# A PARACONSISTENT LOGIC: $\mathbb{J}_3$

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**RELATÓRIO TÉCNICO Nº 12/87**

**ABSTRACT.** The three-valued paraconsistent modal logic $\mathbb{J}_3$ is introduced. We axiomatize $\mathbb{J}_3$ using the basic implication of the system, and suggest that paraconsistent logics are inconsistent only with respect to classical semantics, not with respect to their own formal or informal semantic notions. We give a "set-assignment semantics" for $\mathbb{J}_3$, prove a strong Completeness Theorem, and discuss its "truth-default semantics". This work is the Chapter IX of the book "The semantic Foundation of Logic, Vol. I, Propositional Logics", by R. L. Epstein.

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Março – 1987

# IX. A Paraconsistent Logic: $J_3$

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A paraconsistent logic is one in which non-trivial theories may include both a proposition and its negation. We will introduce the general notion of a paraconsistent logic and then analyze in detail the 3-valued paraconsistent logic $J_3$. We will axiomatize $J_3$ and in doing so will suggest that paraconsistent logics are inconsistent only with respect to classical semantics, not with respect to their own formal or informal semantic notions. An analysis of set-assignment semantics for $J_3$ will highlight the way in which the general framework for semantics of Chapter IV uses falsity as a default truth-value.

* The work of the both authors of this paper was supported in part by a grant of the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP - 84).

## Chapter IX: A Paraconsistent Logic

### A. Paraconsistent logics

How does one proceed when faced with apparently contradictory sentences both of which seem equally plausible?

'It is raining' and 'It is not raining'.

Taking $z = \{x : x \notin x\}$, '$z \in z$' and '$z \notin z$'.

The classical logician cannot incorporate both into a theory because from a proposition and its negation one can deduce in classical logic any other proposition of the semi-formal language. There is only one classically inconsistent formal theory, and that is the trivial one consisting of all wffs.

The classical logician resolves the matter, perhaps, by building separate theories based on first one and then the other proposition, comparing the consequences of each. Or he may say that the difficulty in the first pair is that the word 'raining' is vague, and he will strive to reach agreement on what that word means, making it sufficiently precise that one of the sentences is definitively true, the other false. But for the latter example such options won't work, and the only course left is to exclude them as being incoherent or place restrictions on what formulas define sets.

There is another tradition in logic, however, which embraces contradictions as either accurately representing reality, or as fruitful for study for the syntheses they may generate. In the words of the poet Whitman, 'Do I contradict myself? Very well then . . . . I contradict myself; I am large . . . . I contain multitudes.'

Jaśkowski, 1948, proposed constructing logical systems which would allow for non-trivial theories containing (apparent) contradictions. The motives for doing so, he said, were: to systematize theories which contain contradictions, particularly as they occur in dialectics; to study theories in which there are contradictions caused by vagueness; and to study empirical theories whose postulates or basic assumptions could be considered contradictory. He proposed the following problem.

> . . . : the task is to find a system of the sentential calculus which : 1) when applied to the contradictory systems would not always entail their [triviality] , 2) would be rich enough to enable practical inference, 3) would have an intuitive justification.
>
> Jaśkowski, 1948, p.145

Jaśkowski himself devised a propositional calculus to satisfy these criteria which he called 'discursive' .

Jaśkowski's work first appeared in Polish in 1948 and was translated into English only in 1969. Quite independently of him da Costa in 1963 had developed a sequence of logics called $C_n$ which allowed for non-trivial theories based on (apparent) contradictions. His motives were very similar to Jaśkowski's, and are described in da Costa, 1974, where he summarizes the systems, their extensions to first-order logic, and his investigations in set-theory based on them. Those logics are presented entirely syntactically: no explanations of the connectives are given, though we can assume that they are formal versions of 'not', 'if…then…', 'and', 'or' .

Due primarily to his influence much work has been done on these and other systems which allow for non-trivial theories which may contain (apparent) contradictions, dubbed paraconsistent logics by F. Miró Quesada. Arruda, 1980 and 1987, surveys this work and the history of the subject.

We have already seen a paraconsistent logic in this volume: Johansson's minimal intuitionistic logic J (§VII.E) . In this chapter we will study the paraconsistent logic $J_3$ which was first proposed by D'Ottaviano and da Costa, 1970, as a solution to Jaśkowski's problem, and which was later developed by D'Ottaviano, 1985 A,B and 1987 .

## B. The semantics of $J_3$

### 1. D'Ottaviano on the semantic basis of $J_3$

The semantic intuitions behind $J_3$ are described by D'Ottaviano and da Costa as follows.

> '... in the preliminary phase of the formulation of a theory (mathematical, physical, etc. ) contradictions can appear which, in the definitive formulation, are eliminated; 0, 1, ½ are the truth-values, where 0 represents the "false", 1 the "truth" and ½ the provisional value of a proposition A, such that A and ¬A are theses of the theory under consideration in its provisional formulation; in the definitive form of the theory, the value ½ will be reduced, at least in principle, to 0 or 1 ...
>
> The calculus $J_3$ can also be used as a foundation for inconsistent and non-trivial systems ... In this case, ½ represents the logical value of a formula which is, really, true and false at the same time.
>
> ... in the elaboration of a logic suitable to handle "exact concepts" and "inexact concepts" ... $J_3$ also constitutes a solution.'
>
> D'Ottaviano and da Costa, 1970 p.1351

Thus we will consider a 3-valued logic whose truth-values will be 0, ½, 1. However, unlike the other many-valued logics studied in this volume, two of the truth-values will be designated: 1 and ½. D'Ottaviano explains the idea behind this as follows:

> Łukasiewicz, in comparison, introduced the many valued logics $L_3$, $L_4$, ..., $L_\aleph$ but he required that only the value 1 represents truth. In fact, he didn't open up the possibility of characterizing more truth, or degrees or levels of truth. He characterized only different degrees of falsity. The

idea of absolute truth (the value 1) was maintained and, in general, this might not be the case in nature.

I believe that not only absolute truth (value 1) and absolute falsity (value 0) but different degrees, levels, or grades of truth and falsity must be assumed by the underlying logic for theories which represent reality.

$J_3$ has only 3 truth-values. The aim is to work with these three values, trying to understand the mechanism underlying the existence of two designated truth-values. But a further motivation is to generalize $J_3$ to logics with n designated truth-values and m undesignated.

If a sentence such as 'Chimpanzees can reason' is given value ½ in a model it is because we wish to treat is as a provisional truth. Its negation, however, is no less likely, probable, or reasonable to assume true. Hence it, too, is assigned value ½.

Differing from Łukasiewicz, a proposition which is possible is taken as suitable to proceed on as the basis of reasoning, to build theories with. But its negation is no less suitable.

From a classical perspective we would build one theory based on a proposition which is possible, and another on its negation, comparing the consequences of each. But here it is not a matter of knowing whether the proposition is true or false, or which of the proposition and its negation is most fruitful to be taken as the basis of a theory. Rather, as with paradoxical sentences, the proposition is neither absolutely true nor absolutely false, and it and its negation are inseparable. The appropriate methodology is to base one theory on both the proposition and its negation.

## 2. The truth-tables

The original presentation of $J_3$ by D'Ottaviano and da Costa, 1970, was in terms of 3 primitive connectives: negation, disjunction, and a possibility operator. In this section we will follow that approach in the main, differing only in taking conjunction rather than disjunction as primitive. In the next section we will give a presentation which reflects a radically different view of paraconsistency and the relation of $J_3$ to classical logic using quite different primitives. In that presentation the possibility operator will be derived as a meta-logical abbreviation, avoiding the question of whether it involves a use-mention confusion if used as a connective.

We begin by introducing a new symbol, ~, for negation, the reasons for which we will explain in §C,§D, and particularly in §G. The table for this <u>weak negation</u> is

| A | ~A |
|---|---|
| 1 | 0 |
| ½ | ½ |
| 0 | 1 |

This is the same table as for '¬' in $L_3$. But Łukasiewicz, taking ½ as non-designated, treats a proposition which is possible as provisionally false and its negation also as false. Here the import of the table is that both are treated as true.

The tables for conjunction and disjunction are

| | A∧B | B: 1 | ½ | 0 |
|---|---|---|---|---|
| | 1 | 1 | ½ | 0 |
| A | ½ | ½ | ½ | 0 |
| | 0 | 0 | 0 | 0 |

| | A∨B | B: 1 | ½ | 0 |
|---|---|---|---|---|
| | 1 | 1 | 1 | 1 |
| A | ½ | 1 | ½ | ½ |
| | 0 | 1 | ½ | 0 |

We may take either as primitive; we choose conjunction, and then

$$A \vee B \equiv_{Def} \sim(\sim A \wedge \sim B).$$

We also have that ~(~A∨~B) has the same table as A∧B. These two tables have their usual classical meanings in the sense that if the designated values 1 and ½ are replaced by T, and 0 is replaced by F then we have (with repetitions) the classical tables for ∧ and ∨.

Because of the significance of the notion of possibility in the semantic motivation, we symbolize it in the language with the operator ◇. Its table is

| A | ◇A |
|---|---|
| 1 | 1 |
| ½ | 1 |
| 0 | 0 |

We then define the necessity operator as $\Box A \equiv_{Def} \sim(\Diamond \sim A)$ with table

| A | □A |
|---|---|
| 1 | 1 |
| ½ | 0 |
| 0 | 0 |

Though all these tables above are the same as for $L_3$, their interpretations are quite different due to both 1 and ½ being designated. That difference makes the table for '→' of $L_3$ inappropriate here. We take instead

| | A→B | B: 1 | ½ | 0 |
|---|---|---|---|---|
| | 1 | 1 | ½ | 0 |
| A | ½ | 1 | ½ | 0 |
| | 0 | 1 | 1 | 1 |

Here we acknowledge that if the antecedent is false then A→B is definitely true; if the antecedent is true, absolutely or provisionally, and the consequent is false, then A→B is false. Hence the table is again classical in the sense that if 1 and ½ are replaced by T and 0 is replaced by F we have the classical table. Because of this we have that the rule of <u>modus ponens</u> is valid. The remaining cases are when the antecedent is true, either definitely or provisionally, and the consequent is provisionally true (possible). In that case it is correct to ascribe only provisional truth (possibility) to A→B.

We choose not to take → as primitive, defining it as

$A \to B \equiv_{Def} \sim(\Diamond A \wedge \sim B)$.

Several further connectives are important for $J_3$. First we define $A \leftrightarrow B \equiv_{Def} (A \to B) \wedge (B \to A)$. Its table is

| | | B | | |
|---|---|---|---|---|
| | $A \leftrightarrow B$ | 1 | ½ | 0 |
| | 1 | 1 | ½ | 0 |
| A | ½ | ½ | ½ | 0 |
| | 0 | 0 | 0 | 1 . |

If this seems puzzling, recall that a proposition which is provisionally true (possible) cannot be equivalent to a false one, while it can be provisionally equivalent to a true one.

In most paraconsistent logics, and in particular $J_3$ and da Costa's 1974 systems $C_n$, two negations are distinguished. The first we have already seen, ~, which is called weak negation. The other is (in this development) a defined connective, ¬, which is called strong or classical negation. In $J_3$ it is defined as $\neg A \equiv_{Def} \sim \Diamond A$ with table

| A | ¬A |
|---|---|
| 1 | 0 |
| ½ | 0 |
| 0 | 1 . |

This is classical in the sense that, unlike weak negation, the strong negation of a "true" proposition (one with designated value) is false, and of a false one is true.

This is not the first time in this volume that we have seen two different formalizations of a single English connective. All the modal logics of Chapter II used both '→' and '⊃'.

Finally, a metalogical abbreviation which is useful in axiomatizing $J_3$ is $\circledcirc A \equiv_{Def} \neg(\Diamond A \wedge \Diamond \sim A)$. This has table

| A | ⊚A |
|---|---|
| 1 | 1 |
| ½ | 0 |
| 0 | 1 |

With this we can assert that A has a classical (absolute) truth-value. A strained reading of this as a connective might be 'It is classically (absolutely) true or false that ...'. An alternative definition which could be used for ⊙A is □A ∨ □~A.

<u>Note on notation</u>: In <u>D'Ottaviano and da Costa, 1970</u>, and in D'Ottaviano's later work ◇A and □A are written as ▽A and △A, and → is written as ⊃. What we symbolize here as ~ is written there as ¬, and what we write as ¬ they symbolize as ¬*.

We now define for the language $L(p_0, p_1, \ldots \sim, \wedge, \Diamond)$ a $J_3$-<u>evaluation</u> to be a map $e: PV \to \{0, \frac{1}{2}, 1\}$ which is extended to all wffs by the tables above for ~, ∧, ◇. <u>The designated values are 1 and ½</u>, so that $e \vDash A$ means $e(A)=1$ or ½. And $\vDash A$ means that $e \vDash A$ for all $J_3$-evaluations e. Finally, <u>$\Gamma \vDash A$</u> means that for every $J_3$-evaluation e, if $e \vDash B$ for all $B \in \Gamma$, then $e \vDash A$. The collection of valid wffs we refer to as $J_3$, or the $J_3$-<u>tautologies</u>. When we compare this notion of validity and consequence to others we will subscript it as '$\vDash_{J3}$'.

Note that the Semantic Deduction Theorem holds in $J_3$:

$$\Gamma \cup \{A\} \vDash B \text{ iff } \Gamma \vDash A \to B .$$

## 3. <u>Interdefinability of the connectives</u>

The choice of which primitives are used in the development of $J_3$ and how we symbolize them strongly reflects the way in which we understand paraconsistency, the relation of $J_3$ to classical logic, and the adequacy of the general form of semantics of Chapter IV. Before we can explain why this is so, we need to know which choice of primitives we <u>can</u> use.

By 'definable' in the next theorem we mean the strong notion that for the connective in question there is a schema built from the other connectives which has the same 3-valued table. For instance, we show that a unary connective which is evaluated always as $\frac{1}{2}$ cannot be defined. However, there is a schema semantically equivalent to such a connective, e.g. $A \to A$. It always takes a designated value, and since both 1 and $\frac{1}{2}$ are designated the schema and connective are semantically equivalent.

<u>Theorem 1</u>: <u>i</u>. <u>From</u> $\sim, \wedge, \Diamond$ <u>we can define</u> $\vee, \to, \neg, \circledcirc$.

<u>ii</u>. <u>From</u> $\sim, \vee, \Diamond$ <u>we can define</u> $\wedge, \to, \neg, \circledcirc$.

<u>iii</u>. <u>From</u> $\neg, \to, \wedge, \sim$ <u>we can define</u> $\vee, \Diamond, \circledcirc$.

<u>iv</u>. $\sim$ <u>cannot be defined from</u> $\wedge, \vee, \to, \Diamond$.

<u>v</u>. $\sim$ <u>cannot be defined from</u> $\wedge, \vee, \to, \Diamond, \neg$.

<u>vi</u>. $\Diamond$ <u>cannot be defined from</u> $\sim, \wedge, \vee, \to$

<u>vii</u>. $\circledcirc$ <u>cannot be defined from</u> $\sim, \wedge, \vee, \to$

<u>viii</u>. $\neg$ <u>cannot be defined from</u> $\sim, \wedge, \vee, \to$

<u>ix</u>. <u>No schema built from</u> $\sim, \wedge, \Diamond$, <u>or any of the connectives above takes value</u> $\frac{1}{2}$ <u>only</u>.

<u>Proof</u>: i. Recall

$$A \vee B \equiv_{Def} \sim(\sim A \wedge \sim B)$$
$$A \to B \equiv_{Def} \sim(\Diamond A \wedge \sim B)$$
$$\neg A \equiv_{Def} \sim \Diamond A$$
$$\Box A \equiv_{Def} \sim \Diamond \sim A$$
and $$\circledcirc A \equiv_{Def} \Box A \vee \Box \sim A .$$

ii. Observe that $A \wedge B$ has the same table as $\sim(\sim A \vee \sim B)$.

iii. First note that for any B, $B \wedge \neg B$ takes value 0. Exploiting this we may define

$$\Diamond A \equiv_{Def} \neg(A \to (A \wedge \neg A))$$
and $$\circledcirc A \equiv_{Def} \neg[\neg(A \to A \wedge \neg A)) \wedge \neg(\sim A \to A \wedge \neg A)]$$

Note that the latter is $\neg(\Diamond A \wedge \Diamond \sim A)$.

iv. By a straightforward induction on the length of a schema S(A) built from $\wedge, \vee, \to, \Diamond$ we can show that if $e(A) = 1$ or $\frac{1}{2}$ then $e(S(A)) = 1$ or $\frac{1}{2}$. So $\sim$ cannot be defined.

v. Here we prove by induction on the length of a schema S(A) built from ∧, ∨, →, ◇, ¬ that for every A if e(A) = 1 or ½, then e(S(A)) = 1 or ½ or if e(A) = 1 or ½, then e(S(A)) = 0. That is, we cannot separate the values 1 and ½. So ~ cannot be defined.

vi. For any schema S(A) built from ~, ∧, ∨, →, if e(A) = ½, then e(S(A)) = ½.

vii. Were ◎ definable from ~, ∧, ∨, → then we could define □A as ~(◎A→~A) and ◇A as ~□~A, contradicting vi.

viii. Were ¬ definable from ~, ∧, ∨, → then by iii, ◇ would be, too, contradicting vi.

ix. If e is any $J_3$-evaluation such that e: PV→{0,1}, then its extension satisfies e: Wffs→{0,1}. So no connective taking only value ½ can be defined. ▪

There are two very different ways we may present $J_3$. The first is to use ~, ∧, ◇ or ~, ∨, ◇ as primitives. This is what we did above and is in accord with D'Ottaviano and da Costa's original motivation.

The alternative is to take ¬, →, ∧, ~ as primitives. As we will see in §C and §D, to do so is to view $J_3$ as an extension of classical logic. In what follows, we will assume that the definitions of validity and semantic consequence for $J_3$ are made with respect to either $L(p_0, p_1, \ldots \sim, \wedge, \Diamond)$ or $L(p_0, p_1, \ldots \neg, \to, \wedge, \sim)$ as appropriate to the discussion at hand. Though we now have two different semantic consequence relations for two distinct languages we will use the same symbol, '$\models_{J3}$' or '$\models$', for both: we view them as "the same logic" formulated in two different languages (see the discussion in §II.G.4 and X.E.5).

Note: We can also define a connective in $J_3$ whose table is that of → in $L_3$, namely $A \twoheadrightarrow B \equiv_{Def} (\Diamond{\sim}A \vee B) \wedge (\Diamond B \vee {\sim}A)$. Because of the definability of the tables for ∨, ◇ from ~ and ↠ which was shown in the discussion of $L_3$ (§VIII.C.1.a), we could take ~ and » as primitives for $J_3$. Besides being counter-intuitive, we have that $\{A, A \twoheadrightarrow B\} \vDash B$ is not a valid rule. Nonetheless, D'Ottaviano, 1985 A, has exploited this close relation between $L_3^*$ and $J_3$ to give an axiomatization of $J_3$ in $L(p_0, p_1, \ldots \sim, \twoheadrightarrow)$. However, it does not appear to be strongly complete.

C. The relation between $J_3$ and classical logic

The way we see the relation between $J_3$ and classical logic depends on how we understand the role of weak negation, ~, in $J_3$.

1. ~ as standard negation

If we identify ~ with negation in **PC**, then the fragment of $J_3$ in the language of ~,∧,∨,→ is contained in **PC**. This is because the $J_3$-tables for these connectives restricted to 0 and 1 are the classical tables, with 1 read as T, and 0 as F. However, this fragment ≠ **PC**. We have that

$$(A \wedge \sim A) \rightarrow B$$

is not a $J_3$-tautology, for A may take value ½ and B value 0. This is in accord with the design of $J_3$ as a paraconsistent logic, ensuring that $\{A, \sim A\} \vDash_{J3} B$ is not a valid rule.

Other noteworthy classical tautologies which fail to be $J_3$-tautologies when ~ is identified with negation in **PC**, are

$$\sim A \rightarrow (A \rightarrow B)$$
$$(A \rightarrow B) \rightarrow ((A \rightarrow \sim B) \rightarrow \sim A)$$
$$(A \rightarrow B) \rightarrow (\sim B \rightarrow \sim A)$$
$$((A \vee B) \wedge \sim A) \rightarrow B.$$

Perhaps surprisingly, however, both

$$\sim(A \wedge \sim A) \quad \text{and} \quad (A \vee \sim A)$$

are tautologies. Viewed individually each proposition apparently obeys these principles of bivalence, but not so in terms of its consequences. Instead of $(A \wedge {\sim}A) \to B$ we have

$$(A \wedge {\sim}A \wedge \circledcirc A) \text{ is a } J_3\text{-tautology;}$$

instead of $B \to (A \vee {\sim}A)$, we have

$$B \to (A \vee {\sim}A \vee \circledcirc A) \text{ is a } J_3\text{-tautology.}$$

So long as we restrict our attention to propositions which are "classical" we can reason classically in $J_3$. Define a map * from the language $L(p_0, p_1, \ldots \neg, \to, \wedge, \vee)$ of PC to the language $L(p_0, p_1, \ldots {\sim}, \wedge, \Diamond)$ of $J_3$.

$A^*$ is A with $\neg$ replaced by ~ everywhere,

with the understanding that $\vee$ and $\to$ are the defined connectives of $J_3$

<u>Theorem 2</u>: <u>The map</u>

$$A' = (\bigwedge_{(p_i \text{ in } A)} \circledcirc p_i) \to A^*$$

<u>is a translation of PC to $J_3$</u>.

That is, $\Gamma \models_{PC} A$ iff $\Gamma' \models_{J3} A'$.

<u>Proof</u>: $PC \models A$ iff for every PC-model $v$, $v(A)=T$

iff for every $J_3$-evaluation $e: PV \to \{0,1\}$, $e(A^*)=1$

iff for every $J_3$-evaluation $e$, $e(A')=1$,

the last line being because if $e(p) = \frac{1}{2}$ for some p in A, then the antecedent of $A'$ is evaluated to be 0. ▪

2. ¬ as standard negation

We may view $J_3$ as an extension of PC if we present it in the language of ¬, →, ∧, ~.

Theorem 3: The fragment of $J_3$ in the language of ¬, →, ∧ is PC.

That is, for such wffs $\Gamma \vDash_{PC} A$ iff $\Gamma \vDash_{J3} A$.

Proof: This follows from the observations we made in §B.2 that the tables for ¬, →, ∧ are the classical ones if we identify the designated values with T, and 0 with F. ■

From this point of view $J_3$ arises by adding an intensional connective, ~, to classical logic.

We also have that ~(~A∧~B) is semantically equivalent in $J_3$ to ¬(¬A∧¬B), which you can check.

D. Consistency vs. paraconsistency

1. Definitions of completeness and consistency for $J_3$ theories

A theory is consistent if it contains no contradiction. A proposition is a contradiction if it is false due to its form only, or semantically, if the corresponding wff is false in all models.

A theory is complete if it is as full a description as possible of "the way the world is" relative to the atomic propositions we've assumed and the semantics, informal or formal, which we employ. It might be inconsistent; but if not, it will contain as many complex propositions as possible relative to the atomic ones while still being consistent.

So a complete and consistent theory corresponds to a possible description of the world, relative to our semantic intuitions and choice of atomic propositions. Therefore, if we have formal semantics, a complete and consistent theory should correspond to the collection of propositions true in a model.

In classical logic we took the standard form for a contradiction to be $A \wedge \neg A$. A theory (collection of sentences closed under deduction) is then said to be consistent (with respect to classical logic) if for no A does it contain $A \wedge \neg A$; or equivalently, for no A does it contain both A and $\neg A$. This reflects the classical semantic assumption that for no A can both A and $\neg A$ be true.

If we understand negation to be formalized by ~, then this is clearly not applicable to $J_3$. And the associated criterion of consistency is inappropriate for $J_3$ in which we specifically assumed that it is acceptable to build a theory on the basis of both A and ~A. By the semantic assumptions of $J_3$, a theory which contains both A and ~A for some A is not necessarily contradictory, for it can reflect a possible way the world could be. To formulate an appropriate criterion of consistency for $J_3$ we first note the following.

<u>Lemma 4</u>: <u>If e is a $J_3$-evaluation, then</u>

$$e \models A \quad \text{iff} \quad e(A) = 1 \text{ or } \tfrac{1}{2}$$

$$e \models {\sim}A \quad \text{iff} \quad e(A) = 0 \text{ or } \tfrac{1}{2}$$

$$e \models \circledcirc A \quad \text{iff} \quad e(A) = 0 \text{ or } 1$$

Thus if all three of A, ~A, ©A are assumed by a theory then we can have no model of it. Such a theory is inconsistent: it would contain the contradiction $A \wedge {\sim}A \wedge \circledcirc A$.

In classical logic we took a theory to be complete if for every A it contains at least one of A, $\neg A$. That is, it must decide between these, and hence stipulate which is assumed to be true, or else embrace them both and be inconsistent.

That criteria is inappropriate for $J_3$ if we understand negation to be formalized by ~ : by choosing one of A,~A we have not stipulated the truth-value of A . To do that we need to choose two of A,~A,⊗A .

Definition: Γ is consistent relative to $J_3$ if for every A at most two of A,~A,⊗A are syntactic consequences of Γ.

Γ is complete relative to $J_3$ if for every A at least two of A,~A,⊗A are in Γ.

Here the notion of syntactic consequence for $J_3$ is either one of those which are formalized in §E which, with hindsight, we know are strongly complete.

In Theorem 7 we demonstrate that these are the appropriate definitions for $J_3$: Γ is complete and consistent relative to $J_3$ iff there is a $J_3$-evaluation which validates exactly Γ.

However, we may view $J_3$ formulated in the language of ¬,→,∧,~ as an extension of classical logic (Theorem 3). In that case the standard way to formalize 'not' is with ¬, and a theory is classically consistent means that at most one of A,¬A is a consequence of Γ; a theory is classically complete means that at most one of A,¬A is in Γ. Because we have

$$\models_{J3} (A \wedge {\sim}A \wedge \otimes A) \leftrightarrow (A \wedge \neg A)$$

we can demonstrate in the next section that for $J_3$-theories, Γ is complete and consistent relative to $J_3$ iff Γ is classically complete and consistent.

## 2. The status of negation in $J_3$

Relative to the connectives {∧,∨,→}, weak negation, ~, and strong negation, ¬, are both primitive (Theorem 1).

The approach favored by paraconsistent logicians is to view weak negation as primitive. Then possibility is formalized as a connective, with appropriate deference made to the use-mention controversy surrounding that decision. Strong negation, ¬, is taken as merely a defined connective. Only under favorable circumstances where are all the atomic propositions under discussion are classically (absolutely) true or false can we view negation, ~, as classical (Theorem 2). This is reflected by an alternative definition we can give of ¬A as ~A∧◇A. We may interpret 'not' as ¬ only for those propositions which satisfy □A∨□~A.

From a more classical point of view we might argue that we never suggested that all logically significant uses of 'not' can be properly modeled by classical negation. The simple example of two assertions about a die, 'Three faces are even numbered', and 'Three faces are not even numbered', should convince us of that. The symbol ¬ should be reserved for formalizing 'not' in those cases where the proposition and the proposition with 'not' deleted cannot both be true (at the same time), and this is the analysis given in Chapter IV. We may choose to introduce a new connective, ~, to formalize other uses of 'not' where the truth of the proposition and the truth of the proposition with 'not' deleted are (apparently) inseparable. Such propositions might be ones using vague terms, such as 'It is not raining', or paradoxical ones such as 'This sentence is not true.' The division of logical uses of 'not' is the same as the paraconsistent logician's, but comes from a very different perspective.

Paraconsistent logics contain "inconsistent non-trivial theories" only from the application of classical criteria to an admittedly non-classical connective, ~. From their own semantic point of view such theories are consistent, corresponding to a possible description of the world. To call a theory inconsistent if it contains, for some A, both A and ~A is tantamount to understanding negation in its usual sense as assumed by all other logics in this volume: it cannot be that both A and ~A are true. That cannot be how we understand ~ in paraconsistent logics, for it would preclude building a theory based on both A and ~A.

For $J_3$ this strong view of negation is expressed by the table for $\neg$. Using this connective the classical criterion of consistency is apt, and we cannot have a non-trivial inconsistent theory: $\vDash_{J3} (A \wedge \neg A) \rightarrow B$.

We will continue this discussion of the status of negation in §F and §G.

## E. Axiomatizations of $J_3$

In this section we will give two axiomatizations. In the first we take the viewpoint of the paraconsistent logician that ~ is the standard interpretation of 'not', and treat $J_3$ as a modal logic. Strong negation, $\neg$, does not appear in the axiomatization, and the only notions of completeness and consistency used are relative to $J_3$.

In the second axiomatization we view $J_3$ as an extension of classical logic in the language of $\neg, \rightarrow, \wedge$ with an intensional connective, ~. The notions of completeness and consistency will be the classical ones.

<u>For both axiomatizations there are no inconsistent non-trivial theories.</u>

## Chapter IX: A Paraconsistent Logic

### 1. As a modal logic

$$J_3$$

in $L(p_0, p_1, \ldots \sim, \wedge, \Diamond)$

$A \vee B \equiv_{Def} \sim(\sim A \wedge \sim B)$ ; $\Box A \equiv_{Def} \sim\Diamond\sim A$

$A \to B \equiv_{Def} \sim\Diamond A \vee B$ ; $\circledcirc A \equiv_{Def} \neg(\Diamond A \wedge \Diamond\sim A)$

$A \leftrightarrow B \equiv_{Def} (A \to B) \wedge (B \to A)$

axiom schemas

1. $B \to (A \to B)$
2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
3. $(B \to (A \to C)) \to ((A \wedge B) \to C)$
4. $A \to (B \to (A \wedge B))$
5. $(A \wedge \sim A \wedge \circledcirc A) \to B$
6. $((\sim A \wedge \circledcirc A) \to A) \to A$
7. $\sim\sim A \leftrightarrow A$
8. $\circledcirc A \leftrightarrow \circledcirc \sim A$
9. $\sim\Diamond A \leftrightarrow (\sim A \wedge \circledcirc A)$
10. $\circledcirc(\Diamond A)$
11. $(\sim(A \wedge B) \wedge \circledcirc(A \wedge B)) \wedge B \to (\sim A \wedge \circledcirc A)$
12. $(\sim A \wedge \circledcirc A) \to [\sim(A \wedge B) \wedge \circledcirc(A \wedge B)]$
13. $(A \wedge B) \wedge \circledcirc(A \wedge B) \leftrightarrow (A \wedge \circledcirc A) \wedge (B \wedge \circledcirc B)$

rule $\dfrac{A, A \to B}{B}$

We denote by $\vdash_{J3,\Diamond}$ the consequence relation of this axiom system. In this section only (§E.1) we will write $\vdash$ for $\vdash_{J3,\Diamond}$.

Recall from the last section that $\Gamma$ is consistent relative to $J_3$ ($J_3$-consistent) if for every A at most two of $A, \sim A, \circledcirc A$ are consequences of $\Gamma$. $\Gamma$ is complete relative to $J_3$ ($J_3$-complete) if for every A at least two of $A, \sim A, \circledcirc A$ are in $\Gamma$. As usual, $\Gamma$ is a theory if $\Gamma$ is closed under deduction.

Lemma 5: i. (The Syntactic Deduction Theorem)

$\Gamma \cup \{A\} \vdash B$ iff $\Gamma \vdash A \rightarrow B$

ii. $\vdash A \wedge B \rightarrow B$

iii. $\vdash A \wedge B \rightarrow B \wedge A$

iv. $\Gamma \cup \{A, B\} \vdash A \wedge B$

v. $\Gamma \cup \{A, B\} \vdash C$ iff $\Gamma \vdash (A \wedge B) \rightarrow C$

Proof: The proof of part i is as for PC (Theorem II.8) due to the presence of axioms 1 and 2.

Part ii follows from axiom 3 taking B for C, and axiom 1. Part iii uses axiom 3 taking $B \wedge A$ for C. Part iv uses axiom 4 and part v uses axioms 3 and 4. ■

Lemma 6: i. $\Gamma$ is $J_3$-inconsistent iff for every B, $\Gamma \vdash B$.

ii. If $\Gamma$ is $J_3$-complete and $J_3$-consistent and $A \notin \Gamma$, then for every B, $\Gamma \cup \{A\} \vdash B$.

iii. If $\Gamma$ is $J_3$-complete and $J_3$-consistent, then $\Gamma$ is a theory.

iv. If $\Gamma \nvdash A$ then $\Gamma \cup \{\sim A, \circledcirc A\}$ is $J_3$-consistent.

v. If $\Gamma$ is $J_3$-consistent, then one of $\Gamma \cup \{A, \sim A\}$, $\Gamma \cup \{A, \circledcirc A\}$, $\Gamma \cup \{\sim A, \circledcirc A\}$ is consistent.

Proof: i. From right to left is immediate.

In the other direction, if $\Gamma$ is $J_3$-inconsistent then for some A, $\Gamma \vdash A$, $\Gamma \vdash \sim A$, and $\Gamma \vdash \circledcirc A$ , Hence by axiom 5, for every B, $\Gamma \vdash B$ .

ii. This is immediate from i.

iii. Suppose $\Gamma$ is $J_3$-complete and $J_3$-consistent and $\Gamma \vdash A$ . If $A \notin \Gamma$ then by completeness $\Gamma \cup \{A\}$ is $J_3$-inconsistent. But $Th(\Gamma) = Th(\Gamma \cup \{A\})$ so

$\Gamma$ is $J_3$-inconsistent, a contradiction. So $A \in \Gamma$.

iv. Suppose $\Gamma \cup \{\sim A, \circledcirc A\}$ is $J_3$-inconsistent. Then $\Gamma \cup \{\sim A, \circledcirc A\} \vdash A$. So by Lemma 5, $\Gamma \vdash (\sim A \wedge \circledcirc A) \rightarrow A$, and hence by axiom 6 and Lemma 5, $\Gamma \vdash A$.

v. Suppose $\Gamma \cup \{\sim A, \circledcirc A\}$ is $J_3$-inconsistent. Then $\Gamma \vdash (\sim A \wedge \circledcirc A) \rightarrow A$, hence by axiom 6, $\Gamma \vdash A$. Now suppose $\Gamma \cup \{A, \circledcirc A\}$ is also $J_3$-inconsistent. Then by axiom 7, $\Gamma \cup \{\sim\sim A, \circledcirc A\}$ is $J_3$-inconsistent. Hence $\Gamma \vdash (\sim\sim A \wedge \circledcirc A) \rightarrow \sim A$, so using axiom 6, $\Gamma \vdash \sim A$. Hence, $\Gamma \cup \{A, \sim A\}$ is $J_3$-consistent. ∎

<u>Theorem 7</u>: <u>$\Gamma$ is $J_3$-complete and $J_3$-consistent</u>

iff i. <u>there is some $J_3$-evaluation e such that $\Gamma = \{A : e \vDash A\}$</u>

iff ii. <u>there is some $J_3$-evaluation e such that</u>

$e(A) = 1$ iff $A, \circledcirc A \in \Gamma$

$e(A) = \frac{1}{2}$ iff $A, \sim A \in \Gamma$

$e(A) = 0$ iff $\sim A, \circledcirc A \in \Gamma$

iff iii. <u>there is some $J_3$-evaluation e such that</u>

$e(A) = 1$ iff $\sim A \notin \Gamma$

$e(A) = \frac{1}{2}$ iff $\circledcirc A \notin \Gamma$

$e(A) = 0$ iff $A \notin \Gamma$.

<u>Proof</u>: The equivalence of i, ii, and iii comes from Lemma 4. We will show that $\Gamma$ is $J_3$-complete and $J_3$-consistent iff ii.

First suppose an e as in ii exists. Then for every A exactly two of $A, \sim A, \circledcirc A \in \Gamma$. Hence $\Gamma$ is $J_3$-complete. By the equivalence of i and ii, the Semantic Deduction Theorem, and the fact that all the axioms are $J_3$-tautologies, $\Gamma$ is a theory. Hence $\Gamma$ is $J_3$-consistent.

Now suppose $\Gamma$ is $J_3$-complete and $J_3$-consistent. Then by Lemma 6.iii, $\Gamma$ is a theory. Define e as in ii. It remains to show that e is a $J_3$-evaluation.

$e(\sim A) = 1$ iff $\sim A, \circledcirc(\sim A) \in \Gamma$

iff $\sim A, \circledcirc A \in \Gamma$ by axiom 8

iff $e(A) = 0$.

$e(\sim A) = \frac{1}{2}$ iff $\sim A, \sim\sim A \in \Gamma$

iff $\sim A, A \in \Gamma$ by axiom 7

iff $e(A) = \frac{1}{2}$.

Therefore, by process of elimination, $e(\sim A) = 0$ iff $e(A) = 1$, so e evaluates ~ correctly.

$e(\Diamond A) = 0$ iff $\sim\Diamond A, \circledcirc(\Diamond A) \in \Gamma$

iff $\sim A, \circledcirc A \in \Gamma$ by axioms 9 and 10

iff $e(A) = 1$.

Since $\Gamma$ is a theory, by axiom 10 $\circledcirc(\Diamond A) \in \Gamma$, so we cannot have $e(\Diamond A) = \frac{1}{2}$. Thus e evaluates $\Diamond$ correctly.

Finally, we turn to conjunction.

$e(A \wedge B) = 0$ iff $\sim(A \wedge B), \circledcirc(A \wedge B) \in \Gamma$.

If $e(A \wedge B) = 0$ and $e(B) \neq 0$, then $B \in \Gamma$. So by axiom 11, $\sim A \wedge \circledcirc A \in \Gamma$, so $e(A) = 0$.

If $e(A) = 0$, then $\sim A, \circledcirc A \in \Gamma$, so by axiom 12 $\sim(A \wedge B), \circledcirc(A \wedge B) \in \Gamma$, so $e(A \wedge B) = 0$. Using Lemma 5.iii, the same reasoning establishes that if $e(B) = 0$ then $e(A \wedge B) = 0$.

$e(A \wedge B) = 1$ iff $A \wedge B, \circledcirc(A \wedge B) \in \Gamma$

iff $A, \circledcirc A$, and $B, \circledcirc B \in \Gamma$ by axiom 13

iff $e(A) = 1$ and $e(B) = 1$.

By process of elimination $e(A \wedge B) = \frac{1}{2}$ iff neither of $e(A), e(B) = 0$ and not both $e(A), e(B) = 1$. Hence e evaluates $\wedge$ correctly. ∎

<u>Lemma 8: If $\Delta \nvdash A$ then there is some $J_3$-complete and $J_3$-consistent theory $\Gamma$ such that $\Delta \subseteq \Gamma$ and $A \notin \Gamma$.</u>

<u>Proof</u>: Suppose $\Delta \nvdash A$.

Define $\Gamma_0 = \Delta \cup \{\sim A, \circledcirc A\}$.

This is $J_3$-consistent by Lemma 6.iv. Numbering the wffs of the language as $A_0, A_1, \ldots$ define

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{A_n, \sim A_n\} & \text{if this is } J_3\text{-consistent; if not then} \\ \Gamma_n \cup \{A_n, \circledcirc A_n\} & \text{if this is } J_3\text{-consistent; if not then} \\ \Gamma_n \cup \{\sim A_n, \circledcirc A_n\} & \end{cases}$$

$\Gamma = \bigcup_n \Gamma_n$

By Lemma 6.v, for every n, $\Gamma_{n+1}$ is $J_3$-consistent. Hence $\Gamma$ is $J_3$-consistent, and by construction it is $J_3$-complete. As $\Gamma_0 \subseteq \Gamma$, $A \notin \Gamma$. ■

<u>Theorem 9</u>: (Strong completeness of $\vdash_{J3,\Diamond}$)

$$\Gamma \vdash_{J3,\Diamond} A \text{ iff } \Gamma \models_{J3} A .$$

The proof is standard using Theorem 7 and Lemma 8 .

## <u>2. As an extension of classical logic</u>

In Chapter II.K.4 we gave an axiomatization of PC in the language of $\neg, \to, \wedge, \vee$ . You can check that the first seven axioms and the rule of that system give a strongly complete axiomatization of PC in the language of $\neg, \to, \wedge$ . If we allow any formula of the language $L(p_0, p_1, \ldots \neg, \to, \wedge, \sim)$ to be an instance of A, B, or C in those schema, then we have <u>an axiomatization of PC based on $\neg, \to, \wedge$ in the language of $J_3$</u> .

## $J_3$

in $L(p_0, p_1, \ldots \neg, \to, \wedge, \sim)$

$©A \equiv_{Def} \neg[\neg(A \to (A \wedge \neg A)) \wedge \neg(\sim A \to (A \wedge \neg A))]$

$\Diamond A \equiv_{Def} \neg(A \to (A \wedge \neg A))$

axiom schemas PC based on $\neg, \to, \wedge$

1. $(\sim A \wedge ©A) \leftrightarrow \neg A$
2. $\sim\sim A \leftrightarrow A$
3. $©(\neg A)$
4. $(A \wedge B) \wedge ©(A \wedge B) \leftrightarrow (A \wedge ©A) \wedge (B \wedge ©B)$
5. $(\sim A \wedge ©A) \to ©(A \to B)$
6. $(B \wedge ©B) \to ©(A \to B)$

rule $\dfrac{A, A \to B}{B}$

We denote by $\vdash_{J3,\neg}$ the consequence relation of this axiom system. In this section only (§E.2) we will write $\vdash$ for $\vdash_{J3,\neg}$.

We call a collection of wffs $\Gamma$ classically consistent if not both $A, \neg A$ are consequences of $\Gamma$. We say that $\Gamma$ is classically complete and consistent if for every $A$, exactly one of $A, \neg A$ is in $\Gamma$.

Throughout the following we will liberally use results from PC, justified by this axiomatization. In particular, the Syntactic Deduction Theorem holds.

Lemma 10: i. $\Gamma$ is classically consistent iff for every $A$ at most two of $A, \sim A, ©A$ are in $\Gamma$.

ii. $\Gamma$ is classically complete and consistent iff for every $A$ exactly two of $A, \sim A, ©A$ are in $\Gamma$.

Proof: i. $\Gamma$ is classically inconsistent iff (by PC) $\Gamma \vdash B$ for every $B$.

So if $\Gamma$ is classically inconsistent, for every A, all three of A, ~A, ©A are consequences of $\Gamma$. If for some A all three of A, ~A, ©A are consequences of $\Gamma$, then by axiom 1 both A and $\neg A$ are consequences of $\Gamma$, hence $\Gamma$ is classically inconsistent.

ii. Suppose $\Gamma$ is classically complete and consistent. If $A \notin \Gamma$ then $\neg A \in \Gamma$, so by axiom 1, ~A, ©A $\in \Gamma$. If $A \in \Gamma$, then suppose ©A $\notin \Gamma$. In that case $\neg$©A $\in \Gamma$. Using PC and the definition of ©A, we have $\vdash(A \wedge \neg$©A$) \rightarrow$ ~A. So ~A $\in \Gamma$. ■

Lemma 6 as above now follows if we replace '$J_3$-consistent' by 'classically consistent' and '$J_3$-complete and $J_3$-consistent' by 'classically complete and consistent.' We only need to establish $\vdash(($~A$\wedge$©A$)\rightarrow A) \rightarrow A$. But by PC $\vdash(\neg A \rightarrow A) \rightarrow A$ and $\vdash[((B \rightarrow C) \wedge ((C \rightarrow D) \rightarrow D)] \rightarrow ((B \rightarrow D) \rightarrow D)$, so using axiom 1 we have the desired theorem.

We now prove the analogue to Theorem 7 above.

<u>Theorem 11</u>: <u>$\Gamma$ is classically complete and consistent</u>

iff i. <u>there is some $J_3$-evaluation e such that</u> $\Gamma = \{A : e \models A\}$

iff <u>ii</u>. <u>there is some $J_3$-evaluation e such that</u>

$e(A) = 1$ iff A, ©A $\in \Gamma$

$e(A) = \frac{1}{2}$ iff A, ~A $\in \Gamma$

$e(A) = 0$ iff ~A, ©A $\in \Gamma$

iff <u>iii</u>. <u>there is some $J_3$-evaluation e such that</u>

$e(A) = 1$ iff ~A $\notin \Gamma$

$e(A) = \frac{1}{2}$ iff ©A $\notin \Gamma$

$e(A) = 0$ iff $A \notin \Gamma$ .

<u>Proof</u>: Because of Lemma 10 the proof follows as for Theorem 7. We only need to establish that the connectives are evaluated correctly if e is defined as in ii.

We will leave to you to check that by PC and axiom 2 we have

$\vdash \circ A \leftrightarrow \circ{\sim}A$ . So the proof that ~ is evaluated correctly is the same as in Theorem 7.

$e(A) = 0$ iff ${\sim}A, \circ A \in \Gamma$

iff $\neg A \in \Gamma$ by axiom 1

iff $e(\neg A) = 1$ by axiom 3 .

Now we will show that if $e(\neg A) \neq 1$ then $e(\neg A) = 0$ . So suppose that $e(\neg A) \neq 1$ . Then one of $\neg A, \circ\neg A \notin \Gamma$ ; so by axiom 3, $\neg A \notin \Gamma$, so $e(\neg A) = 0$. Thus $\neg$ is evaluated correctly.

$e(A \wedge B) = 0$ iff ${\sim}(A \wedge B), \circ(A \wedge B) \in \Gamma$

iff $\neg(A \wedge B) \in \Gamma$ by axiom 1

iff (by PC) $A \notin \Gamma$ or $B \notin \Gamma$ .

iff $e(A) = 0$ or $e(B) = 0$ .

$e(A \wedge B) = 1$ iff $A \wedge B, \circ(A \wedge B) \in \Gamma$

iff $A, \circ A$, and $B, \circ B \in \Gamma$ by axiom 4

iff $e(A) = 1$ and $e(B) = 1$ .

So $\wedge$ is evaluated correctly, the other case following by process of elimination.

$e(A \to B) = 0$ iff ${\sim}(A \to B), \circ(A \to B) \in \Gamma$

iff $\neg(A \to B) \in \Gamma$ by axiom 1

iff $A \wedge \neg B \in \Gamma$ by PC

iff $A, \neg B \in \Gamma$ by PC

iff $e(A) \neq 0$ and ${\sim}B, \circ B \in \Gamma$ by axiom 1

iff $e(A) \neq 0$ and $e(B) = 0$ .

Suppose now $e(A \to B) = 1$ . Then $A \to B$, $\circ(A \to B) \in \Gamma$. Suppose $e(B) \neq 1$ . Then $B \notin \Gamma$, so $\neg B \in \Gamma$. But by PC $\vdash ((A \to B) \wedge \neg B) \to \neg A$ so $\neg A \in \Gamma$. And then by axiom 1, $e(A) = 0$ .

Suppose $e(A) = 0$ . Then $\neg A \in \Gamma$, so by PC, $A \to B \in \Gamma$ . And by axiom 5, $\circ(A \to B) \in \Gamma$, so $e(A \to B) = 1$ .

If $e(B) = 1$ then $B, \circ B \in \Gamma$, so by PC, $A \to B \in \Gamma$. And by axiom 6, $\circ(A \to B) \in \Gamma$, so $e(A \to B) = 1$ .

Thus $\to$ is evaluated correctly, the other case following by process of elimination. ⊠

**Lemma 12**: If $\Delta \nvdash A$ then there is a classically complete and consistent $\Gamma$ such that $\Delta \subseteq \Gamma$ and $A \notin \Gamma$.

The proof is as for **PC**.

Now it's routine to prove the following using Theorem 11 and Lemma 12.

**Theorem 13**: (Strong completeness of $\vdash_{J3,\neg}$)

$$\Gamma \vdash_{J3,\neg} A \text{ iff } \Gamma \vDash_{J3} A.$$

Note: For both axiomatizations the rule of substitution of provable equivalents, $\dfrac{\vdash A \leftrightarrow B}{\vdash C(A) \leftrightarrow C(B)}$, fails. We have

$\vDash_{J3} (A \leftrightarrow A) \leftrightarrow (B \leftrightarrow B)$ whereas $\nvDash_{J3} \sim(A \leftrightarrow A) \leftrightarrow \sim(B \leftrightarrow B)$

for the latter may fail when $e(A) = 1$ and $e(B) = \frac{1}{2}$. D'Ottaviano, 1985 a, defines a stronger equivalence

$$(A \equiv^* B) \equiv_{Def} (A \leftrightarrow B) \wedge (\sim A \leftrightarrow \sim B).$$

Then the relation $A \approx B$ iff $\vDash_{J3} (A \equiv^* B)$ defines a congruence relation on the set of wffs.

## F. Set-assignment semantics for $J_3$

In giving set-assignment semantics for $J_3$ we must decide which negation, $\sim$ or $\neg$, should be modeled by the usual set-assignment table for negation. In this section we will choose $\neg$, which is most in keeping with the discussion in §D.2 and Chapter IV. In the next section we will consider taking $\sim$ as primary, giving set-assignment semantics from that perspective.

Accordingly, in this section we first take as our language for $J_3$ $L(p_0, p_1, \ldots \neg, \rightarrow, \wedge, \sim)$.

We say that $\langle \upsilon, s \rangle$ is a $J_3$-model for $L(p_0, p_1, \ldots \neg, \rightarrow, \wedge, \sim)$ if

$\upsilon(\neg A)=T$ iff $\upsilon(A)=F$

$\upsilon(A \wedge B)=T$ iff $\upsilon(A)=T$ and $\upsilon(B)=T$

$\upsilon(A \rightarrow B)=T$ iff not both $\upsilon(A)=T$ and $\upsilon(B)=F$

$\upsilon(\sim A)=T$ iff $s(A) \neq S$

and

1. $s(p) \neq \emptyset$ iff $\upsilon(p)=T$
2. for all A, B $s(A) \subseteq s(B)$ or $s(B) \subseteq s(A)$
3. $s(A \wedge B) = s(A) \cap s(B)$
4. $s(A \rightarrow B) = \begin{cases} s(B) & \text{if } \emptyset \subset s(A) \subset S \\ \overline{s(A)} \cup s(B) & \text{otherwise} \end{cases}$
5. $s(\neg A) = \begin{cases} S & \text{if } s(A)=\emptyset \\ \emptyset & \text{otherwise} \end{cases}$
6. $s(\sim A) = \begin{cases} s(A) & \text{if } \emptyset \subset s(A) \subset S \\ \overline{s(A)} & \text{otherwise} \end{cases}$ .

From this point of view $J_3$ is classical logic with one wholly intensional connective, ~, added. To give a model we first assign each propositional variable (proposition) a truth-value (T or F), and then choosing any collection of sets linearly ordered by inclusion, assign one set to each propositional variable in accord with condition 1, $s(p) \neq \emptyset$ iff $\upsilon(p)=T$. Conditions 3-6 then allow an inductive definition of the set-assignment on wffs, and hence of the valuation on all wffs.

Which are the false propositions? Those with no content. If A has some content, but not the full content $S$, then both A and its weak negation, ~A, are true.

The proof of the following lemma is routine.

Lemma 14:

i. $\upsilon(A)=T$ iff $s(A)\neq\emptyset$

ii. $\upsilon(A\vee B)=T$ iff $\upsilon(A)=T$ or $\upsilon(B)=T$

iii. If there are only three possible content sets, $\emptyset\subset\mathcal{U}\subset\mathcal{S}$ then $s(A\vee B)=s(A)\cup s(B)$. Otherwise,

$$s(A\vee B)=\begin{cases} s(A)\cap s(B) & \text{if both } \emptyset\subset s(A)\subset\mathcal{S} \text{ and } \emptyset\subset s(B)\subset\mathcal{S} \\ s(A)\cup s(B) & \text{otherwise} \end{cases}$$

iv. $\upsilon(\Diamond A)=T$ iff $\upsilon(A)=T$

v. $$s(\Diamond A)=\begin{cases} \mathcal{S} & \text{if } s(A)\neq\emptyset \\ \emptyset & \text{if } s(A)=\emptyset \end{cases}$$

vi. $$s(\Diamond A)=\begin{cases} \mathcal{S} & \text{if } \upsilon(A)=T \\ \emptyset & \text{if } \upsilon(A)=F \end{cases}$$

vii. $\upsilon(\Diamond A)=F$ iff $\upsilon(A)=T$ and $s(A)=\mathcal{S}$

viii. Let $e$: Wffs $\rightarrow\{0,\frac{1}{2},1\}$ be a $J_3$-evaluation. Define $s(A)=\{x : x<e(A) \text{ and } x\in[0,1]\}$ and $\upsilon(p)=T$ iff $e(p)=1$ or $\frac{1}{2}$. Then $\langle\upsilon,s\rangle$ is a $J_3$-model and $\langle\upsilon,s\rangle\vDash A$ iff $e(A)=1$ or $\frac{1}{2}$.

ix. Given a $J_3$-model $\langle\upsilon,s\rangle$, define $e$: Wffs $\rightarrow\{0,\frac{1}{2},1\}$ by

$$e(A)=\begin{cases} 1 & \text{if } s(A)=\mathcal{S} \\ \frac{1}{2} & \text{if } \emptyset\subset s(A)\subset\mathcal{S} \\ 0 & \text{if } s(A)=\emptyset . \end{cases}$$

Then $e$ is a $J_3$-evaluation and $e(A)=1$ or $\frac{1}{2}$ iff $\langle\upsilon,s\rangle\vDash A$.

From parts viii and ix we have that the consequence relation for these set-assignment semantics is the same as for the $J_3$-matrix, which by Theorem 13 coincides with the syntactic consequence relation in this language.

Theorem 15: (Strong completeness of the set-assignment semantics)

$\Gamma\vdash_{J3} A$ iff every set-assignment $J_3$-model which validates $\Gamma$ also validates A

iff $\Gamma\vdash_{J3,\neg} A$.

If we take $J_3$ to be formulated in the language $L(p_0, p_1, \ldots \sim, \wedge, \Diamond)$ then we can define a $J_3$-model $\langle \upsilon, s \rangle$ to be one satisfying

$\upsilon(A \wedge B) = T$ iff $\upsilon(A) = T$ and $\upsilon(B) = T$

$\upsilon(\sim A) = T$ iff $s(A) \neq S$

$\upsilon(\Diamond A) = T$ iff $\upsilon(A) = T$

and

1. $s(p) \neq \emptyset$ iff $\upsilon(p) = T$
2. for all A, B $s(A) \subseteq s(B)$ or $s(B) \subseteq s(A)$
3. $s(A \wedge B) = s(A) \cap s(B)$
4. $s(\sim A) = \begin{cases} s(A) & \text{if } \emptyset \subset s(A) \subset S \\ \overline{s(A)} & \text{otherwise} \end{cases}$
5. $s(\Diamond A) = \begin{cases} S & \text{if } s(A) \neq \emptyset \\ \emptyset & \text{if } s(A) = \emptyset \end{cases}$

You can check that for these semantics we have

$\upsilon(\neg A) = T$ iff $\upsilon(A) = F$

$\upsilon(\sim A) = F$ iff $\upsilon(A) = T$ and $s(A) = S$

$\upsilon(A \to B) = T$ iff $\upsilon(A) = F$ or $\upsilon(B) = T$

$\upsilon(A \vee B) = T$ iff $\upsilon(A) = T$ or $\upsilon(B) = T$.

And as above we can establish the following theorem.

Theorem 16: $\Gamma \models_{J3} A$ iff every set-assignment $J_3$-model in $\sim, \wedge, \Diamond$ which validates $\Gamma$ also validates A iff $\Gamma \vdash_{J3,\Diamond} A$.

## G. Truth-default semantics

Let us take the paraconsistent logician's point of view that the English 'not' is to be formalized as $\sim$; $\neg$ is simply a defined connective which happens to correspond to classical negation. In that case there is no way to give set-assignment semantics in the format of the general framework of Chapter IV in such a way that we can translate $J_3$-evaluations to set-assignment models and vice-versa while satisfying $e \models A$ iff $\langle v,s \rangle \models A$. This is because we do not have that if $e \models A$ then $e \not\models \sim A$. The following table cannot be realized.

| A | N(A) | ~A |
|---|---|---|
| any value | fails | F |
| T | | F |
| F | holds | T |

The tables of the general framework of Chapter IV are based on the view that for a proposition to be true it must pass certain tests: if it fails any it is false. I have argued in Chapter IV and really throughout this volume, as well as in Epstein, 198?, that this is correct: we analyze what it means for a proposition to be true, and every proposition which is not true is false.

The semantic intuitions behind $J_3$ are a mirror image of this: we analyze what it means for a proposition to be false, and every proposition which is not false is true.

The general form of semantics of Chapter IV can be viewed as using falsity as the default truth-value. In $J_3$ truth is taken as the default truth-value. Previously we have said: we cannot have both A and its negation false. For $J_3$ we say: we cannot have both A and its negation true. It is precisely because these views are incompatible and yet are both represented in $J_3$, albeit one of them derivatively, that has led us to use a different symbol, ~, for this negation. The appropriate form of the set-assignment table for negation then is the followng.

(1)

| A | $\mathcal{N}(A)$ | $\sim A$ |
|---|---|---|
| any value | fails | T |
| T | | F |
| F | holds | T |

Because $\wedge, \vee$, and $\rightarrow$ are evaluated classically in $J_3$ they can be presented by tables which take either truth or falsity as the default value. Truth-default tables for these connectives have the following form.

(2)

| A | B | $\mathcal{B}(A,B)$ | $A \rightarrow B$ |
|---|---|---|---|
| any values | | fails | T |
| T | T | | T |
| T | F | holds | F |
| F | T | | T |
| F | F | | T |

(3)

| A | B | $\mathcal{C}(A,B)$ | $A \wedge B$ |
|---|---|---|---|
| any values | | fails | T |
| T | T | | T |
| T | F | holds | F |
| F | T | | F |
| F | F | | F |

(4)

| A | B | $\mathcal{A}(A,B)$ | $A \vee B$ |
|---|---|---|---|
| any values | | fails | T |
| T | T | | T |
| T | F | holds | T |
| F | T | | T |
| F | F | | F |

# Chapter IX: A Paraconsistent Logic

Set-assignment semantics using tables (1)-(4) we call truth-default set-assignment semantics (though we would normally use the symbol ¬ for the connective of table (1)). We will continue to call semantics of the general form of Chapter IV simply 'set-assignment semantics', but when there is a need to distinguish them from this alternate form we will call them falsity-default set-assignment semantics. Similar definitions apply for relation based semantics.

Consider now the set-assignment semantics for $J_3$ in the language of ~,∧,◇ given in the last section. These are truth-default semantics if we interpret ~ as the formalization of 'not' because the evaluation of the connectives can be expressed equivalently as

$$\nu(A \wedge B)=F \text{ iff } \nu(A)=F \text{ or } \nu(B)=F$$

$$\nu(\sim A)=F \text{ iff } \nu(A)=T \text{ and } s(A)\neq S$$

$$\nu(\Diamond A)=F \text{ iff } \nu(A)=F$$

and the defined connectives as

$$\nu(A \rightarrow B)=F \text{ iff } \nu(A)=T \text{ and } \nu(B)=F$$

$$\nu(A \vee B)=F \text{ iff } \nu(A)=F \text{ and } \nu(B)=F .$$

This points out that for all the set-assignment semantics for logics previously considered in this volume it is essential that the truth-value conditions "tag along" in the evaluation of the connectives even though the connective often could be evaluated as dependent only on the content of the constituent propositions.

We can generally classify many-valued semantics as being either falsity-default or truth-default. We say that a table for ¬,∧,∨, or → of a many-valued matrix is standard if when all the designated values are renamed T and the undesignated values are renamed F, then the table is (with repetitions) the classical table for that connective.

We say we have a falsity-weighted table if renaming as above, any row of the classical table which takes value F also takes value F in this table, whereas a row which takes value T in the classical table may take value F in the renamed many-valued table.

A truth-weighted table is one in which any row of the classical table which takes value T also takes value T in the re-named many-valued table.

Every standard table is both falsity-weighted and truth-weighted. Every many-valued logic of Chapter VIII uses standard or falsity-weighted tables for each of ¬,→,∧,¬. Only the table for ~ in $J_3$ is truth-weighted.

A many-valued logic which uses truth-weighted tables for ¬,→,∧,∨, at least one of which is not standard cannot be given the usual falsity-default set-assignment semantics in such a way that we can translate many-valued models to set-assignment ones and vice-versa while preserving validity in a model. It seems likely to us, though, that every truth-weighted many-valued matrix can be presented in terms of truth-default semantics, and every falsity weighted many-valued matrix can be presented as falsity-default semantics.