

ON BEST APPROXIMATION BY RATIONAL  
AND HOLOMORPHIC MAPPINGS  
BETWEEN BANACH SPACES

*A. O. Chiacchio*

*M. C. Matos*

and

*M. S. M. Roversi*

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**ABSTRACT.** After proving a generalized version of a Garkavi's theorem, we give as applications proofs of existence results on best approximation by polynomial, fractional linear and holomorphic operators between Banach spaces. We also get theorems on best approximation by some types of rational functions defined in open subsets of Banach spaces. By considering a natural non-normable distance we prove that every mapping bounded on the bounded subsets of a Banach space has best approximation by polynomials of degree less than or equal to a fixed natural number  $n$ .

Universidade Estadual de Campinas  
Instituto de Matemática, Estatística e Ciência da Computação  
IMECC – UNICAMP  
Caixa Postal 6065  
13.081 - Campinas, SP  
BRASIL

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## 1. INTRODUCTION

It is classical the result of Ch. de La Vallée Poussin [13] that every real continuous function in  $[-1,1]$  has a best approximation in the set of all functions of the form  $\frac{P(x)}{Q(x)}$  where  $P$  and  $Q$  are polynomials of degree less than or equal to  $m$  and  $n$  respectively and  $Q(x) > 0$  for every  $x \in [-1,1]$ . Walsh [14] proved this type of best rational approximation for continuous functions in a perfect subset of the complex plane. Cheney and Loeb [3] considered the problem of best approximation by rational trigonometric functions. Newman and Shapiro [10], Rice [11] and Bohem [2] considered the existence of best approximation by quotients of finite linear combinations of real continuous functions in topological spaces  $X$ . Other aspects of the theory of best approximation by such rational functions have been studied by many authors.

When  $U$  is a non void open bounded subset of a complex Banach space  $E$  and  $F$  is another complex Banach space it makes sense to consider polynomials from  $U$  into  $F$  defined through continuous multilinear mappings from  $E$  into  $F$ . In this article we study the existence of best approximation of bounded mappings from  $U$  into  $F$  by certain quotients of polynomials from  $U$  into  $F$  by polynomials from  $U$  into  $\mathbb{C}$  (i.e.: rational mappings from  $U$  into  $F$ ). The proof of our results depends on the compactness of certain subsets of holomorphic (i.e., Gâteaux-differentiable and continuous) mappings from  $U$  into  $F$  and they also provide proofs of results on best approximation of bounded mappings from  $U$  into  $F$  by holomorphic and polynomials mappings from  $U$  into  $F$ .

We denote by  $\ell^\infty(U;F)$  the vector space of all bounded mappings from  $U$  into  $F$  with the norm

$$\|f\|_\infty = \sup\{\|f(x)\|; x \in U\} \quad f \in \ell^\infty(U;F).$$

The vector subspace of  $\ell^\infty(U;F)$  formed by all bounded holomorphic mappings from  $U$  into  $F$  is denoted by  $\mathcal{H}^\infty(U;F)$ . We prove in paragraph 3 that, when  $F$  is a dual space, every  $f \in \ell^\infty(U;F)$  has



a best approximation in  $\mathcal{H}^\infty(U;F)$  and, as a corollary to this result, that  $f$  has a best approximation in the set of all continuous polynomials from  $E$  into  $F$  with degree less than or equal to  $n$ .

A mapping  $f \in \mathcal{H}^\infty(U;F)$  is called a rational mapping of type  $(m,n)$  if there are continuous polynomials  $P$  from  $E$  into  $F$  and  $Q$  from  $E$  into  $\mathbb{C}$  of degrees less than or equal to  $m$  and  $n$  respectively such that  $f(x)Q(x) = P(x)$  for every  $x$  in  $U$  and  $Q$  is not identically zero in  $U$ . We denote the set of all such mappings by  $\mathcal{R}_{(m,n)}^\infty(U;F)$ . In paragraph 4 we prove the existence of best approximations of  $f \in \mathcal{L}^\infty(U;F)$  by elements of  $\mathcal{R}_{(m,n)}^\infty(U;F)$  when  $\dim(E) < +\infty$  and  $F$  is a dual space. We also prove that when  $\dim(E) = +\infty$  and  $F$  is  $\mathbb{C}$  there exist best approximations of  $f \in \mathcal{L}^\infty(U;\mathbb{C})$  by elements of  $\mathcal{R}_{(0,n)}^\infty(U;\mathbb{C})$  and  $\mathcal{R}_{(1,1)}^\infty(U;\mathbb{C})$ . The problem is open for the other values of  $m$  and  $n$ , but we conjecture that at least for the cases  $m = 1$ ,  $n \in \mathbb{N}$ ,  $F = \mathbb{C}$ , we should have results of existence on best approximation by rational functions of this type.

We denote by  $F_b(E;F)$  the vector space of all mappings from  $E$  into  $F$  which are bounded over the bounded subsets of  $E$ . The locally convex topology  $\tau_b$  in  $F_b(E;F)$  of the uniform convergence over the bounded subsets of  $E$  is metrizable but non-normable in general. In paragraph 5 we prove results of best approximation of  $f \in F_b(E;F)$  by polynomial mappings from  $E$  into  $F$  with respect to a metric defining  $\tau_b$ .

It is well known that the vector space of all compact linear mappings from  $E$  into  $F$  may be antiproximinal in the Banach space of all continuous linear mappings from  $E$  into  $F$  (see Holmes and Kripke [7]). However Deutsch, Mach and Saatkamp proved in [4] that, when  $F$  is a dual space, the set of continuous linear mappings from  $E$  into  $F$  of finite rank  $N$  (i.e., mappings whose images are contained in vector subspaces of dimension  $N$ ) is proximinal in the Banach space of all bounded linear mappings from  $E$

into  $F$ . In paragraph 6, with the help of a result communicated to us by J. Mujica and a result of K. Floret [5], we show how theorems of this type are easily proved for holomorphic, rational and polynomials mappings of finite rank  $N$ .

The lemma (and its corollary) proved in paragraph 2 is fundamental for the proofs of our results. It generalizes a result of Garkavi [6] and it is stated in greater generality than it is necessary for our applications but, since it is interesting in it self, we felt we should state and prove it in this way.

## 2. THE FUNDAMENTAL LEMMA

If  $X$  is a separated topological vector space over  $\mathbb{K}$  ( $\mathbb{C}$  or  $\mathbb{R}$ ) with topology  $\tau$  we consider the set  $S(X)$  of all function  $\varphi$  from  $X$  into  $\mathbb{R}$  such that (i)  $\varphi(x) \geq 0$  for every  $x \in X$ , (ii)  $\varphi$  is continuous in  $X$ , (iii) for every bounded subset  $B$  of  $X$  we have  $\|\varphi\|_B \leq \text{diam } \varphi$ , where  $\|\varphi\|_B = \sup \{\varphi(t); t \in B\}$  and  $\text{diam } \varphi = \sup \{\varphi(x); x \in X\} = \sup \{\varphi(y-x); y, x \in X\}$ .

### 2.1. EXAMPLES

(a) If  $p \in (0,1]$  and  $g$  is a non-zero continuous  $p$ -seminorm in  $(X;\tau)$  then  $g \in S(X)$  with  $\text{diam } g = +\infty$ .

(b) If  $p \in (0,1]$  and the topology  $\tau$  of  $X$  is defined by a sequence  $(q_n)_{n=1}^{\infty}$  of  $p$ -seminorms in  $X$ , then we may consider

$$\varphi(x) = \sum_{n=1}^{\infty} 2^{-n} \frac{q_n(x)}{1 + q_n(x)}, \quad \forall x \in X$$

and

$$d(x,y) = \varphi(y - x), \quad \forall x,y \in X.$$

Then  $d$  is a metric defining the topology  $\tau$  of  $X$ . It is clear that  $\varphi$  is continuous in  $X$  and that  $\text{diam } \varphi = 1$ . In order to show



that  $\varphi \in S(X)$  it is enough to prove that for  $n = 1, 2, \dots$  and for every non-empty bounded subset  $B$  of  $(X, \tau)$

$$\sup \left\{ \frac{q_n(x)}{1 + q_n(x)} ; x \in B \right\} < 1 \quad (1)$$

which implies  $\|\varphi\|_B < 1 = \text{diam } \varphi$ . If (1) were not true there would be a positive integer  $n$  such that for every  $k = 1, 2, \dots$  we could find  $x_k \in B$  satisfying

$$\frac{q_n(x_k)}{1 + q_n(x_k)} > 1 - \frac{1}{k}.$$

This would give  $q_n(x_k) > k-1$  for every  $k = 1, 2, \dots$ . But this is impossible since  $q_n$  has to be bounded over  $B$ .

(c) If  $q$  is a non-zero continuous quasi-seminorm in  $(X, \tau)$  then  $q \in S(X)$  with  $\text{diam } q = +\infty$

2.2. DEFINITION - If  $(X, \tau)$  is a separated topological vector space over  $\mathbb{K}$  and  $\varphi \in S(X)$ , a non-empty subset  $Y$  of  $X$  is said to have the Chebyshev center property in  $X$  relative to  $\varphi$  if for every non-empty bounded subset  $B$  of  $X$  there is  $f \in Y$  such that

$$\sup_{x \in B} \varphi(f - x) = \inf_{g \in Y} \sup_{x \in B} \varphi(g - x). \quad (2)$$

In this case  $f$  is called a Chebyshev center of  $B$  relative to  $Y$  and  $\varphi$ , and the right-hand side of (2) is called the radius of Chebyshev of  $B$  relative to  $Y$  and  $\varphi$ . If  $B = \{x\}$  we get (2) written in the form

$$\varphi(f - x) = \inf_{g \in Y} \varphi(g - x) \quad (3)$$

and we say that  $f$  is a best approximation of  $x$  in  $Y$  relative

to  $\varphi$ . If this happens for all  $x \in X$  we say that  $Y$  is proximal in  $X$  relative to  $\varphi$ . When there is no doubt about the  $\varphi$  which is being considered we drop out the reference to  $\varphi$  (e.g.:  $Y$  has the relative Chebyshev center property in  $X$ ,  $f$  is a Chebyshev center of  $X$  relative to  $Y$ , etc).

When  $Y$  is proximal in  $X$  relative to  $\varphi$  and  $\varphi^{-1}(\{0\}) = \{0\}$ , then it is quite simple to prove that  $Y$  is a closed subset of  $X$  for the topology  $\tau$ .

2.3. LEMMA - Let  $(X, \tau)$  be a separated topological vector space over  $\mathbb{K}$  and let  $\varphi$  be an element of  $S(X)$ . If  $\Delta_r = \{t \in \mathbb{R}; |t| \leq r\}$  we consider a separated topology  $\sigma$  in  $X$  compatible with the vector space structure such that  $\varphi^{-1}(\Delta_r)$  is  $\sigma$ -closed for every  $r \in [0, \text{diam } \varphi)$ . We denote

$$K_{r, \varphi}(B) = \{x \in X; x \in b + \varphi^{-1}(\Delta_r) \quad \forall b \in B\}.$$

If  $Y$  is a non-empty subset of  $X$  such that  $Y \cap K_{r, \varphi}(B)$  is  $\sigma$ -countably compact for every  $r \in [0, \text{diam } \varphi)$  and every non-empty bounded subset  $B$  of  $(X, \tau)$ , then  $Y$  has the Chebyshev center property in  $X$  relative to  $\varphi$ .

PROOF - For a non-empty bounded subset  $B$  of  $(X, \tau)$  we consider

$$r_B = \inf_{y \in Y} \sup_{t \in B} \varphi(y - t) < \text{diam } \varphi$$

and we define  $f_B(x) = \sup\{\varphi(x - t); t \in B\}$  for every  $x$  in  $X$ . For  $\rho \in \mathbb{R}$  we have

$$\{x \in X; \varphi(x - t) \leq \rho\} = t + \varphi^{-1}(\Delta_\rho)$$

$\sigma$ -closed for every  $t \in B$ . Hence  $f_B$  is  $\sigma$ -lower semicontinuous in  $X$  and, consequently, for  $n = 1, 2, \dots$  and  $\delta = \min\{1, \text{diam } \varphi - r_B\}$



$$C_n = \{y \in Y; f_B(y) \leq r_B + \frac{\delta}{n}\} \neq \emptyset$$

is relatively  $\sigma$ -closed in  $Y$ . We also have  $C_{n+1} \subset C_n$  and  $Y \cap K_{r_B + 2^{-1}\delta, \varphi}(B) \supset C_n$  for every  $n \geq 2$ . Since  $Y \cap K_{r_B + 2^{-1}\delta, \varphi}(B)$  is  $\sigma$ -countably compact, it follows that  $\bigcap_{n=2}^{\infty} C_n \neq \emptyset$ . Hence we have  $r_B = \sup\{\varphi(f-t); t \in B\}$  for each  $f \in \bigcap_{n=2}^{\infty} C_n$ . This means that each element of  $\bigcap_{n=2}^{\infty} C_n$  is a Chebyshev center of  $B$  relative to  $Y$  and  $\varphi$ .

Q.E.D.

2.4. COROLLARY - Let  $X$  be a vector space over  $IK$  and let  $q$  be either a  $p$ -norm ( $p \in (0,1]$ ) or a quasi-norm in  $X$ . If  $\sigma$  is a topology in  $X$  compatible with the vector space structure such that  $\bar{B}_{q,1}(0) = \{x \in X; q(x) \leq 1\}$  is  $\sigma$ -closed and  $Y$  is a non-empty subset of  $X$  such that  $\{y \in Y; q(y) \leq r\}$  is  $\sigma$ -countably compact for every  $r > 0$ , then  $Y$  has the Chebyshev center property in  $X$  relative to  $q$ .

PROOF - First we note that  $q^{-1}(\Delta_r) = \{x \in X; q(x) \leq r\} = \bar{B}_{q,r}(0)$  is the closed ball of center  $0$  and radius  $r$  with respect to  $q$ . If  $q$  is a  $p$ -norm  $q^{-1}(\Delta_r) = r^{\frac{1}{p}} \bar{B}_{q,1}(0)$ , and, if  $q$  is a quasi-norm  $q^{-1}(\Delta_r) = r \bar{B}_{q,1}(0)$ . In any case  $q^{-1}(\Delta_r)$  is  $\sigma$ -closed. If  $B$  is a non-empty bounded subset of  $(X, q)$ , then there is  $\rho \geq 0$  such that  $\sup\{q(t); t \in B\} \leq \rho$ . If  $r \geq 0$  and  $q$  is a  $p$ -norm we have

$$Y \cap K_{r,q}(B) = \bigcap_{b \in B} \{y \in Y; q(y-b) \leq r\}$$

$$\subset \{y \in Y; q(y) \leq r + \rho\}.$$

If  $q$  is a quasi-norm, we know that there is  $M \geq 0$  such that

$q(z+t) \leq Mq(z) + Mq(t)$  for all  $z$  and  $t$  in  $X$ . Hence

$$Y \cap K_{r,q}(B) \subset \{y \in Y; q(y) \leq M(r+\rho)\}.$$

In any case we get  $Y \cap K_{r,q}(B)$   $\sigma$ -countably compact. Now we apply Lemma 2.3 with  $\varphi = q$ .

Q.E.D.

### 3. BEST APPROXIMATION BY HOLOMORPHIC OPERATORS

In this paragraph we consider  $E$  a complex Banach space,  $U$  a non-void bounded open subset of  $E$  and  $F = G^*$  a dual Banach space. We denote by  $\ell^\infty(U;F)$  the vector space of all bounded mappings from  $U$  into  $F$  normed by

$$\|f\|_\infty = \sup\{\|f(x)\|; x \in U\} \quad \forall f \in \ell^\infty(U;F).$$

The Banach subspace of  $\ell^\infty(U;F)$  formed by all bounded holomorphic (i.e., Gâteaux-differentiable and continuous) mappings from  $U$  into  $F$  will be denoted by  $\mathcal{H}^\infty(U;F)$ . The locally convex topology in  $\ell^\infty(U;F)$  generated by the seminorms

$$p_{K,y}(f) = \sup\{|f(x)(y)|; x \in K\}$$

for  $f \in \ell^\infty(U;F)$ ,  $K$  a compact subset of  $U$  and  $y \in G$ , is denoted by  $\tau_0^*$ . The compact-open topology in  $\ell^\infty(U;F)$  is indicated by  $\tau_0$  and it is clear that  $\tau_0 = \tau_0^*$  when  $F$  is a finite dimensional Banach space.

**3.1. THEOREM** - (1) If  $V$  is a vector subspace of  $\ell^\infty(U;F)$  containing  $\mathcal{H}^\infty(U;F)$ , then  $\mathcal{H}^\infty(U;F)$  has the relative Chebyshev center property (hence, it is proximal) in  $V$ .

(2) If  $W$  is a  $\tau_0^*$ -closed subset of  $\mathcal{H}^\infty(U;F)$  and  $V$  is a vector subspace of  $\ell^\infty(U;F)$  containing  $W$ , then  $W$  has the relative



Chebyshev center property in  $V$ .

PROOF - (1) is a consequence of Corollary 2.4 if we prove that

$$B_r = \{f \in \mathcal{H}^\infty(U; F); \|f\|_\infty \leq r\}$$

is  $\tau_0^*$ -compact for every  $r \geq 0$ . By the generalized Montel's Theorem (see Barroso, Matos and Nachbin [1])  $B_r$  is  $\tau_0^*$ -relatively compact in  $\mathcal{H}(U; (F, \sigma(F; G)))$ . Here  $\mathcal{H}(U; (F, \sigma(F; G)))$  denotes the vector space of all holomorphic (i.e.: Gâteaux-differentiable and continuous) mappings from  $U$  into  $(F, \sigma(F; G))$  and  $\sigma(F; G)$  denotes the weak topology in  $F$  defined by  $G$ . If  $f$  is in the  $\tau_0^*$ -closure of  $B_r$  in  $\mathcal{H}(U; (F, \sigma(F; G)))$  there is a net  $(f_\alpha)_{\alpha \in I}$  in  $B_r$  which is  $\tau_0^*$ -convergent to  $f$ . It follows that  $(|f_\alpha(x)(z)|)_{\alpha \in I}$  converges to  $|f(x)(z)|$  for every  $x \in U$  and  $z \in G$ . Hence

$$\|f\|_\infty = \sup_{\substack{z \in G, \|z\| \leq 1 \\ x \in U}} |f(x)(z)| \leq r$$

and  $f \in \mathcal{H}^\infty(U; F)$ . Since  $\mathcal{H}^\infty(U; F)$  is  $\tau_0^*$ -closed in  $\mathcal{H}(U; (F, \sigma(F; G)))$  it follows that  $f \in B_r$ . Hence  $B_r$  is  $\tau_0^*$ -compact.

(2) is a consequence of Corollary 2.4 since

$$W \cap B_r = \{f \in W; \|f\|_\infty \leq r\}$$

is  $\tau_0^*$ -compact.

Q.E.D.

Part (2) of this theorem gives results of best approximation by polynomial operators. In order to give the precise results of this type we fix the notations we are going to use. If  $n = 1, 2, \dots$  we consider the complex vector space  $L(^n E; F)$  of all continuous  $n$ -linear mappings from  $E^n$  into  $F$ . We denote by  $P(^n E; F)$  the

vector space formed by all mappings  $P$  from  $E$  into  $F$  such that there is  $A \in L(^nE; F)$  satisfying  $P(x) = A(x, \dots, x) = Ax^n$  for all  $x \in E$ . For  $n = 0$  the vector space  $P(^0E; F)$  is formed by all constant mappings from  $E$  into  $F$ . The elements of  $P(^nE; F)$ ,  $n = 0, 1, \dots$ , are called  $n$ -homogeneous continuous polynomials from  $E$  into  $F$ . If we set

$$\|P\| = \sup\{\|P(x)\|; \|x\| \leq 1\} \quad \forall P \in P(^nE; F)$$

then  $P(^nE; F)$  is a Banach space and it is not difficult to show that  $\|\cdot\|_\infty$  is an equivalent norm in this space. Hence we may consider  $P(^nE; F)$  as a Banach subspace of  $\ell^\infty(U; F)$  through the restriction mapping to  $U$ . A mapping  $P: E \rightarrow F$  is called a continuous polynomial of degree less than or equal to  $m \in \mathbb{N} = \{0, 1, \dots\}$  if  $P = P_0 + P_1 + \dots + P_m$  for some  $P_j \in P(^jE; F)$ ,  $j = 0, 1, \dots, m$ . The vector space of all such mappings will be denoted by  $P_m(E; F)$ . For all  $n, m \in \mathbb{N}$  the subspaces  $P(^nE; F)$  and  $P_m(E; F)$  are  $\tau_0^*$ -closed in  $\mathcal{H}^\infty(U; F)$ . Hence, from Theorem 3.1, part (2), it follows that the following results are true for all  $n, m \in \mathbb{N}$ .

3.2. COROLLARY - (1) The vector space  $P_m(E; F)$  of all continuous polynomials from  $E$  into  $F$  of degree less than or equal to  $m$  has the relative Chebyshev center property (hence, it is proximal) in  $\ell^\infty(U; F)$ .

(2) The vector space  $P(^nE; F)$  of all continuous  $n$ -homogeneous polynomials from  $E$  into  $F$  has the relative Chebyshev center property (hence, it is proximal) in  $\ell^\infty(U; F)$ .

The special case of part (2) in Corollary 3.2 was proved by Roversi in [12].

#### 4. BEST APPROXIMATION BY RATIONAL MAPPINGS

In this paragraph  $E$  is a complex Banach space,  $U$  is a



non-empty bounded open subset of  $E$  and  $F$  is a complex dual Banach space.

We denote by  $\mathcal{R}_{(m,n)}^{\infty}(U;F)$  the set of all  $f \in \mathcal{K}^{\infty}(U;F)$  such that there are polynomials  $P \in \mathcal{P}_m(E;F)$ ,  $Q \in \mathcal{P}_n(E;\mathbb{C})$  satisfying  $Q(x)f(x) = P(x)$  for every  $x \in U$  with  $Q$  not identically zero in  $U$ . The elements of  $\mathcal{R}_{(m,n)}^{\infty}(U;F)$  are called *bounded rational mappings of type  $(m,n)$  from  $U$  into  $F$* . We note that:

(i) The elements of  $\mathcal{R}_{(m,0)}^{\infty}(U;F)$  are the restrictions to  $U$  of the polynomials of degree less than or equal to  $m$ .

(ii) If  $f \in \mathcal{R}_{(0,n)}^{\infty}(U;F)$  there are  $c \in F$ ,  $Q \in \mathcal{P}_n(E;\mathbb{C})$  such that  $Q(x)f(x) = c$  for every  $x \in U$  and  $Q$  is not identically zero in  $U$ . If  $f$  is not the constant mapping 0 we have  $Q \cdot f$  not identically zero in an open dense subset of  $U$ . Hence  $c \neq 0$  and it follows that  $f(x) \neq 0$  and  $Q(x) \neq 0$  for every  $x \in U$ . Therefore  $f(x) = \frac{c}{Q(x)}$  for every  $x \in U$ .

(iii) The elements of  $\mathcal{R}_{(1,1)}^{\infty}(U;F)$  are called *bounded linear fractional mappings from  $U$  into  $F$* .

Next lemma is fundamental in the proof of the results we get on best approximation by rational mappings.

4.1. LEMMA - Let  $B_r$  be the subset of  $\mathcal{R}_{(m,n)}^{\infty}(U;F)$  formed by those mappings  $f$  such that  $\|f\|_{\infty} \leq r$ . If  $(f_j)_{j=1}^{\infty}$  is a sequence of elements of  $B_r$ , then there are  $f \in \mathcal{K}^{\infty}(U;F)$ ,  $x_0 \in U$ , a subnet  $(f_{j_{\alpha}})_{\alpha \in I}$  of  $(f_j)_{j=1}^{\infty}$  such that  $(f_{j_{\alpha}})_{\alpha \in I}$  converges to  $f$  in the sense of the topology  $\tau_0^*$  and, for every finite-dimensional vector subspace  $S$  of  $E$  with  $x_0 \in S$ , we have  $f|_{U \cap S}$  as an element of  $\mathcal{R}_{(m,n)}^{\infty}(U \cap S;F)$ .

PROOF - For every  $j = 1, 2, \dots$  there are  $P_j \in \mathcal{P}_m(E;F)$ ,  $Q_j \in \mathcal{P}_n(E;\mathbb{C})$

such that  $Q_j(x)f_j(x) = P_j(x)$  for every  $x \in U$  and  $Q_j$  is not identically zero in  $U$ . With no loss of generality we may take  $\|Q_j\|_\infty = 1$  for every  $j = 1, 2, \dots$ . Hence  $\|P_j\|_\infty \leq r$  for every  $j = 1, 2, \dots$ . By the generalized version of Montel's Theorem we can get  $f \in \mathcal{H}^\infty(U; F)$ ,  $P \in \mathcal{P}_m(E; F)$ ,  $Q \in \mathcal{P}_n(E; F)$  and a subnet  $(f_{j_\alpha})_{\alpha \in I}$  of  $(f_j)_{j=1}^\infty$  such that  $(f_{j_\alpha})_{\alpha \in I}$ ,  $(P_{j_\alpha})_{\alpha \in I}$  and  $(Q_{j_\alpha})_{\alpha \in I}$  converge respectively to  $f, P$  and  $Q$  in the sense of the topology  $\tau_0^*$ . It is clear that  $f(x)Q(x) = P(x)$  for every  $x \in U$ . If  $f$  is identically zero in  $U$  the lemma is already proved. If  $f$  is not identically zero we consider the sets  $A_j = \{x \in U; Q_j(x) \neq 0\}$ ,  $j = 1, 2, \dots$  and  $A = \{x \in U; f(x) \neq 0\}$ . These sets are open dense subsets of  $U$ . Hence, by Baire's Theorem,  $B = A \cap (\bigcap_{j=1}^\infty A_j)$  is dense in  $U$  and there is  $x_0 \in U$  such that  $f(x_0) \neq 0$  and  $Q_j(x_0) \neq 0$  for every  $j = 1, 2, \dots$ . If  $S$  is a finite dimensional vector subspace of  $E$  with  $x_0 \in S$ , then  $U \cap S$  is relatively compact in  $S$  and we have  $\|Q_j\|_{U \cap S} = \sup\{\|Q_j(x)\|; x \in U \cap S\} = \sup\{\|Q_j(x)\|; x \in \overline{U \cap S}\} = \|Q_j\|_{\overline{U \cap S}}$ . Since  $(Q_{j_\alpha})_{\alpha \in I}$  converges to  $Q$  for  $\tau_0^*$  in  $E$  we have  $(\|Q_{j_\alpha}\|_{\overline{U \cap S}})_{\alpha \in I}$  converging to  $\|Q\|_{\overline{U \cap S}}$ . By dividing  $Q_j$  and  $P_j$  by  $\|Q_j\|_{\overline{U \cap S}}$  we may consider  $\|Q_j\|_{\overline{U \cap S}} = 1$  for every  $j = 1, 2, \dots$ . It follows that  $\|Q\|_{\overline{U \cap S}} = 1$  and  $Q$  is not identically zero in  $U \cap S$ . Hence  $f|_{U \cap S}$  is an element of  $\mathcal{R}_{(m,n)}^\infty(U \cap S; F)$ .

Q.E.D.

We remark that the above proof does not provide us with  $Q$  not identically zero in  $U$  since we modified the  $Q_j$ 's when we divided them by  $\|Q_j\|_{\overline{U \cap S}}$ . Now we can prove the following results

4.2. THEOREM - When  $\dim(E) < \infty$

(i) If  $V$  is a vector subspace of  $\ell^\infty(U; F)$  containing



$\mathcal{R}_{(m,n)}^\infty(U;F)$ , then  $\mathcal{R}_{(m,n)}^\infty(U;F)$  has the relative Chebyshev center property (hence, it is proximal) in  $V$ .

(ii) If  $\mathcal{W}$  is a non-empty  $\tau_0^*$ -closed subset of  $\mathcal{R}_{(m,n)}^\infty(U;F)$  and  $V$  is a vector subspace of  $\ell^\infty(U;F)$  containing  $\mathcal{W}$ , then  $\mathcal{W}$  has the relative Chebyshev center property in  $V$ .

4.3. THEOREM - (i) If  $V$  is a vector subspace of  $\ell^\infty(U;\mathbb{C})$  containing  $\mathcal{R}_{(0,n)}^\infty(U;\mathbb{C})$  (respectively,  $\mathcal{R}_{(1,1)}^\infty(U;\mathbb{C})$ ), then  $\mathcal{R}_{(0,n)}^\infty(U;\mathbb{C})$  (respectively,  $\mathcal{R}_{(1,1)}^\infty(U;\mathbb{C})$ ) has the relative Chebyshev center property in  $V$ .

(ii) If  $\mathcal{W}$  is a  $\tau_0$ -closed non-empty subset of either  $\mathcal{R}_{(0,n)}^\infty(U;\mathbb{C})$  or  $\mathcal{R}_{(1,1)}^\infty(U;\mathbb{C})$  and  $V$  is a vector-subspace of  $\ell^\infty(U;\mathbb{C})$  containing  $\mathcal{W}$ , then  $\mathcal{W}$  has the relative Chebyshev center property in  $V$ .

PROOF OF THEOREM 4.2 - (i) follows from Corollary 2.4 since, when  $E$  has finite dimension, Lemma 4.1 implies that  $B_r = \{f \in \mathcal{R}_{(m,n)}^\infty(U;F); \|f\|_\infty \leq r\}$  is  $\tau_0^*$ -countably compact. Part (ii) follows from the fact that  $B_r \cap \mathcal{W} = \{f \in \mathcal{W}; \|f\|_\infty \leq r\}$  is  $\tau_0^*$ -countably compact.

PROOF OF THEOREM 4.3 - (i) will be proved as a consequence of Corollary 2.4 if we show that  $B_r = \{f \in \mathcal{R}_{(m,n)}^\infty(U;\mathbb{C}); \|f\|_\infty \leq r\}$  is  $\tau_0$ -countably compact when (a)  $m = 0$  and (b)  $m = n = 1$ .

Case (a): Let  $(f_j)_{j=1}^\infty$  be a sequence in  $B_r$ . By Lemma 4.1 we know that there are  $f \in \ell^\infty(U;\mathbb{C})$ ,  $x_0 \in U$ , a subnet  $(f_{j_\alpha})_{\alpha \in I}$  of  $(f_j)_{j=1}^\infty$  such that  $(f_{j_\alpha})_{\alpha \in I}$  converges to  $f$  in the sense of the topology  $\tau_0$  and, for every finite-dimensional vector subspace  $S$  of  $E$  with  $x_0 \in S$ , we have  $f|_{U \cap S} \in \mathcal{R}_{(0,n)}^\infty(U \cap S;\mathbb{C})$ . Since  $\|f\|_\infty \leq r$  it is enough to show that  $f \in \mathcal{R}_{(0,n)}^\infty(U;\mathbb{C})$ . If  $f = 0$  this is trivial. We suppose  $f \neq 0$ . For each above mentioned  $S$  we

can find  $c_S \in P_0(S; \mathbb{C}) = \mathbb{C}$  and  $Q^S \in P_n(S; \mathbb{C})$  such that  $f(x) = \frac{c_S}{Q^S(x)}$  for every  $x \in U \cap S$  and  $Q^S(x) \neq 0$  for every  $x \in U \cap S$ .

(See remark about  $R_{(0,n)}^\infty(U; F)$  made in the beginning of this paragraph). By examining the proof of Lemma 4.1 it is clear we may consider  $f(x_0) \neq 0$ . Hence  $c_S \neq 0$  for every  $S$  and we may consider  $c_S = 1$  for every  $S$ . We consider the Taylor series developments of  $Q_S$  and  $f$  around  $x_0$  and we write

$$Q^S = \sum_{j=0}^n Q_j^S$$

$$f = \sum_{j=0}^{\infty} f_j$$

where  $f_j \in P(j; E; \mathbb{C})$  for  $j \in \mathbb{N}$ ,  $Q_j^S \in P(j; S; \mathbb{C})$  for  $j = 0, 1, \dots, n$ , and the equality holds true in a neighborhood of  $x_0$  in  $U \cap S$ . Since  $f \cdot Q^S = 1$  in  $U \cap S$  the unicity of the Taylor series development implies that in  $S$  we have:

$$f_0 Q_0^S = 1$$

$$f_1 Q_0^S + f_0 Q_1^S = 0$$

.....

$$f_n Q_0^S + f_{n-1} Q_1^S + \dots + f_0 Q_n^S = 0$$

$$f_k Q_0^S + f_{k-1} Q_1^S + \dots + f_{k-n} Q_n^S = 0, \quad \text{for } k \geq n+1.$$

Hence, since  $f_0 = f(x_0) \neq 0$ , we have  $f_0^{n+1} \neq 0$  and the first  $n+1$  above equations have a unique solution  $Q_0^S, \dots, Q_n^S$  defined in  $S$  by expressions in terms of  $1, f_0, \dots, f_n$  (by the so-called Cramer's rule). If we define  $Q_0, \dots, Q_n$  in  $E$  by the same expressions (it makes sense to do it because  $f_1, \dots, f_n$  are defined in  $E$  and  $1, f_0$  are non-zero constants) we get  $Q = Q_0 + \dots + Q_n \in P_n(E; \mathbb{C})$



satisfying  $Q$  not identically zero in  $U$  and  $Q \cdot f = 1$  in  $U$  (since  $Q|_S \cdot f = Q^S \cdot f = 1$  in  $U \cap S$  for every  $S$ ). Thus  $f$  is in  $\mathcal{H}_{(0,n)}^\infty(U; \mathbb{C})$ .

Case (b): Let  $(f_j)_{j=1}^\infty$  be a sequence in  $B_r$ . By Lemma 4.1 we know that there are  $f \in \mathcal{H}^\infty(U; \mathbb{C})$ ,  $x_0 \in U$ , a subnet  $(f_{j_\alpha})_{\alpha \in I}$  of  $(f_j)_{j=1}^\infty$  such that  $(f_{j_\alpha})_{\alpha \in I}$  converges to  $f$  in the sense of the topology  $\tau_0$  and, for every finite dimensional vector subspace  $S$  of  $E$  with  $x_0 \in S$ , we have  $f|_{S \cap U}$  in  $\mathcal{H}_{(1,1)}^\infty(U; \mathbb{C})$ . Since  $\|f\|_\infty \leq r$  it is enough to show that  $f \in \mathcal{H}_{(1,1)}^\infty(U; \mathbb{C})$ . This is trivial if  $f$  is identically zero in  $U$ . We suppose that this is not the case. For every  $S$  we consider  $P^S, Q^S \in \mathcal{P}_1(S; \mathbb{C})$  such that  $f(x) \cdot Q^S(x) = P^S(x)$  for each  $x \in S \cap U$  and  $Q^S$  not identically zero in  $U \cap S$ . Now we consider the Taylor series developments of  $P^S, Q^S$  and  $f$  in a neighborhood of  $x_0$  in  $S \cap U$ :

$$P^S = P_0^S + P_1^S, \quad Q^S = Q_0^S + Q_1^S, \quad f = \sum_{j=0}^{\infty} f_j.$$

Here  $P_0^S, Q_0^S \in \mathbb{C}$ ,  $P_1^S, Q_1^S \in \mathcal{P}^1(S; \mathbb{C})$ ,  $f_j \in \mathcal{P}^j(E; \mathbb{C})$ ,  $j \in \mathbb{N}$ . By the unicity of the Taylor series development the equality  $P^S = Q^S \cdot f$  in a neighborhood of  $x_0$  in  $U \cap S$  implies the following equalities in  $S$ :

$$\begin{aligned} f_0 Q_0^S &= P_0^S \\ f_1 Q_0^S + f_0 Q_1^S &= P_1^S \\ f_j Q_0^S + f_{j-1} Q_1^S &= 0 \quad \text{for } j \geq 2 \end{aligned}$$

As we saw in the proof of Lemma 4.1 we can always consider  $f(x_0) = f_0 \neq 0$ . We have two possibilities to consider:

(1) For every finite dimensional vector subspace  $S$  of  $E$  such that  $x_0 \in S$  there is another such vector subspace  $S' \supset S$  satisfying  $P_0^{S'} = 0$ .

(2) There is a finite dimensional vector subspace  $S_0$  of  $E$  such that  $x_0 \in S_0$  and for every other such subspace  $S$  of  $E$ ,  $S \supset S_0$  we have  $P_0^S \neq 0$ .

In case (1), if  $p_0^{S'} = 0$  it follows that  $Q_0^{S'} = 0$  since  $f_0 \neq 0$ . Then  $f_0 Q_1^{S'} = p_1^{S'}$  and  $f_{j-1} Q_1^{S'} = 0$  for  $j \geq 2$ . Since  $Q^{S'} \neq 0$  and  $Q_0^{S'} = 0$ , we must have  $Q_1^{S'} \neq 0$  in an open dense subset of  $U \cap S'$ . Thus  $f_{j-1}$  is identically zero in this set and  $f_{j-1}|_{S'} = 0$  for  $j \geq 2$ . Therefore  $f$  is constant in  $S' \cap U$ . But under our hypothesis of case (1) it follows that  $f$  is constant in  $U$  and hence  $f \in \mathcal{R}_{(0,0)}^\infty(U; \mathbb{C}) \subset \mathcal{R}_{(1,1)}^\infty(U; \mathbb{C})$ .

In case (2) with no loss of generality we may suppose that  $p_0^S = 1$  for every finite dimensional vector subspace  $S$  of  $E$  containing  $S_0$ . It follows that  $Q_0^S = \frac{1}{f_0}$  and  $Q_1^S = \frac{p_1^S}{f_0} - \frac{f_1}{f_0^2}$  in  $S$ . If we replace

these values in the equations  $f_j Q_0^S + f_{j-1} Q_1^S = 0$  for  $j \geq 2$  we get in  $S$

$$\frac{f_{j-1}}{f_0} p_1^S = \frac{f_{j-1} \cdot f_1}{f_0^2} - \frac{f_j}{f_0}.$$

If for some  $j \geq 2$ ,  $f_{j-1} \neq 0$  in  $E$  we have  $f_{j-1}(x) \neq 0$  for every  $x$  in an open dense subset  $V$  of  $E$ . Hence

$$p_1^S(x) = \frac{f_1(x)}{f_0} - \frac{f_j(x)}{f_{j-1}(x)}$$

for every  $x \in S \cap V$ . For all those  $S$  such that  $S \cap V \neq \emptyset$  the right-hand side of the above equation defines a continuous function in an open dense subset  $V \cap S$  of  $U \cap S$  and (by the left-hand side) it has a continuous linear extension to  $S$  equals to  $p_1^S$ . Since the right-hand side is independent of the  $S$  we consider, by defining

$$p_1(x) = \frac{f_1(x)}{f_0} - \frac{f_j(x)}{f_{j-1}(x)}$$

for  $x \in V$  we get a continuous function in  $V$  which has a linear extension  $p_1 \in P(^1E; \mathbb{C})$ . We may take

$$Q_1 = \frac{p_1}{f_0} - \frac{f_1}{f_0^2} \in P(^1E; \mathbb{C})$$



and we get  $Q_1|_S = Q_1^S$ . Hence  $f \cdot (Q_1 + Q_0) = P_1 + P_0$  where  $Q_0 = \frac{1}{f_0}$  and  $P_0 = 1$  in  $U$  with  $Q_0 + Q_1 \neq 0$ . Hence  $f \in \mathcal{R}_{(1,1)}^\infty(U; \mathbb{C})$ .

If for any  $j \geq 2$  we have  $f_{j-1} = 0$  in  $E$ , then  $f$  is constant in  $U$  and it belongs to  $\mathcal{R}_{(0,0)}^\infty(U; \mathbb{C}) \subset \mathcal{R}_{(1,1)}^\infty(U; \mathbb{C})$ .

Q.E.D.

The question of density of rational functions in the set of holomorphic functions over compact subsets of Banach spaces was examined by Matos in [8]

##### 5. BEST NON-NORMABLE METRIC APPROXIMATION BY POLYNOMIAL OPERATORS

In this paragraph  $E, F$  and  $G$  are complex Banach spaces and  $F = G^*$ . We denote  $F_b(E; F)$  the complex vector space of all mappings from  $E$  into  $F$  which are bounded over the bounded subsets of  $E$ . The set of all bounded subsets of  $E$  is indicated by  $b(E)$ . If  $B \in b(E)$  and  $f \in F_b(E; F)$  we set

$$\|f\|_B = \sup \{ \|f(t)\| ; t \in B \}$$

The locally convex topology  $\tau_b$  in  $F_b(E; F)$  generated by the family of seminorms  $(\|\cdot\|_B)_{B \in b(E)}$  is metrizable. A corresponding metric defining this topology is given by

$$|f - g| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_{B_n}}{1 + \|f - g\|_{B_n}}$$

where  $(B_n)_{n=1}^{\infty}$  is an increasing sequence of elements of  $b(E)$  such that  $E = \bigcup_{n=1}^{\infty} B_n$  and every  $B \in b(E)$  is contained in some  $B_n$  (e.g.,  $B_n = \{x \in E; \|x\| \leq n\}$ ,  $n = 1, 2, \dots$ ). It is obvious that

this metric depends on the sequence  $(B_n)_{n=1}^{\infty}$  we take, but it is quite simple to see that all the results we are going to prove will be true for any one of these metrics. In order to simplify our writing we choose  $B_n = \{x \in E; \|x\| \leq n\}$ ,  $n = 1, 2, \dots$ . As it was shown in Example 2.1(b) (with  $p=1$ ) the function  $\varphi(f) = |f| = |f-0|$  for  $f \in F_b(E;F)$  is an element of  $S((F_b(E;F), \tau_b))$  with  $\text{diam}|\cdot| = 1$ . We denote by  $\tau_0^*$  the locally convex topology in  $F_b(E;F)$  generated by the seminorms  $p_{K,z}$ , where  $K$  is a compact subset of  $E$  and  $z \in G$  (see §3 where we first considered  $p_{K,z}$ ). Hence  $\tau_0^* \subset \tau_b$ . It is clear that the topology  $\omega^*$  in  $F_b(E;F)$  generated by the seminorms  $p_{\{x\},z}$  with  $x \in E$  and  $z \in G$  is such that  $\omega^* \subset \tau_0^*$ .

5.1. LEMMA. For  $r \in [0, 1)$  the set

$$\mathcal{D}_r = \{f \in F_b(E;F); |f| \leq r\}$$

is  $\omega^*$ -closed, hence  $\tau_0^*$ -closed in  $F_b(E;F)$ .

PROOF. First we suppose that there is  $f$  in the  $\omega^*$ -closure of  $\mathcal{D}_r$  not belonging to  $\mathcal{D}_r$ . Hence  $|f| > r$  and there is a net  $(f_\alpha)_{\alpha \in A}$  in  $\mathcal{D}_r$  converging to  $f$  for the  $\omega^*$  topology. We consider

$\rho = 2^{-1}(|f| - r) > 0$  and  $k \in \mathbb{N}$  such that

$$\sum_{n=1}^k 2^{-n} \frac{\|f\|_{B_n}}{1 + \|f\|_{B_n}} > r + \rho \quad (4)$$

If  $n \in \{1, \dots, k\}$  and  $\delta_n > 0$ , since

$$\|f\|_{B_n} = \sup\{|f(x)(t)|; x \in B_n, t \in G, \|t\| \leq 1\}$$

and

$$t \in \mathbb{R}^+ \longrightarrow \frac{t}{1+t} \in \mathbb{R}^+$$



is continuous and increasing, there are  $x_n \in B_n$ ,  $t_n \in G$ ,  $\|t_n\| \leq 1$  such that

$$2^{-n} \frac{\|f\|_{B_n}}{1 + \|f\|_{B_n}} - 2^{-n} \frac{|f(x_n)(t_n)|}{1 + |f(x_n)(t_n)|} < \delta_n. \quad (5)$$

Since  $\lim_{\alpha \in A} |f_\alpha(x_n)(t_n)| = |f(x_n)(t_n)|$ , for a given  $\rho_n > 0$  there is  $\alpha_n \in A$  such that  $\alpha \in A$ ,  $\alpha \geq \alpha_n$  implies

$$2^{-n} \frac{|f(x_n)(t_n)|}{1 + |f(x_n)(t_n)|} - 2^{-n} \frac{|f_\alpha(x_n)(t_n)|}{1 + |f_\alpha(x_n)(t_n)|} < \rho_n. \quad (6)$$

Hence, for  $\alpha \geq \alpha_n$ , it follows from (5) and (6) that

$$2^{-n} \frac{|f_\alpha(x_n)(t_n)|}{1 + |f_\alpha(x_n)(t_n)|} > 2^{-n} \frac{\|f\|_{B_n}}{1 + \|f\|_{B_n}} - (\rho_n + \delta_n). \quad (7)$$

Now if we consider  $\delta_n$  and  $\rho_n$  such that

$$\sum_{n=1}^k (\delta_n + \rho_n) < \rho$$

and  $\alpha_0 \in A$  such that  $\alpha_0 \geq \alpha_n$  for  $n = 1, \dots, k$ , it follows from (7) and (4) that

$$\sum_{n=1}^k 2^{-n} \frac{|f_\alpha(x_n)(t_n)|}{1 + |f_\alpha(x_n)(t_n)|} > r$$

for every  $\alpha \geq \alpha_0$ . Thus

$$\begin{aligned}
|f_\alpha| &\geq \sum_{n=1}^k 2^{-n} \frac{\|f_\alpha\|_{B_n}}{1 + \|f_\alpha\|_{B_n}} \\
&\geq \sum_{n=1}^k 2^{-n} \frac{|f_\alpha(x_n)(t_n)|}{1 + |f_\alpha(x_n)(t_n)|} > r
\end{aligned}$$

for all  $\alpha \geq \alpha_0$ . But this is impossible since  $f_\alpha \in \mathcal{D}_r$  for every  $\alpha \in A$ . Hence we must have  $f \in \mathcal{D}_r$ .

Q.E.D.

5.2. THEOREM. (a) If  $U$  is a vector subspace of  $F_b(E;F)$  containing  $P_m(E;F)$ , then  $P_m(E;F)$  has the Chebyshev center property (hence, it is proximal) in  $U$  relative to  $|\cdot|$ .

(b) if  $W$  is a  $\tau_0^*$ -closed subset of  $P_m(E;F)$  and  $U$  is a vector subspace of  $F_b(E;F)$  containing  $W$ , then  $W$  has the Chebyshev center property (hence, it is proximal) in  $U$  relative to  $|\cdot|$ .

PROOF. (a) will follow from Lemma 2.3 and Lemma 5.1 if we show that for each  $\tau_b$ -bounded subset  $B$  of  $F_b(E;F)$  and each  $r \in [0,1)$ , the set

$$K_r(B) = \bigcap_{Q \in B} \{P \in P_m(E;F); P \in Q + \mathcal{D}_r\}$$

is  $\tau_0^*$ -compact. Let  $k > 0$  be such that  $\frac{k}{1+k} > r$ . If  $P \in P_m(E;F)$  and  $Q \in B$  are such that  $\|P - Q\|_{B_j} \geq k$  for all  $j = 1, 2, \dots$ , then

$$|P - Q| \geq \sum_{j=1}^{\infty} 2^{-j} \frac{k}{1+k} = \frac{k}{1+k} > r$$

and  $P \notin Q + \mathcal{D}_r$ . Hence, if  $P \in K_r(B)$  and  $Q \in B$ , there is  $j \in \{1, 2, \dots\}$  such that



$$\|P - Q\|_{B_1} \leq \|P - Q\|_{B_j} \leq k.$$

It follows that

$$\|P\|_{B_1} \leq k + \|Q\|_{B_1} \leq k + \sup_{Q \in B} \|Q\|_{B_1} = k + C < +\infty$$

and

$$\sup_{P \in K_r(B)} \|P\|_{B_1} \leq k + C < +\infty.$$

Thus  $K_r(B)$  is contained in the closed ball of center 0 and radius  $k+C$  in  $P_m(E;F)$  with respect to the norm  $\|\cdot\|_{B_1}$ . We denote this ball by  $\mathcal{D}$ . We know that for every  $P \in P_m(E;F)$

$$P = \sum_{j=0}^m P_j \quad (P_j \in P({}^jE;F), j=0, \dots, m)$$

where

$$P_j(x) = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{P(\lambda x)}{\lambda^{j+1}} d\lambda$$

for every  $x \in B_1$  (see Nachbin [9]). It follows that

$$\|P_j\|_{B_1} \leq \|P\|_{B_1} \leq k + C$$

for all  $P \in \mathcal{D}$  and  $j = 0, \dots, m$ . Hence

$$\sup_{P \in \mathcal{D}} \|P(x)\| \leq (k+C) \sum_{j=0}^m \|x\|^j < +\infty$$

and  $\mathcal{D}(x) = \{P(x); P \in \mathcal{D}\}$  is  $\sigma(F,G)$ -relatively compact for every  $x \in E$ . If  $K$  is a compact subset of  $E$  and  $z \in G$  we have

$$\sup_{p \in \mathcal{D}} p_{K,z}(p) \leq (k + C) \|z\| \sum_{j=0}^m \sup_{t \in K} \|t\|^j < +\infty.$$

Hence  $\mathcal{D}$  is  $\tau_0^*$ -bounded. By the generalized Montel's Theorem  $\mathcal{D}$  is  $\tau_0^*$ -relatively compact in  $\mathcal{K}(E; (F, \sigma(F, G)))$ . In order to prove that  $\mathcal{D}$  is  $\tau_0^*$ -compact it is enough to show that  $\mathcal{D}$  is  $\tau_0^*$ -closed in  $\mathcal{K}(E; (F, \sigma(F, G)))$ . Let  $(p_\alpha)_{\alpha \in I}$  be a net in  $\mathcal{D}$   $\tau_0^*$ -convergent to

$f \in \mathcal{K}(E; (F, \sigma(F, G)))$ . We have  $\lim_{\alpha \in I} p_\alpha(x)(z) = f(x)(z)$  for every

$x \in E$  and  $z \in G$ . This implies that  $f$  is of the form

$$f(x) = \sum_{j=0}^m Q_j(x) \quad \forall x \in E$$

with  $Q_j(x) = A_j(x, \dots, x) = A_j x^j$ ,  $\forall x \in E$ , where  $A_j$  is a  $j$ -linear mapping from  $E^j$  into  $F$ . Hence  $f$  is a polynomial (not necessarily continuous) of degree less than or equal to  $m$ . Since  $|p_\alpha(x)(z)| \leq k + C$  for  $x \in B_1$ ,  $z \in G$ ,  $\|z\| = 1$ , we get  $|f(x)(z)| \leq k + C$  for  $x \in B_1$ ,  $z \in G$ ,  $\|z\| = 1$ . This means that  $\sup_{x \in B_1} \|f(x)\| \leq k + C$ . But we know that a polynomial bounded over the unit ball is continuous. Therefore  $f \in P_m(E; F)$  and  $f \in \mathcal{D}$ .

Since  $K_r(B)$  is contained in the  $\tau_0^*$ -compact subset  $\mathcal{D}$  of  $P_m(E; F)$  it is enough to prove that  $K_r(B)$  is  $\tau_0^*$ -closed in  $P_m(E; F)$  in order to show that  $K_r(B)$  is  $\tau_0^*$ -compact. If  $p \in P_m(E; F)$  is the  $\tau_0^*$ -limit of a net  $(p_\alpha)_{\alpha \in I}$  of  $K_r(B)$ , we have  $p_\alpha - q \in \mathcal{D}_r$  for every  $\alpha \in I$  and  $q \in B$ . Since  $\mathcal{D}_r$  is  $\tau_0^*$ -closed in  $F_b(E; F)$  by Lemma 5.1, it follows that  $p - q = \lim_{\alpha \in I} p_\alpha - q \in \mathcal{D}_r$ . Hence

$p \in K_r(B)$  as it was our objective.

(b) follows from Lemma 2.3, Lemma 5.1 and from the fact that for every  $B$   $\tau_b$ -bounded in  $F_b(E; F)$  and every  $r \in [0, 1)$ , the set



$$\bigcap_{Q \in \mathcal{B}} \{P \in \mathcal{W}; P \in Q + \mathcal{V}_r\} = \mathcal{W} \cap K_r(\mathcal{B})$$

is  $\tau_0^*$ -compact.

Q.E.D.

We note that for every  $n \leq m$  the vector subspace  $P(^nE; F)$  is  $\tau_0^*$ -closed in  $P_m(E; F)$ . Hence part (b) of Theorem 5.2 implies that  $P(^nE; F)$  is proximal in  $F_b(E; F)$  with respect to  $|\cdot|$ .

With the methods of this paper we cannot prove results of best approximation by holomorphic or rational mappings relative to  $|\cdot|$ . The problem is that, in general, the set of holomorphic mappings, corresponding to  $K_r, |\cdot|(\mathcal{B})$  of Lemma 2.3, is not  $\tau_0^*$ -compact.

## 6. BEST APPROXIMATION BY FINITE RANK OPERATORS

As we have considered before  $E, F$  and  $G$  are complex Banach spaces with  $F = G^*$  and  $U$  is a non-empty bounded open subset of  $E$ . In  $\ell^\infty(U; F)$  and in  $F_b(E; F)$  we consider their subsets  $\ell_N^\infty(U; F)$  and  $F_{b,N}(E; F)$  of all mappings whose images are contained in vector subspaces of  $F$  with finite dimension  $\leq N$ . Then we consider:

$$\mathcal{H}_N^\infty(U; F) = \mathcal{H}^\infty(U; F) \cap \ell_N^\infty(U; F)$$

$$P^N(^nE; F) = P(^nE; F) \cap \ell_N^\infty(U; F)$$

$$P_m^N(E; F) = P_m(E; F) \cap \ell_N^\infty(U; F)$$

$$\mathcal{R}_{(m,n),N}^\infty(U; F) = \mathcal{R}_{(m,n)}^\infty(U; F) \cap \ell_N^\infty(U; F)$$

We recall the following results

6.1. THEOREM -  $P_N(^1E; F) = L_N(E; F)$  is  $\tau_0^*$ -closed in  $P(^1E; F) = L(E; F)$ .

This result is due to K. Floret, see [5].

6.2. THEOREM - For an open subset  $U$  of  $E$  there are a complex Banach space  $H_U^\infty$  and  $\varepsilon_U \in \mathcal{K}^\infty(U; H_U^\infty)$  with the following universal property: for every complex Banach space  $H$  and every  $f \in \mathcal{K}^\infty(U; H)$  there is a unique  $T_f \in P({}^1H_U^\infty; H) = L(H_U^\infty; H)$  such that  $T_f \circ \varepsilon_U = f$ .

This result has been communicated to us by Jorge Mujica and it will be published later on.

It is easy to prove the following Corollary to these two Theorems:

6.3. COROLLARY:  $P^N({}^nE; F)$ ,  $P_m^N(E; F)$ ,  $\mathcal{R}_{(m,n),N}^\infty(U; F)$ ,  $\mathcal{K}_N^\infty(U; F)$  are  $\tau_0^*$ -closed in  $P({}^nE; F)$ ,  $P_m(E; F)$ ,  $\mathcal{R}_{(m,n)}^\infty(U; F)$ ,  $\mathcal{K}^\infty(U; F)$  respectively.

With this Corollary and Theorems 3.1, 4.1, 4.2, 5.2 we get immediately the following results

6.4. THEOREM - (i)  $P^N({}^nE; F)$  and  $P_m^N(E; F)$  have the relative Chebyshev center property in  $\ell^\infty(U; F)$ .

(ii)  $P^N({}^nE; F)$  and  $P_m^N(E; F)$  have the relative Chebyshev center property in  $F_b(E; F)$  with respect to  $|\cdot|$ .

(iii)  $\mathcal{R}_{(m,n),N}^\infty(U; F)$  has the relative Chebyshev center property in  $\ell^\infty(U; F)$  when  $\dim(E) < +\infty$ .

(iv)  $\mathcal{K}_N^\infty(U; F)$  has the relative Chebyshev center property in  $\ell^\infty(U; F)$ .

Part (i) of this theorem was proved by Roversi [12] for  $P^N({}^nE; F)$ ,  $n \in \mathbb{N}$ , and by Deutsch, Mach, Saatkamp [4] for  $P^N({}^1E; F)$  with direct proofs.



## REFERENCES

- [1] BARROSO, J.A.; MATOS, M.C.; NACHBIN, L. - On holomorphy versus linearity in classifying locally convex spaces. Infinite Dimensional Holomorphy and applications. North-Holland Mathematics Studies, vol. 12, Amsterdam, (1977), 31-74.
- [2] BOHEM, B. - Existence of best rational Tchebycheff approximations. Pacific Journal of Mathematics, vol. 15, Nº 1, (1965), 19-28.
- [3] CHENEY, E.W.; LOEB, H.L. - Generalized rational approximation, J. Soc. Indust. Appl. Math., Series B (Numerical Analysis) 1(1964), 11-25.
- [4] DEUTSCH, M.; MACH, J.; SAATKAMP, K. - Approximation by finite rank operators. J. Approx. Theory, 33 (1981), 199-213.
- [5] FLORET, K. - Elementos de posto finito em produtos tensoriais topológicos, 24º Seminário Brasileiro de Análise. São Paulo (1986) 189-196.
- [6] GARKAVI, A.L. - The best possible net and the best possible cross section of a set in normed spaces. Amer. Math. Soc. Translations 39 (1964), 111-132.
- [7] HOLMES, R.B.; KRIPKE, B.R. - Best approximation by compact operators. Indiana University Mathematics Journal, 21 (1971), 255-263.
- [8] MATOS, M.C. - Approximation of analytic functions by rational functions in Banach spaces. Journal of Functional Analysis, 56 (1984), 251-264.

[9] NACHBIN, L. - Topology on spaces of holomorphic mappings. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 47, Springer-Verlag, New York (1969).

[10] NEWMAN, D.L.; SHAPIRO, H.S. - Approximation by generalized rational functions. *Proceedings of Conference on Approximation*. Birkhäuser Verlag (1964).

[11] RICE, J.R. - On the existence of best Tchebycheff approximations by general rational functions, Abstract 63T-331. *Notices Amer. Math. Soc.* 10 (1963), 576.

[12] ROVERSI, M.S.M. - Best approximation by polynomial operators in Banach spaces. *Numer. Funct. Anal. and Optimiz.* 7(1), (1984-1985), 63-74.

[13] VALLÉE POUSSIN, Ch. DE LA. - Sur les polynômes d'approximation et la représentation approchée d'un angle. *Bull. acad. Belgique* (1910), 808-844.

[14] WALSH, J.L. - The existence of rational functions of best approximation. *Trans. Amer. Math. Soc.* 33 (1931), 668-689.