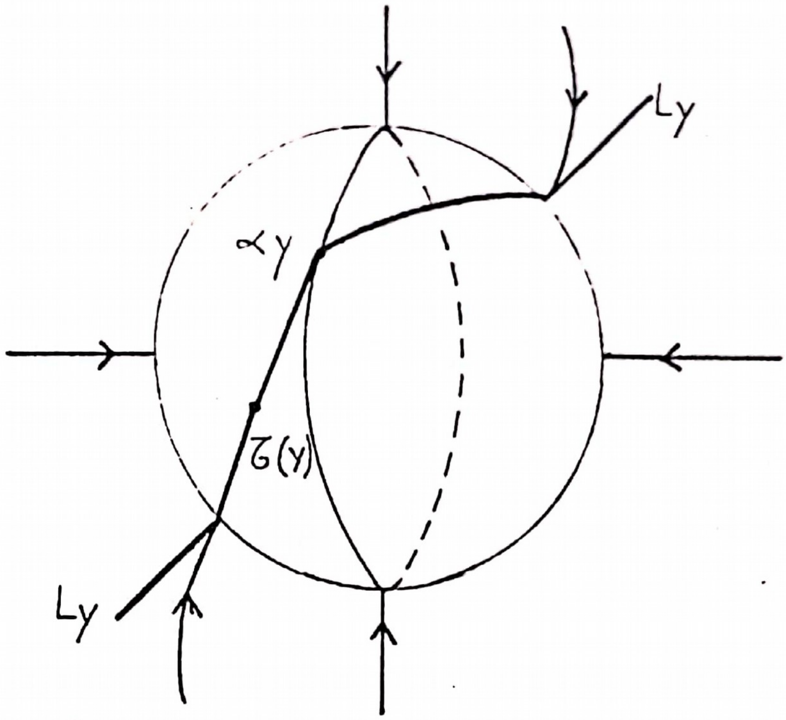

Figure 3.7. G(Y) $\neq$ 0 and p_Y is not a saddle.

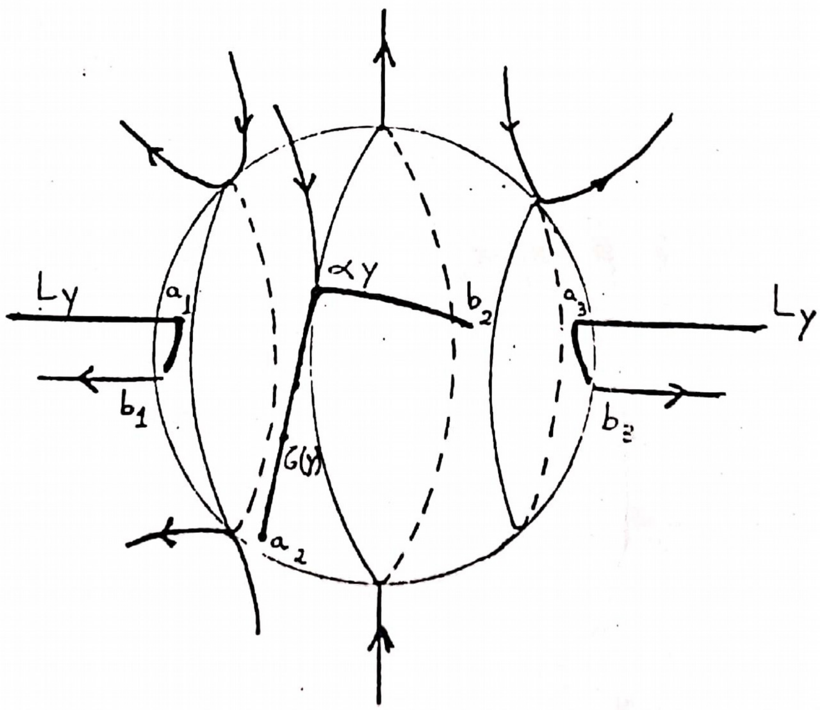

Figure 3.8. G(Y) $\neq$ 0 and p_Y is a saddle.

3.12. REMARK: NORMAL FORMS OF A S-HYPERBOLIC CRITICAL POINT.

1. $X(x_1,x_2,x_3) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)$ and $f(x_1,x_2,x_3) = x_1+x_2+x_3$ with $\lambda_1, \lambda_2, \lambda_3$ are real, nonzero and distinct.

2. $X(x_1,x_2,x_3) = (ax_1+x_2, x_1-ax_2, \lambda x_3)$ and $f(x_1,x_2,x_3) = x_1+$ $+ x_2 + x_3$ with $a \neq 0$, $\lambda \neq 0$ and $a \neq \lambda$.

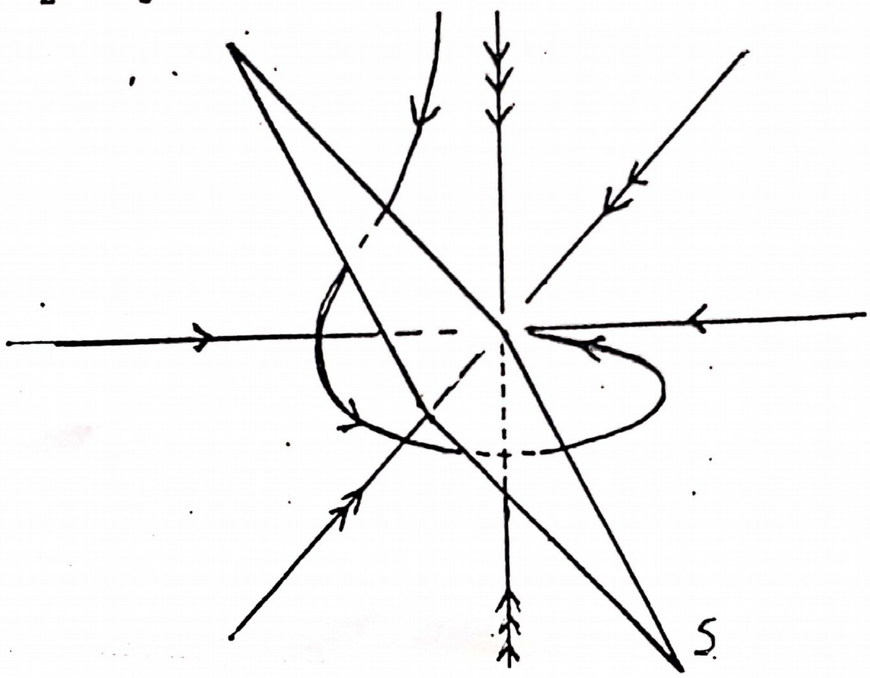

Figure 3.9. Node

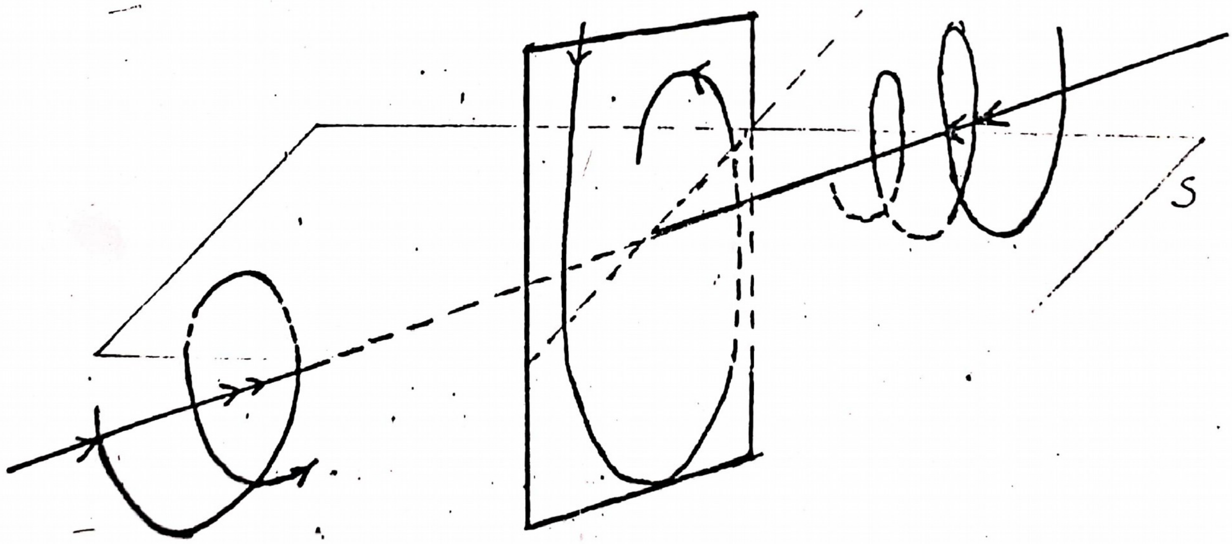

Figure 3.10. Node Focus

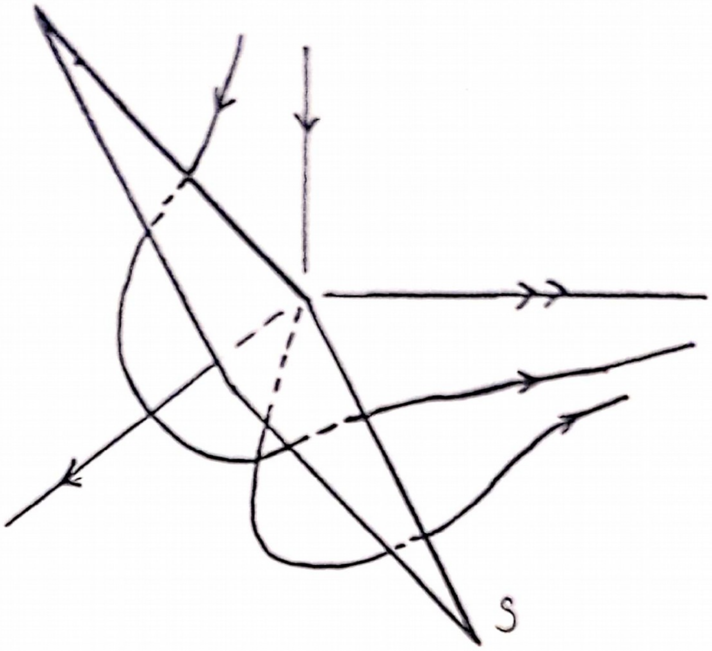

Figure 3.11. Node-Saddle

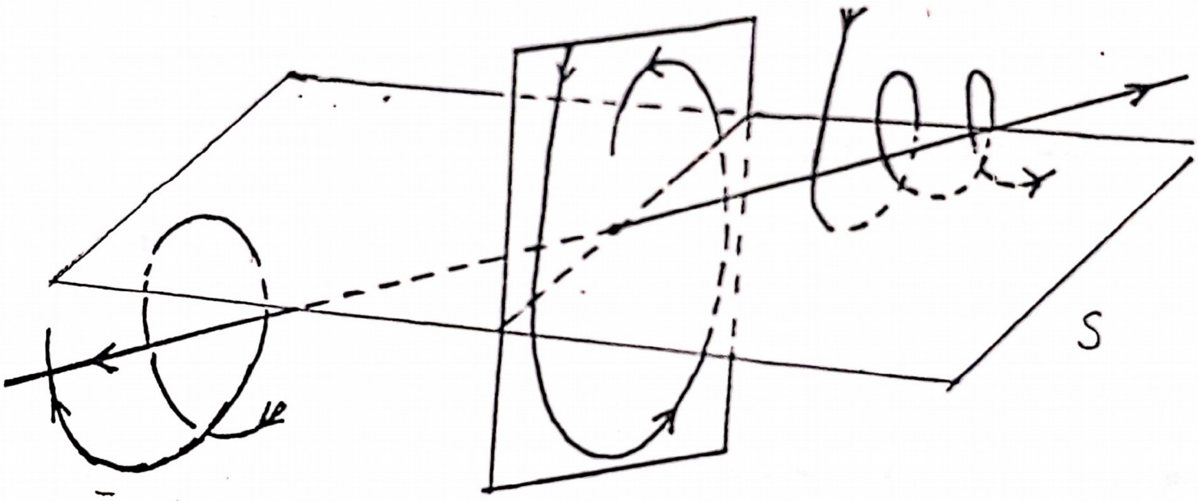

Figure 3.12. Focus-Saddle

3.13. REMARK. UNFOLDING OF A SINGULARITY OF TYPE Σ^1(a)-
-NORMAL FORMS.

1. $X(x_1,x_2,x_3) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)$ and $f_\mu(x_1,x_2,x_3) = \mu + x_1 + x_2 + x_3$ with $\lambda_1, \lambda_2, \lambda_3$ are real, nonzero and distincts and $\mu \in \mathbb{R}$.

2. $X(x_1,x_2,x_3) = (ax_1 + x_2, x_1 - ax_2, \lambda x_3)$ and $f_\mu(x_1,x_2,x_3) = \mu + x_1 + x_2 + x_3$, with $a \neq 0$, $\lambda \neq 0$, $a \neq \lambda$ and $\mu \in \mathbb{R}$.

§4. LOCAL STABILITY.

In this section we discuss the local structural stability (at S) of $X \in \mathcal{X}^r$.

Call $\Sigma_o(p)$ the set of all $X \in \mathcal{X}^r$ which are locally S-stable at $p \in S$. It is known that (see $[S_2]$): a) $X \in \Sigma_o(p)$ if and only if p is either a regular point or a stable singularity of X; b) $\Sigma_o(p)$ is open and dense in $\mathcal{X}^r$.

Denote $\mathcal{X}_1^r(p) = \mathcal{X}^r - \Sigma_o(p)$.

We may refer to a singularity of type $\Sigma'(a)$ or $\Sigma'(b)$ simply as an S-quasi generic singularity.

Let $\Sigma_1(p)$ be the set of all $X \in \mathcal{X}^r(p)$ which are locally S-stable (at $p \in S$) relative to $\mathcal{X}_1^r(p)$. This set is characterized by the following proposition.

4.1. PROPOSITION: $X \in \Sigma_1(p)$ if and only if p is a quasigeneric singularity of X.

PROOF: The sufficient condition follows from Proposition 3.6 and . .

Let p S be a singularity of $X \in \mathcal{X}_1^r(p)$. We are assuming that p is not a quasigeneric singularity of X.

If $X(p) \neq 0$ then we use results contained in [S.3] to find a sequence (X_n) in $\mathcal{X}_1^r(p)$ converging to X such that each X_n is never C^o- S equivalent to X (at p).

If $X(p) = 0$ and one of the conditions i), ii), iii) and iv) of Definition 1.2 is violated then we see that $X \notin \Sigma_1(p)$. We are going to discuss more the conditions ii) and iii).

Assume that X, at p, satisfies the above conditions i) and iv).

If the Condition ii) is dropped, this means that two eigenvalues of $DX(p)$ are equal then we can pick a suitable invariant 2-dimensional manifold W and restrict our study to $X|_W$. In this way we select a sequence (X_n) in $\mathcal{X}_1^r(p)$ such that W is still an invariant 2-dimensional manifold of each X_n and the eignevalues of $D(X_n|_W)(p)$ are distinct. Now by using the same techniques and arguments as [T] we prove directly that $X \notin \Sigma_1(p)$.

To illustrate the last situation, suppose that the eigenvalues of DX(p) are $\lambda_1 = \lambda_2 < \lambda_3 < 0$. We choose W as being the invariant 2-dimensional manifold of X generated by λ_1. In addition we take local coordinates around p such that $X(x,y,z) = (\lambda_1 x + h_1(x,y,z),\ \alpha x + \lambda_1 y + h_2(x,y,z),\ \lambda_3 z + h_3(x,y,z))$ for some nonzero scalar α. We can choose (X_n), given by

$$X_n(x,y,z) = (\lambda_1 x - (\tfrac{1}{n})^2 y + h_1(x,y,z),\ \alpha x + \lambda_1 y + h_2(x,y,z),$$

$$\lambda_3 z + h_3(x,y,z).$$

If the eigenvalues of DX(p) are $\lambda_1 = a \in \mathbb{R}$ and $\lambda_2 = a \pm ib$ with $b \neq 0$. Then following Remark 3.3 we see that this situation contradicts the Condition iv) of Definition 2.4. This finishes the proof. $\square$

The following result is an immediate consequence of Proposition 2.3, Lemma 3.4, Proposition 3.6 and Proposition 4.1.

4.2. PROPOSITION: a) $\Sigma_1(p)$ is open and dense in $\mathcal{X}_1^r(p)$; b) $\Sigma_1(p)$ is a codimension one submanifold of $\mathcal{X}^r$ and c) $\mathcal{X}_2^r = \mathcal{X}_1^r - \Sigma_1(p)$ is a closed set of codimension greater than one.

4.3. REMARK: Let $I = [-\varepsilon, \varepsilon]$ be a closed interval. Denote by ϕ^r the space of C^1 mappings $\zeta: I \in \mathcal{X}^r(p)$ with the C^1 topology. We say $\lambda_o \in I$ is an ordinary value of $\zeta \in \phi^r$ if there is a neighborhood N of λ_o such that $\zeta(\lambda)$ is C^o equivalent to $\zeta(\lambda_o)$ (at p) for every $\lambda \in N$; if λ_o is not an ordinary value of ζ, it is called a bifurcation value of ζ. We say that ζ_1 and ζ_2 of ϕ^r

are C^o S-equivalent is there is a homeomorphism $h : I \to I$ and a map $H : I \to \text{Homeo}(M)$ such that $H(\lambda)$ is a C^o S-equivalent (at p) between $\zeta_1(\lambda)$ and $\zeta_2(h(\lambda))$. With this concept we get naturally the structural stability definition in ϕ^r. Let us denote by $A^r(p)$ the collection of the elements $\zeta \notin \phi^r$ such that: 1) $\zeta(I) \subset \subset \Sigma_o(p) \cup \Sigma_1(p)$, 2) ζ is transversal to $\Sigma_1(p)$, 3) $\zeta(-\varepsilon)$ and $\zeta(\varepsilon)$ are in $\Sigma_o(p)$. We have that ζ is structurally stable in ϕ^r if and only if $\zeta \in A^r(p)$.

§5. TRANSIENT VECTOR FIELDS.

We clarify the intrinsic features of the proofs of Theorem 1 and Theorem 2 by giving in this section some constructions and preparatory results concerning transient vector fields in suitable neighborhoods of S in M. In [P] the structural stability of transient vector fields on a manifold is studied.

For any compact connected C^∞ 3-manifold P with a non-empty 2-dimensional boundary we say that a vector field is called transient in P if each integral curve of it leaves the manifold in finite positive and negative time.

Let $p \in S$ be either a S-stable or a S-quasigeneric singularity of $X \in \mathcal{X}^r$, B a small neighborhood of X in $\mathcal{X}^r$ and F_ε be a fundamental neighborhoods system of p_Y in S (p_Y given either in Proposition 3.5 or in Lemma 4.4) for each $Y \in B$.

5.1. LEMMA; Let $X(p) \neq 0$. Then for each $Y \in B$ there exists a fundamental neighborhoods system V_ε around P_Y, diffeomorphic to $F_\varepsilon \times [-1,1]$ such that: i) Y is transient in V_ε; ii) Y is transverse to $F_\varepsilon \times \{\pm 1\}$ (this implies in particular that the endpoints of each trajectory of Y in V_ε are outside of L_Y).

PROOF. Consider $I_\varepsilon \times I_\varepsilon \times I_\varepsilon$ a ε-flow box around $p \in S$ with respect to X such that $I = [-\varepsilon,\varepsilon]$; $p = (0,0,0)$ and $X(x,y,z) = (1,0,0)$. The surface can be given locally by the grafic of $z = g(x,y)$ with $(x,y) \in I_\varepsilon \times I_\varepsilon$. This proves the assertions of the lemma for the original vector field X. Now the proof follows by continuity. $\square$

The following lemma is immediate.

5.2. LEMMA: If X satisfies the hypotheses of Theorem 2 and $X(q) \neq 0$ for every $q \in S$ then there exists a neighborhhod B of X in $\mathcal{X}^r$ such that for each $Y \in B$ there is a fundamental neighborhhods system W_ε of S where $X|_{W_\varepsilon}$ is transient.

5.3. REMARK: We recall from 3 that a vector field is transient if and only if it is a gradient field (for some metric) with no critical points.

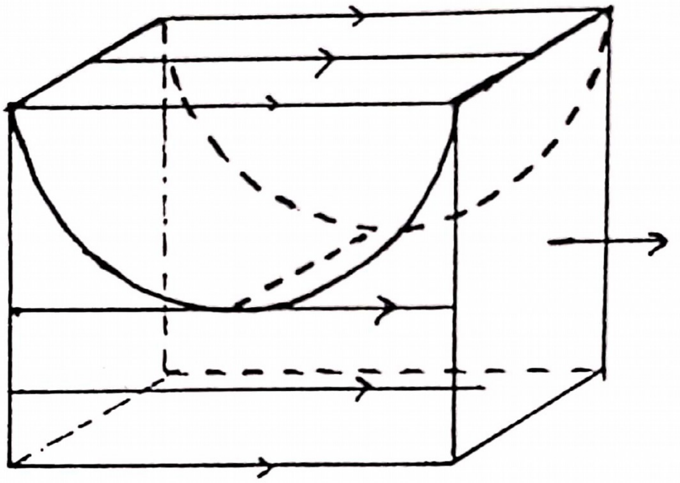

Figure 5.1. The fold singularity or the Dove's Tail singularity.

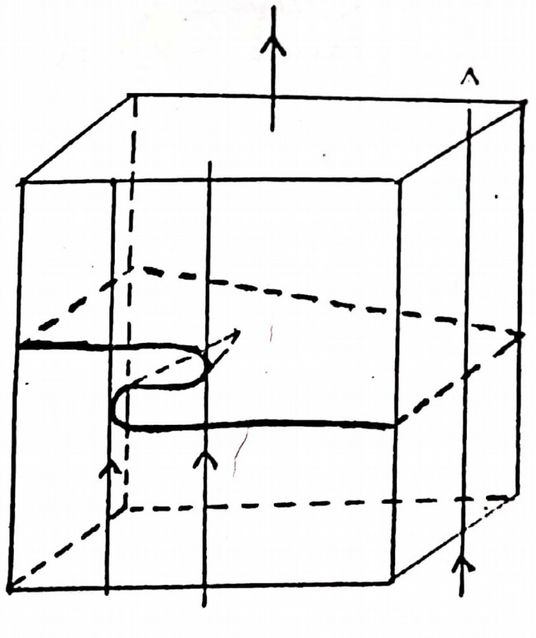

Figure 5.2. The cusp singularity.

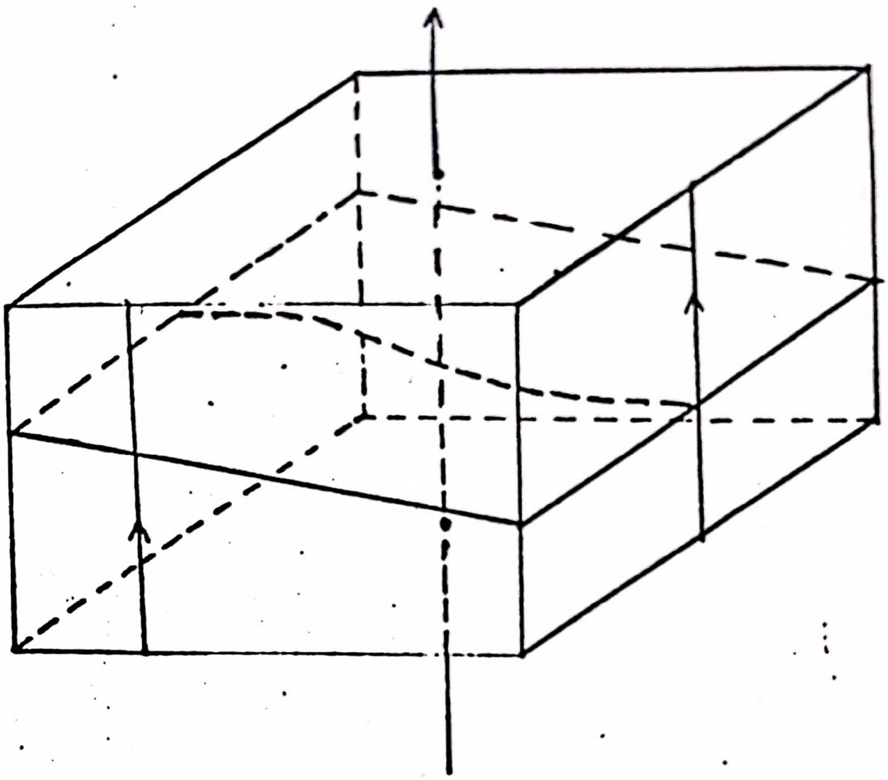

Figure 5.3. The Lips singularity

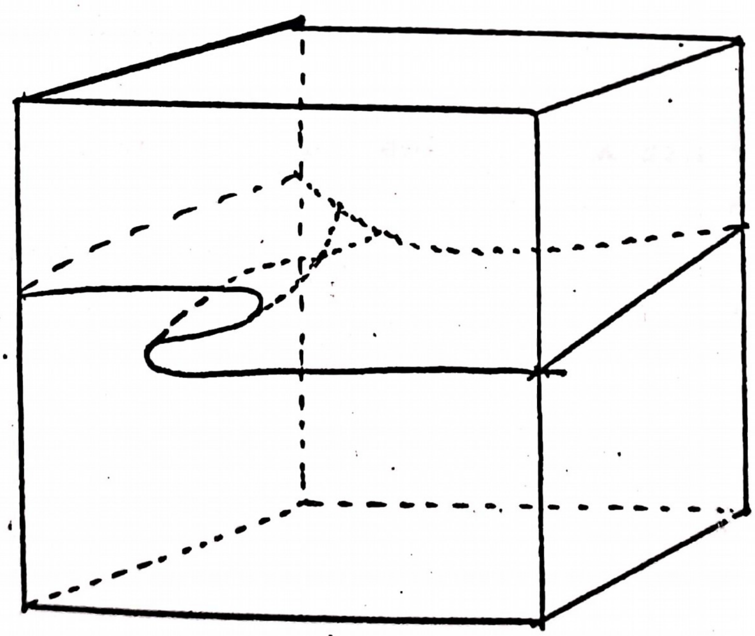

Figure 5.4, The Bec to Bec singularity

Assume for instance that $p \in S$ is a singularity of X of type $\Sigma_1(a)$.

We are considering V the neighborhood of p as in 3.9.

As before, call $L_X(p)$ the connected component of L_X containing p.

If $q \in L_X(p) \cap \partial V$ then consider $V_\varepsilon(q)$ as in Lemma 5.2 and define the sets

$$\Omega_\varepsilon(q,p) = V - \{V_\varepsilon(q) \cap V\}$$

and

$$\Omega_\varepsilon(p) = \bigcup_{q \in L_X \cap V} (\Omega_\varepsilon(p,q))$$

If p is a saddle point of X then ε must be choosen scuh that $0 < \varepsilon < d(L_X, T_X)$ where T_X is the set of external tangencies between X and ∂V.

The proof of the nest lemma is straighforward.

5.5. LEMMA. Assume that X satisfies the hypotheses of Theorem 2 with $X(p) = 0$ and $p \in S$. Then there exists a neighborhood B of X in $\mathcal{X}^r$ such that for each $Y \in B$ there is a fundamental neighborhood system W of $S - \{\Omega_\varepsilon(p) \cap S\}$ such that $Y|_{W_\varepsilon}$ is transient.

§6. PROOF OF THEOREMS.

First of all, observe that the proofs of the theorems are immediate consequence of the results contained in Sections 2, 3, 4 and 5.

6.1. PROOF OF THEOREM 1. This proof follows directly from the characterization of $\Sigma_o(p)$ for every S-stable singularity of a vector field, from Proposition 2. and from Lemma 5.1. .

6.2. PROOF OF THEOREM 2. Part i) follows from Proposition 4.1. and Lemma 5.2. .

Part ii) and iii) are immediate consequence of Proposition 4.2. .

Part iv) follows from Proposition 2.3, Proposition 4.6. and Corollary 3.11. .

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