CORRECT CLIFFORD ALGEBRA REPRESENTATIONS OF PAULI, DIRAC, DOTTED AND UNDOTTED TWO-COMPONENTS SPINORS

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ABSTRACT. In this paper we clearify the relation between Pauli, undotted and dotted two components and Dirac spinors, as defined by physicists, and Spinors - elements of certain minimal left (or right) ideals in appropriated Clifford algebras $\mathbb{R}_{p,q}$. Our approach is based on the notion of the Spinorial-metric in Spinor-space and the fact that $\mathrm{Spin}^+(p,q)$ is the invariant group of the Spinorial-metric for $p+q \leq 5$. Particularly important results are the representations in $\mathbb{R}_{1,3}$ (the space-time algebra) of the undotted and dotted two component spinors and Dirac spinors.

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1. INTRODUCTION

The usual presentation of Spinors as elements of minimal left ideals in Clifford algebras(*) $\mathbb{R}_{p,q}^{(1,2,3)}$ as well as the introduction in this context of the groups $\text{Spin}_+(p,q)$, does not leave clear the relation between these objects and the spinors(**) and the universal covering groups of some groups $SO(p,q)$ used in theoretical physics.

The main purpose of this paper is to clear up the situation, and in the process we will obtain some very interesting results.

To formulate our problem we start by remembering that physicists use the following kind of spinors

(i) Pauli spinors - which are the vectors of a complex 2-dimensional space $\mathbb{C}(2)$ equiped with the spinorial metric

$$\beta_p : \mathbb{C}(2) \times \mathbb{C}(2) \longrightarrow \mathbb{C} \quad , \quad \beta_p(\psi,\varphi) = \psi^*\varphi \ , \tag{1}$$

$$\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} ; \ \varphi = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} , \ z_i, y_i \in \mathbb{C} \ , \ i=1,2 \ \text{and} \quad \psi^* = (\bar{z}_1, \bar{z}_2)$$

(*) for our notation see §2.

(**) We use the notation Spinors for the elements of minimal left ideals in Clifford algebras which represent the spinors defined in §1.

where in this text $\bar{z}$ always means the complex conjugate of $z \in \mathbb{C}$. The spinorial metric is invariant under the action of the group SU(2), ie, if $u \in SU(2)$ then $\beta_p(u\psi, u\varphi) = \beta_p(\psi,\varphi)$. As is well known[4] Pauli spinors carry the fundamental (irreducible) representations $D^{1/2}$ of SU(2)[5].

(ii) undotted and dotted two-components spinors-which are respectively the vectors of two complex 2-dimensional spaces $\mathbb{C}(2)$ and $\dot{\mathbb{C}}(2)$. In both spaces there are defined as spinorial metric $\beta, \dot{\beta}$ such that

$$\begin{aligned} \beta &: \mathbb{C}(2) \times \mathbb{C}(2) \longrightarrow \mathbb{C}\,, \quad \beta(\psi,\varphi) = \psi^t\, C\, \varphi \\ \dot{\beta} &: \dot{\mathbb{C}}(2) \times \dot{\mathbb{C}}(2) \longrightarrow \mathbb{C}\,, \quad \dot{\beta}(\dot{\psi},\dot{\varphi}) = \dot{\psi}^t\, C\, \dot{\varphi} \end{aligned} \tag{2}$$

where $\psi(\dot{\psi})$ are of the type defined in eq.(1) and $\psi^t(\dot{\psi}^t)$ is the transpose of $\psi(\dot{\psi})$ and

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3}$$

is the representations of $\beta(\dot{\beta})$ in the canonical basis of $\mathbb{C}(2)(\dot{\mathbb{C}}(2))$[6,7]

The spinorial metrics $\beta, \dot{\beta}$ are invariant under the action of the group $SL(2,\mathbb{C})$, ie, if $u \in SL(2,\mathbb{C})$ then $\beta(u\psi, u\varphi) = \beta(\psi,\varphi)$ and $\dot{\beta}((u^*)^{-1}\dot{\psi}, (u^*)^{-1}\dot{\varphi}) = \dot{\beta}(\dot{\psi},\dot{\varphi})$. The matrices u and $(u^*)^{-1}$ are the (non-equivalent) representations $D^{(1/2,0)}$ and $D^{(0,1/2)}$ of the group $SL(2,\mathbb{C})$ and we say that the undotted (dotted) two-components spinors are the carriers of the representations $D^{(1/2,0)}$ $(D^{(0,1/2)})$.

(iii) Dirac spinors-these are the vectors of a complex 4-dimensional space $\mathbb{C}(4)$ equiped with the spinorial metric[6,7]

$$\beta_D : \mathbb{C}(4) \times \mathbb{C}(4) \longrightarrow \mathbb{C}\,; \quad \beta_D(\psi,\varphi) = \psi^t\, B\varphi \tag{4}$$

where a Dirac spinor $\psi(\varphi) \in \mathbb{C}(4)$ is defined as

$$\mathbb{C}(2) \oplus \dot{\mathbb{C}}(2)^* = \mathbb{C}(4) \ni \psi = \xi + \dot{\beta}(\dot{\eta},\ ) \tag{5}$$

where $\xi \in \mathbb{C}(2)$ and $\dot{\beta}(\dot{\eta},\ ) \in \dot{\mathbb{C}}(2)^*$, the dual space of $\dot{\mathbb{C}}(2)$.

In the canonical basis of $\mathbb{C}(4)$ obtained through the canonical basis of $\mathbb{C}(2)$ and $\dot{\mathbb{C}}(2)^*$ the matrix B is the representation of β_D and we have

$$\beta_D = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$$

where C is the matrix defined in eq.(3).

Dirac spinors as is well known carry the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $SL(2,\mathbb{C})$.

We now ask the main question: to which minimal ideals, in which Clifford algebras are the spinors described in (i), (ii), (iii) above to be associated?

We are going to give an original answer to the above question by introducing a *unique "natural scalar product"* (see §3) in certain appropriate minimal left ideals of certain Clifford algebras that "mimic" what has been described in (i), (ii), (iii) above. To this end in Section 2 we define and give the main properties of Clifford algebras over the reals[8,9,10] and analyse the structure of the minimal ideals of these algebras[3]. The material presented fixes our notation and is the minimum necessary to permit the formulation of our ideas in a rigorous way.

In section 3 we define Spinors as the elements of minimal left ideals in Clifford algebras. The Spinors of each one of the Clifford algebras *studied in this paper* have a natural *right F-linear space structure* over one of the following fields $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively the real, complex number and quartenion (§2).

We introduce for each Spinor space $I \subset \mathbb{R}_{p,q}$ a *unique natural scalar product* (Spinorial metric), ie, a non-degenerated bilinear application $\Gamma : I \times I \to F$, where $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is the *natural scalar field* associated with the vector space structure of $I \subset \mathbb{R}_{p,q}$. Our approach to the natural scalar product shows that for $p+q \leq 5$ the groups $Spin^+(p,q)$ are the groups that leave invariant the Spinorial metric. Thus our approach to the scalar product is different from the one discussed by Lounesto[3] and as we shall see offers a solution for the main question formulated above.

We analyse in §4 in detail the special cases $SU(2) \simeq Spin(3,0)$

and $SL(2,\mathbb{C}) \simeq Spin^+(1,3)$ and identify respectively the ideals that contain the objects corresponding to Pauli-spinors ($I_p^+ = \mathbb{R}_{3,0}^+ e_{30}$), and undotted and dotted two components spinors ($I_u = \mathbb{R}_{1,3}^+ e_{13}$, $I_d = I_u^*$). Also in §4.3 we show that the minimal left ideals $I_D = \mathbb{R}_{1,3} f(e_{30})$ of $\mathbb{R}_{1,3}$ the space-time, or Minkowski algebra[11]) carry the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $SL(2,\mathbb{C})$, ie, the space-time Spinors are a representation of Dirac spinors.

In §4.4 we show that the original Dirac algebra ($\mathbb{C}(4)$) must be identified for physical reasons with the real Clifford algebra $R_{4,1}$. We then show that the ideals $\bar{I}_D = R_{4,1}^+ g(f(e_{30}))$ carry also the represen- tation of Dirac spinors.

In §5 we present our conclusions and some comments concerning our results on Dirac spinors and the results obtained by Hestenes[1,2].

2. SOME GENERAL FEATURES ABOUT CLIFFORD ALGEBRAS

2.1. CLIFFORD ALGEBRAS C(V,Q)

Let V be a vector space of finite dimension n over the field F together with a nondegenerate quadratic form Q. The Clifford algebra $C(V,Q) = T(V)/I_Q$ where T(V) is the tensor algebra of V ($T(V) = \sum_{i=1}^{\infty} T^i(V)$; $T^{(0)}(V) = F$, $T^1(V) = V$, $T^r(V) = \otimes^r V$) and I_Q is the bilateral ideal generated by the elements of the form $x \otimes x - Q(x)$, $x \in V$. The signature of Q is arbitrary. The Clifford algebra as constructed is an associative algebra with unit. The space V is naturally imbedded in C(V,Q)

$$V \overset{i}{\hookrightarrow} T(V) \xrightarrow{j} T(V)/I_Q = C(V,Q); \; i_Q = j \circ i \quad \text{and} \quad V \equiv I_Q \subset C(V,Q).$$

Let be $C^+(V,Q)$ (respectively $C^-(V,Q)$) the j-image of $\sum_{i=0}^{\infty} T^{2i}(V)$ (respectively $\sum_{i=0}^{\infty} T^{2i+1}(V)$) in $C(V,Q)$. The elements of $C^+(V,Q)$ form a subalgebra of $C(V,Q)$ called the even subalgebra of $C(V,Q)$.

C(V,Q) has the following universal property: "If A is an associative F-algebra with unit, then for all linear mappings $\phi : V \longrightarrow A$ such that $(\phi(x))^2 = Q(x)1$, $\forall x \in V$ can be extended in an unique way to a homomorphism $\Phi : C(V,Q) \longrightarrow A''$.

In $C(V,Q)$ there exist three linear mappings which are quite natural. They are the extension of the mappings

(a) MAIN INVOLUTION - an automorphism $^{\#} : C(V,Q) \longrightarrow C(V,Q)$, extension of $\alpha : V \longrightarrow T(V)/I_Q$, $\alpha(x) = -i_Q(x) = -x$ $\forall x \in V$

(b) REVERSION - an antiautomorphism $^{*} : C(V,Q) \longrightarrow C(V,Q)$, extension of $^{t} : T^r(V) \longrightarrow T^r(V)$, $T^r(V) \ni x = x_{i_1} \otimes \ldots \otimes x_{i_r} \longrightarrow x^t = x_{i_r} \otimes \ldots \otimes x_{i_1}$, $1 \leq r \leq n$

(c) CONJUGATION - $\tilde{} : C(V,Q) \longrightarrow C(V,Q)$, defined by the composition of the automorphism $^{\#}$ with the antiautomorphism *, ie, if $x \in C(V,Q)$, then $\tilde{x} = (x^*)^{\#}$.

$C(V,Q)$ can be described through its generators, ie, if $\{e_i\}$, $i = 1, 2, \ldots, n$ is a Q-orthogonal basis of V, then $C(V,Q)$ is generated by 1 and the e_i's subjected to the conditions $e_i e_i = Q(e_i)1$ and $e_i e_j + e_j e_i = 0$, $i \neq j$, $i,j = 1,\ldots,n$. If V is a n-dimensional real vector space then we can choose a basis $\{e_i\}$ for V such that $Q(e_i) = \pm 1^{( )}$.

2.2. THE REAL CLIFFORD ALGEBRAS $\mathbb{R}_{p,q}$

Let $\mathbb{R}^{p,q}$ be a real vector space of dimension $p+q=n$ equipped with a metric $g : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \longrightarrow \mathbb{R}$. Let be $\{e_i\}$ the canonical basis of $\mathbb{R}^{p,q}$, such that

$$g(e_i,e_j) = g_{ij} = g(e_j,e_i) = g_{ji} = \begin{cases} +1 & i = j = 1,2,\ldots,p \\ -1 & i = j = p+1,\ldots,p+q=n \\ 0 & i \neq j \end{cases} \quad (7)$$

The Clifford algebra $\mathbb{R}_{p,q} = C(\mathbb{R}^{p,q},Q)$, $p+q = n$ is the Clifford algebra over the real field $\mathbb{R}$, generated by 1 and the $\{e_i\}$, $i = 1,\ldots,n$ such that $Q(e_i) = g(e_i,e_i)$. $\mathbb{R}_{p,q}$ is obviously of dimension 2^n and is the direct sum of the vector spaces $\mathbb{R}^k_{p,q}$ of dimensions $\binom{n}{k}$, $0 \leq k \leq n$. The canonical basis for $\mathbb{R}^k_{p,q}$ are the elements $e_A = e_{\alpha_1} \ldots e_{\alpha_k}$, $1 \leq \alpha_1 < \ldots < \alpha_k \leq n$. The element $e_J = e_1 e_2 \ldots e_n \in \mathbb{R}^n_{p,q}$ commutes (n-odd) or anticomutes (n-even) with all vectors $e_1, \ldots, e_n$ in $\mathbb{R}^1_{p,q} = \mathbb{R}^{p,q}$. The center of $\mathbb{R}_{p,q}$ is $\mathbb{R}^o_{p,q} = \mathbb{R}$ if n is even and it is the direct sum $\mathbb{R}^o_{p,q} \oplus \mathbb{R}^n_{p,q}$ if n is odd[3,12].

All Clifford algebras are semi-simple. If $p+q=n$ is even $\mathbb{R}_{p,q}$ is a simple algebra and if $p+q=n$ is odd we have the following possibilities:

(i) $\mathbb{R}_{p,q}$ is simple $\leftrightarrow e_J^2 = -1 \leftrightarrow p-q \neq 1 \pmod 4 \leftrightarrow$ center $\mathbb{R}_{p,q}$ is isomorphic to $\mathbb{C}$

(ii) $\mathbb{R}_{p,q}$ is not simple $\leftrightarrow e_J^2 = +1 \leftrightarrow p-q = 1 \pmod 4 \leftrightarrow$ center $\mathbb{R}_{p,q}$ is isomorphic to $\mathbb{R}^o_{p,q} \oplus \mathbb{R}^n_{p,q}$.

From the fact that all semi-simple algebras are the direct sum of two simple algebras[13] and from

WEDDENBURN'S THEOREM. "If A is a simple algebra then A is equivalent to F(m), where F is a division algebra and m and F are unique (modulo isomorphisms)", we obtain from the point of view of representation theory $\mathbb{R}_{p,q} \simeq F(m)$ or $\mathbb{R}_{p,q} \simeq F(m) \oplus F(m)$ where $F(m)$ is the matrix algebra of dimension $m \times m$ (for some m) with coefficients in $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Table I (where [n/2] means the integral part of n/2) presents the representation of $\mathbb{R}_{p,q}$ as a matrix algebra[12].

p-q(mod8)	0	1	2	3	4	5	6	7
$\mathbb{R}_{p,q}$	$\mathbb{R}(2^{[n/2]})$	$\mathbb{R}(2^{[n/2]}) \oplus \mathbb{R}(2^{[n/2]})$	$\mathbb{R}(2^{[n/2]})$	$\mathbb{C}(2^{[n/2]})$	$\mathbb{H}(2^{[n/2-1]})$	$\mathbb{H}(2^{[n/2]-1}) \oplus \mathbb{H}(2^{[n/2]-1})$	$\mathbb{H}(2^{[n/2-1]})$	$\mathbb{C}(2^{[n/2]})$

Table I - Representation of the real Clifford algebra $\mathbb{R}_{p,q}$ as a matrix algebra

2.3. MINIMAL LATERAL IDEALS OF $\mathbb{R}_{p,q}$

The minimal left ideals of a semi-simple algebra A are of the type Ae, where e is a primitive idempotent of A, ie, $e^2 = e$, and e is not the sum of two mutually non trivial orthogonal idempotentes, ie, $e = \hat{e} + \check{e}$ with $\hat{e}^2 = \hat{e}$, $\check{e}^2 = \check{e}$ and $\hat{e}\check{e} = \check{e}\hat{e} = 0$[13].

A minimal left ideal of $\mathbb{R}_{p,q}$ is of the type $I = \mathbb{R}_{p,q}e_{pq}$ where $e_{pq} = \frac{1}{2}(1+e_{\alpha_1}) \ldots \frac{1}{2}(1+e_{\alpha_k})$ where $e_{\alpha_1},\ldots,e_{\alpha_k}$ is a set of commuting elements of the canonical basis of $\mathbb{R}_{p,q}$ such that $(e_{\alpha_i})^2 = 1$, $i = 1,\ldots,k$ that generates a group of order $k = q - r_{q-p}$, and r_i are the Radon-Hurwitz numbers, defined by the recurrence formula $r_{i+8} = r_i + 4$[3] and

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

Table II - Radon-Hurwitz numbers

If we have a linear mapping $L_a : \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}$, $L_a(x)$, $\forall x \in \mathbb{R}_{p,q}$ and where $a \in \mathbb{R}_{p,q}$, then, as I is invariant under left multiplication with arbitrary elements of $\mathbb{R}_{p,q}$, we can consider $L_{a|I} : I \to I$. If $p+q=n$ is even or odd with $p-q \neq 1 \pmod 4$ then $\mathbb{R}_{p,q} \simeq \mathcal{L}_F(I) \simeq F(m)$, where $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathcal{L}_F(I)$ is the algebra of linear transformations in I over the field F, $m = \dim_F(I)$ and $F \simeq eF(m)e$[13]. If $p+q=n$ either odd, with $p-q = 1 \pmod 4$ then $\mathbb{R}_{p,q} \simeq \mathcal{L}_F(I) \oplus \mathcal{L}_F(I) \simeq F(m) \oplus F(m)$, $m = \dim_F(I)$ and $e_{pq}\mathbb{R}_{p,q}e_{pq} \simeq \mathbb{R} \oplus \mathbb{R}$ or $\mathbb{H} \oplus \mathbb{H}$.

With the above isomorphisms we can identify the minimal left ideals of $\mathbb{R}_{p,q}$ with the column-matrices of $F(m)$.

3. SPINORS, THE SPINORIAL METRIC AND THE SPIN GROUP

3.1. SPINORS. Given a real Clifford algebra $\mathbb{R}_{p,q}$ with primitive idempotent e_{pq} we call Spinors the elements of the minimal left ideal $\mathbb{R}_{p,q}e_{pq}$ or the elements of $\mathbb{R}^+_{p,q}e_{pq}$.

3.2. THE SPIN GROUP-SPIN(p,q)

The invertible elements $s \in \mathbb{R}_{p,q}$ such that $\forall x \in \mathbb{R}^1_{p,q} = \mathbb{R}^{p,q}$ we have $s^{*}xs^{-1} \in \mathbb{R}^1_{p,q}$, form a multiplicative group called the Clifford group of $\mathbb{R}_{p,q}$ which we denote by Γ_x. This group is generated by the vectors $x \in \mathbb{R}^{p,q}$ such that $g(x,x) \neq 0$.

Consider now the mapping $N : \mathbb{R}_{p,q} \longrightarrow \mathbb{R}_{p,q}$ defined by $N(s) = \tilde{s}s$. If $s \in \Gamma_x$, then N is a homomorphism of the group Γ_x into the multiplicative group of the non null multiples of **1** of $\mathbb{R}_{p,q}$.

We define the groups $\mathrm{Pin}(p,q) = \{s \in \Gamma_x \; ; \; N(s) = \pm 1\}$, $\mathrm{Spin}(p,q) = \mathrm{Pin}(p,q) \cap \mathbb{R}^+_{p,q}$ and $\mathrm{Spin}^+(p,q) = \{s \in \Gamma_x , \tilde{s}s = +1\} \cap \mathbb{R}^+_{p,q}$ as the connected component of Spin(p,q) that contains the identity.

3.3. SCALAR PRODUCT OF SPINORS. THE SPINORIAL METRIC

In §2.3 we saw that when $\mathbb{R}_{p,q}$ is simple, a minimal left ideal I of $\mathbb{R}_{p,q}$ is of the form $I = \mathbb{R}_{p,q}e_{pq}$ where e_{pq} is a primitive idempotent of $\mathbb{R}_{p,q}$ and $F = e_{pq}\mathbb{R}_{p,q}e_{pq}$ with $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ depending on $p-q = 0,1,2 \pmod 8$, $p-q = 3,7 \pmod 8$ or $p-q = 4,5,6 \pmod 8$ respectively (Table I). We can then define a right action F in I, $I \times F \longrightarrow I$, by $I \times F \ni (\psi,\alpha) \longrightarrow \psi\alpha \in I$. In this way I has a natural linear vector space structure over the field F, whose elements are the natural "scalars" of the vector space I.

These remarks suggest us to search for a *natural "scalar product"* in I, ie, a non-degenerated bilinear mapping $\Gamma : I \times I \longrightarrow F$. To this end we observe that if f and g are $\tilde{}$ F-endomorphims in $\mathbb{R}_{p,q}$ then we can define a bilinear mapping Γ in $\mathbb{R}_{p,q}$ using f and g. We simply take $\Gamma(\psi,\varphi) = f(\psi)g(\varphi)$, $\psi,\varphi \in \mathbb{R}_{p,q}$. Considering that $I = \mathbb{R}_{p,q}e_{pq}$ has a natural structure of vector space over F we can take the restriction of Γ to I, and ask the following question:

For $\psi,\varphi \in I$ when does $\Gamma(\psi,\varphi) \in F$?

As we saw in §2.1 we have three natural isomorphisms defined in $\mathbb{R}_{p,q}$, the main involution, the reversion and the conjugation, denoted respectively by $^{\blacksquare}$, * and $\sim$. Combining these isomorhisms with the identity mapping we can define the following bilinear mappings

$$\Gamma_i : I \times I \longrightarrow \mathbb{R}_{p,q}, \quad i = 1,2,3$$

$$\Gamma_1(\psi,\varphi) = \psi^{\blacksquare}\varphi \ ; \ \Gamma_2(\psi,\varphi) = \psi^{\star}\varphi, \ \Gamma_3(\psi,\varphi) = \tilde{\psi}\varphi, \quad \forall\psi,\varphi \in I$$

As already observed in §2.1 the main involution is an automorphism whereas the reversion and conjugation are antiautomorphisms. An automorphism (antiautomorphism) transforms an element of a minimal left ideal in an element of a minimal left ideal (minimal right ideal).

To see the validity of these statements it is enough to observe that the image of a primitive idempotent under an isomorphism is a primitive idempotent and that if $\psi \in I_{p,q} = \mathbb{R}_{p,q} e_{pq}$ then $\psi = xe_{pq}$ with $x \in \mathbb{R}_{p,q}$ and

$$\psi^{\blacksquare} = (xe_{pq})^{\blacksquare} = x^{\blacksquare}e^{\blacksquare}_{pq} \Longrightarrow \psi^{\blacksquare} \in I'_{p,q} = \mathbb{R}_{pq} e^{\blacksquare}_{pq}$$

$$\psi^{\star} = (xe_{pq})^{\star} = e^{\star}_{pq} x^{\star} \Longrightarrow \psi^{\star} \in I^{\star}_{p,q} = e^{\star}_{pq} \mathbb{R}_{p,q} \tag{8}$$

$$\tilde{\psi} = (xe_{pq})^{\sim} = \tilde{e}_{pq} \tilde{x} \Longrightarrow \tilde{\psi} \in \tilde{I}_{p,q} = \tilde{e}_{pq} \mathbb{R}_{p,q} .$$

Using the isomorphisms $\mathbb{R}_{p,q} \simeq \mathcal{L}_F(I) \simeq F(m)$, $m = \dim_F I_{p,q}$ (when $\mathbb{R}_{p,q}$ is simple, cf. §2.3) we identify the elements of the minimal left ideals of $\mathbb{R}_{p,q}$ with the column matrices of $F(m)$. Then if $\psi \in I_{p,q}$ has a representation as a column matrix of $F(m)$ then $\psi^{\star}$ and $\tilde{\psi}$ have representation as row matrices of $F(m)$, and we get that $\psi^{\star}\varphi$ and $\tilde{\psi}\varphi$ are elements of F.

We also observe that Γ_2 and Γ_3 are non-degenerate bilinear mappings. Indeed, $\Gamma_i(\psi,\varphi) = 0$, $i=2,3$ then $\psi = 0$ or $\varphi = 0$, since $\varphi^{\star}$ and $\tilde{\psi}$ are isomorphisms.

We identity the scalars of the vector space structure of $I_{p,q}$ with multiples of

$$e_{pq} \equiv 1 = \begin{bmatrix} 1 & 0 & \ldots \\ 0 & 0 & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix} \tag{9}$$

ie, as matrices in $F(m)$ multiples of the matrix in eq.(9). Through isomorphisms of $\mathbb{R}_{p,q}$ (multiplication by a convenient invertible element $u \in \mathbb{R}_{p,q}$) we can transport $\psi^{*}\varphi$ or $\tilde{\psi}\varphi$ to the position (1,1) in the matrix representation of these operations. We then conclude that the *natural scalar products* in $I_{p,q}$ are

$$\beta_i : I_{p,q} \times I_{p,q} \longrightarrow F \quad , \quad i=1,2 \tag{10}$$

$\beta_1(\psi,\varphi) = u\psi^{*}\varphi$ and $\beta_2(\psi,\varphi) = u\tilde{\psi}\varphi$, $\forall\psi,\varphi \in I_{p,q}$ and $u \in \mathbb{R}_{p,q}$ is a convenient invertible element.

Lounesto[3] obtains the scalar products in eq(10) using similar arguments and immediately proceeds to the classification of the groups of automorphisms of these scalar products, ie, the homomorphisms of right F-modules, $I_{p,q} \longrightarrow I_{p,q}$, $\psi \longrightarrow s\psi$, $s \in \mathbb{R}_{p,q}$ which preserve the products in eq.(10). Observe that from $\beta_1(s\psi,s\varphi) = \beta_1(\psi,\varphi)$ we get $s^{*}s = 1$ and from $\beta_2(s\psi,s\varphi) = \beta_2(\psi,\varphi)$ we get $\tilde{s}s = 1$ $(\psi,\varphi \in I_{p,q})$. Lounesto[3] calls $G_1 = \{s \in \mathbb{R}_{p,q} \; ; \; s^{*}s = 1\}$, $G_2 = \{s \in \mathbb{R}_{p,q} \; , \; \tilde{s}s = 1\}$.

So in Lounesto paper there does not appear in principle any relationship between the groups Spin(p,q) and the groups G_1 and G_2 with the consequence that we do not have a clear basis to mimic within the Clifford algebras $\mathbb{R}_{p,q}$ (for appropriate p and q) the results described in (i), (ii), (iii) of §1. We can mimic these results within some Clifford algebras by introducing the concept of Spinorial metric.

Observe that since $\mathrm{Spin}(p,q) \subset \mathbb{R}^{+}_{p,q}$ it seems interesting to define a scalar product in an ideal $I^{+}_{p,q} = \mathbb{R}^{+}_{p,q}e_{pq}$. The reason is that such a scalar product is now *unique*, since if $s \in \mathbb{R}^{+}_{p,q}$, then $s^{*} = \tilde{s}$. This unique scalar product will be called in what follows the Spinorial metric

$$\beta : I^{+}_{p,q} \times I^{+}_{p,q} \longrightarrow F \; , \tag{11}$$

defined by $\beta(\psi,\varphi) = u\tilde{\psi}\varphi$. We see that $G = \{s \in \mathbb{R}^+_{p,q} \mid \tilde{s}s = 1\}$ is the group of automorphisms of the Spinorial metric just defined, and $G \subset G_1$, $G \subset G_2$.

We now recall a result firstly obtained by Porteous[14]: "$\mathrm{Spin}^+(p,q) = \{s \in \mathbb{R}^+_{p,q} \mid \tilde{s}s = 1\}$ for $p+q \leq 5$" [Proposition 13.58].

With this result we get a new interpretation of the groups $\mathrm{Spin}^+(p,q)$ for $p+q \leq 5$, namely, these are the groups that leave the Spinorial metric of eq.(11) invariant. But even more important is the fact the now we know the way to mimic within appropriate Clifford algebras (i), (ii), (iii) of §1 and thus we can make a *correct* representation within Clifford algebras of the Pauli spinors, undotted and dotted bidimensional spinors and Dirac spinors. This is done in §4.

4. CLIFFORD ALGEBRA REPRESENTATION OF PAULI SPINORS, UNDOTTED AND DOTTED TWO-DIMENSIONAL SPINORS AND DIRAC SPINORS

4.1. PAULI SPINORS AND THE GROUP SU(2).

The algebra $\mathbb{R}_{3,0}$ (Pauli algebra) is isomorphic to $\mathbb{C}(2)$ (see Table I), the algebra of complex matrices, generated by 1 and σ_i, $i = 1, 2, 3$ subject to the conditions $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$, $\delta_{ij} = +1$ or 0 depending if $i=j$ or $i \neq j$.

The element $e_{30} = \frac{1}{2}(1 + \sigma_3)$ is a primitive idempotent of $\mathbb{R}_{3,0}$. We have that $\alpha = \{e_{30}, \sigma_1 e_{30}\}$ is a spinorial basis for $I_{3,0} \equiv I_p = \mathbb{R}_{3,0}e_{30}$.

We shall see that the elements of $I^+_p = \mathbb{R}^+_{3,0}e_{30}$ (Pauli Spinors) are the representatives of Pauli spinors ((i) of §1) within the Pauli algebra. The reason is as follows:

In the above basis we have the following matrix representation for

$$x, x^{\blacksquare}, x^{*}, \tilde{x} \in \mathbb{R}_{3,0}$$

$$\mathbb{C}(2) \ni x = \begin{bmatrix} z^1 & z^2 \\ z^3 & z^4 \end{bmatrix} \; ; \; x^{\blacksquare} = \begin{bmatrix} \bar{z}^4 & \bar{z}^3 \\ \bar{z}^2 & \bar{z}^1 \end{bmatrix} \; ; \; x^{\star} = \begin{bmatrix} \bar{z}^1 & \bar{z}^3 \\ \bar{z}^2 & \bar{z}^4 \end{bmatrix} \; ; \; x = \begin{bmatrix} z^4 & -z^2 \\ -z^3 & z^1 \end{bmatrix} \tag{12}$$

Defining $\hat{\beta} : I_p^+ \times I_p^+ \longrightarrow \mathbb{C}$, $\hat{\beta}(\psi,\varphi) = \psi^{\star}\varphi$, for $\psi = \begin{bmatrix} \psi^1 & 0 \\ \psi^2 & 0 \end{bmatrix}$ and $\varphi = \begin{bmatrix} \varphi^1 & 0 \\ \varphi^2 & 0 \end{bmatrix}$

$$\hat{\beta}(\psi,\varphi) = \bar{\psi}^1\varphi^1 + \bar{\psi}^2\varphi^2 \tag{13}$$

We define now the **Spinorial** metric $\beta = \hat{\beta}|I_p^+$. We have, $\beta(\psi,\varphi) = [\beta(\psi,\varphi)]^-$ (hermitian product). As $\alpha = (e_{30}, \sigma_1 e_{30})$ is an orthonormal basis for I_p^+ we have the following representation of β in the α-basis

$$[\beta]_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{1}_2 \tag{14}$$

$\beta(\psi,\varphi) = \beta(s\psi, s\varphi) \iff s^{\star}s = \mathbf{1}_2 \iff s \in U(2)$.

Now, if $x \in \mathbb{R}_{3,0}^+ \simeq \mathbb{R}_{0,2} \simeq \mathbb{H}$ we have the following representation for x in the α-basis

$$x = \begin{bmatrix} z & -\bar{\omega} \\ \omega & z \end{bmatrix} \text{ and } \tilde{x} = x^{\star} = \begin{bmatrix} \bar{z} & \omega \\ -\omega & z \end{bmatrix},$$

Observe now that $N(x) = \tilde{x}x = \det X \mathbf{1}_2$ and we get $N(x) = 1 \iff \det N = 1$. So the element $s \in \mathbb{R}_{3,0}^+$ such that $\beta(s\psi, s\varphi) = \beta(\psi,\varphi)$, $\psi,\varphi \in I_p^+$ satisfy $\tilde{s}s = \mathbf{1}_2$ and $\det s = +1$, that means that $s \in SU(2)$ and $SU(2) \simeq Spin^+(3,0)$, and our assertion that Pauli spinors are represented by the elements of $I_p^+ = \mathbb{R}_{3,0}^+ e_{30}$ is proved. ~

4.2. UNDOTTED AND DOTTED TWO COMPONENTS SPINORS AND THE GROUP SL(2,$\mathbb{C}$).

We have that $\mathbb{R}_{3,0} \overset{f}{\simeq} \mathbb{R}_{1,3}^+$ where f is the linear extension of $f(\sigma_i) = e_i e_o$, $i \neq 0$, $i=1,2,3$, $\sigma_i \in \mathbb{R}^{3,0}$ and e_μ, $\mu = 0,1,2,3$ is an orthonormal basis of $\mathbb{R}^{1,3}$.

As $e_{3,0} = \frac{1}{2}(1+\sigma_3)$ is a primitive idempotent of $\mathbb{R}_{3,0}$ then $f(e_{30}) = \frac{1}{2}(1+e_3e_o)$ is a primitive idempotent of $\mathbb{R}^+_{1,3}$. We have that $I_u = \mathbb{R}^+_{1,3}\, f(e_{30})$ is a minimal left ideal of $\mathbb{R}^+_{1,3}$ with basis $\alpha = \{f(e_{30}),\ e_1e_o f(e_{30})\}$. Using the isomorphism

$$\rho : \mathbb{R}^+_{1,3} \longrightarrow \mathcal{L}_{\mathbb{C}}(I_u)$$
$$u \longmapsto \rho(u) : I_u \longrightarrow I_u \qquad (16)$$
$$\psi \longmapsto u\psi$$

we have the following matrix representation for $u \in \mathbb{R}^+_{1,3}$ in the α-basis

$$\mathbb{C}(2) \ni u = \begin{bmatrix} z^1 & z^2 \\ z^3 & z^4 \end{bmatrix}, \quad u^* = \tilde{u} = \begin{bmatrix} z^4 & -z^2 \\ -z^3 & z^1 \end{bmatrix} \qquad (17)$$

Defining

$$\beta : I_u \times I_u \longrightarrow \mathbb{C}, \quad \beta(\psi,\varphi) = e_1e_o\tilde{\psi}\varphi \qquad (18)$$

we get

$$\beta = (\psi,\varphi) = \psi^1\varphi^2 - \psi^2\varphi^1 \qquad (19)$$

and the representation of β in the α-basis is

$$[\beta]_\alpha = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad (20)$$

Then, $\beta(u\psi,u\varphi) = \beta(\psi,\varphi) \iff \tilde{u}u = \mathbb{1}_2 \iff \det u = 1 \iff u \in SL(2,\mathbb{C}) \simeq Spin^+(1,3)$

We conclude that the elements of I_u (undotted Spinors) can be said to give a representation of undotted two-component spinors within the space-time algebra $\mathbb{R}_{1,3}$. The vector space I_u carries the $D^{(1/2,0)}$ representation of $SL(2,\mathbb{C})$, ie, the group $Spin^+(1,3)$ is the $D^{(1/2,0)}$ representation of $SL(2,\mathbb{C})$ within the space-time algebra.

Now remembering that * is an antiautomorphism in the Clifford algebra $^* : \mathbb{R}^+_{1,3} \to \mathbb{R}^+_{1,3}$ that preserves the $\mathrm{Spin}^+(1,3)$ group we have: If $u \in \mathrm{Spin}^+(1,3) \Longrightarrow u^* \in \mathrm{Spin}^+(1,3) \Longrightarrow (u^*)^{-1} \in \mathrm{Spin}^+(1,3)$.

Consider now the minimal right ideal $I_d = (\mathbb{R}^+_{1,3} f(e_{30}))^*$ and the isomorphism

$$\overset{o}{\rho} : \mathbb{R}^+_{1,3} \to \mathcal{L}_{\mathbb{C}}(I_d) \; ;$$

$$u \to \overset{o}{\rho}(u) : I_d \to I_d \tag{21}$$

$$\overset{o}{\psi} \to \overset{o}{\psi}(u^*)^{-1}$$

We conclude that the elements of I_d (dotted Spinors) can be said to give a representation of the dotted two component spinors (ii) of §1) within the space-time algebra $\mathbb{R}_{1,3}$. The vector space I_d carries the $D^{(0,1/2)}$ representation of $SL(2,\mathbb{C})$.

4.3. REPRESENTATION OF DIRAC SPINORS WITHIN THE SPACE-TIME ALGEBRA $\mathbb{R}_{1,3}$

We have that $\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$ and the idempotent $f(e_{30})$ is also primitive in $\mathbb{R}_{1,3}$. This means that $I_D = \mathbb{R}_{1,3} f(e_{30})$ is a minimal left ideal of $\mathbb{R}_{1,3}$. It is a bi-dimensional quaternion ideal in $\mathbb{R}_{1,3}$.

We can consider I_D as a 4-dimensional complex vector space and is this way we get a complex representation of $\mathbb{R}_{1,3} \simeq \mathbb{R}^+_{4,1} \subset \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$.

Calling $f(e_{30}) = e_{13} \equiv \bar{e}$ we have

$$\begin{aligned} I_D = \mathbb{R}_{1,3}\bar{e} = a_1\bar{e} + a_2 e_o\bar{e} + a_3 e_1\bar{e} + a_4 e_2\bar{e} + a_5 e_o e_1\bar{e} + a_6 e_o e_2\bar{e} \\ + a_7 e_1 e_2\bar{e} + a_8 e_o e_1 e_2\bar{e} \; ; a_i \in \mathbb{R}, \; i=1,\dots,8 \end{aligned} \tag{22}$$

Observing that

$$\bar{e}\mathbb{R}_{1,3}\bar{e} \simeq [\bar{e}, e_1\bar{e}, e_2\bar{e}, e_1e_2\bar{e}] \simeq \mathbb{H} \; ; \; \mathbb{C} \simeq [\bar{e}, e_1e_2\bar{e}] \subset \bar{e}\mathbb{R}_{1,3}\bar{e}$$

we can rewrite eq.(22) as

$$I_D = \mathbb{R}_{1,3}\bar{e} = \bar{e}(a_1\bar{e} + a_7e_1e_2\bar{e}) + e_o\bar{e}(a_2\bar{e} + a_8e_1e_2\bar{e})$$
$$+ e_1\bar{e}(a_3\bar{e} - a_4e_1e_2\bar{e}) + e_oe_1\bar{e}(a_5\bar{e} - a_6e_1e_2\bar{e}). \tag{23}$$

A complex basis for I_D is then $\alpha_D = \{e_o\bar{e}, e_1\bar{e}, \bar{e}, e_oe_1\bar{e}\}$.

Consider now the injection

$$\gamma : \mathbb{R}_{1,3} \longrightarrow \mathcal{L}_{\mathbb{C}}(I_D)$$
$$u \longmapsto \gamma(u) : I_D \longrightarrow I_D \tag{24}$$
$$\psi \longmapsto u\psi$$

we get the following representation for e_μ, $\mu = 0,1,2,3$ in the α_D-basis

$$\gamma(e_o) \equiv \gamma_o = \begin{bmatrix} 0 & 1_2 \\ 1_2 & 0 \end{bmatrix} ; \gamma(e_i) = \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \ i=1,2,3 \tag{25}$$

where σ_i are the Pauli matrices.

In this basis we have the following representation for $x \in \mathbb{R}_{1,3}$

$$\gamma(x) = \left[\begin{array}{cc|cc} x_1 & x_2 & x_5 & x_6 \\ x_3 & x_4 & x_7 & x_8 \\ \hline x_8 & -x_4 & \bar{x}_4 & -\bar{x}_3 \\ -x_6 & x_5 & -\bar{x}_2 & \bar{x}_1 \end{array}\right] \tag{26}$$

Considering the restriction $\gamma|\mathbb{R}^+_{1,3}$ we get for $z \in \mathbb{R}^+_{1,3}$ the following representation in the α_D-basis.

$$\gamma(z) = \left[\begin{array}{cc|cc} z_1 & z_2 & & \\ z_3 & z_4 & & 0 \\ \hline & & \bar{z}_4 & -\bar{z}_3 \\ & 0 & -\bar{z}_2 & \bar{z}_1 \end{array}\right] \tag{27}$$

Now, since $\mathbb{R}_{3,0} \overset{f}{\simeq} \mathbb{R}^+_{1,3}$ there exists an unique $y \in \mathbb{R}_{3,0} \simeq \mathbb{C}(2)$ such that $z = f(y)$ and we have

$$\gamma(f(y)) = \left[\begin{array}{c|c} y & 0 \\ \hline 0 & (y^*)^{-1} \end{array}\right] \quad y \in \mathbb{C}(2) \tag{28}$$

and

$$\gamma_0 f : \mathbb{R}_{3,0} \longrightarrow \mathcal{L}_{\mathbb{C}}(I_D)$$

$$y \longrightarrow \left[\begin{array}{c|c} y & 0 \\ \hline 0 & (y^*)^{-1} \end{array}\right] \quad y \in \mathbb{R}_{3,0} \tag{29}$$

We see that the restriction $\gamma|\mathbb{R}^+_{1,3}$ gives a complex 4-dimensional representation of $\mathrm{Spin}^+(1,3) \simeq SL(2,\mathbb{C})$ — namely the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $SL(2,\mathbb{C})$.

We call the elements $\psi \in I_D$, space-time Spinors. From the above discussion it is quite clear that space-time Spinors represent in $\mathbb{R}_{1,3}$ the Dirac spinors introduced in (iii) of §1.

We also mimic the spinorial metric in $\mathbb{C}(4)$ [(iii) of §1] defining

$$\beta_D : I_D \times I_D \longrightarrow \mathbb{C}, \quad \beta(\psi,\varphi) = b\psi^{*}\varphi \tag{30}$$

for an appropriated $b \in \mathbb{R}_{1,3}$.

4.4. REPRESENTATION OF DIRAC SPINORS WITHIN THE $\mathbb{R}_{4,1}$ ALGEBRA

From table I we see that $\mathbb{R}_{4,1}$, $\mathbb{R}_{3,2}$ and $\mathbb{R}_{0,5}$ are isomorphic to the algebra $\mathbb{C}(4)$ which is the usual Dirac algebra of physicists. In order to identify the algebra that carries the physical interpretation associated with space-time $\mathbb{R}^{1,3}$ we proceed as follows. Let be E_A, $A = 0,1,2,3,4$ an orthonormal basis for $\mathbb{R}^{p,q}$ with $p+q = 5$. The volume element is $E_J = E_o E_1 E_2 E_3 E_4$ and we get $E_J^2 = -1$ for $q = 1,3,5$. Now define

$$e_\mu = E_\mu E_4 \tag{31}$$

and impose that e_μ is an orthonormal basis for $\mathbb{R}^{1,3}$, ie

$$e_o^2 = -E_o^2 E_4^2 = +1, \quad e_k^2 = -E_k^2 E_4^2 = -1, \; k = 1,2,3 \; . \tag{32}$$

Eq.(32) is satisfied when $p = 4$, $q = 1$, ie, $E_4^2 = E_k^2 = -E_o^2 = 1$ and we conclude that the real Clifford algebra associated with space-time ($\mathbb{R}^{1,3}$) and isomorphic to $\mathbb{C}(4)$ is $\mathbb{R}_{4,1}$.

Eq.(31) shows that $\mathbb{R}_{1,3} \overset{g}{\simeq} \mathbb{R}^+_{4,1}$ where g is the linear extension of $g(e_\mu) = E_\mu E_4$, $\mu = 0,1,2,3$. We already saw in §4.2 and §4.3 that $f(e_{30})$ is a primitive idempotent of $\mathbb{R}_{1,3}$ and we have that $g(f(e_{30}))$ is a primitive idempotent of $\mathbb{R}^+_{4,1}$. Then $\underline{I}_D = \mathbb{R}^+_{4,1}\, g(f(e_{30}))$ is a minimal ideal of $\mathbb{R}^+_{4,1}$ which is a 4-dimensional vector-space over the complex field and its elements are Spinors which are representations in $\mathbb{R}_{4,1}$ of the Dirac spinors

5. CONCLUSIONS

Hestenes[11] said about the theory of spinors: "I have not met anyone who was not dissatisfied with his first readings on the subject".

Well, the reasons for such statement are in our view due to two facts

(A) the usual presentation of spinors such as introduced in (i), (ii), (iii) of §1 does not emphasize the geometrical meaning of these objects.

(B) There are not clear connection between the abstract concepts of spinors as introduced in (i), (ii) and (iii) of §1 and the more abstract concept of Spinors as elements of ideals of particular Clifford algebras.

As to (A) we think that the situation has been partially clarified with the presentation by Hestenes of the geometrical meaning of Pauli spinors[15] and of Dirac spinors[1,2] and also by Penrose and Rindler[16] of the geometrical meaning of the undotted and dotted two-component spinors.

We take this opportunity to clear up the relation between our approach to Dirac spinors and the approach by Hestenes[1,2].

In Ref.(1) Hestenes searchs a representation, using the Clifford bundle[17] [over Minkowski space-time (M)] of the usual Dirac equation

$$i\hbar\gamma^{\mu}(\partial_{\mu} - qA_{\mu}(x))\psi_{s}(x) = m\psi_{s}(x) \tag{33}$$

where $M \ni x \longrightarrow \psi_s(x)$ is a section of the spinor bundle[4] (over M), q is the electric charge, A_μ, $\mu = 0,1,2,3$ are the components (is an orthonormal frame in M) of the electromagnetic potential, and γ^μ are the Dirac matrices satisfying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} = 2\,\mathrm{diag}(+1,-1,-1,-1)$. Hestenes choose the specific *standard representation*[6] where the γ-matrices are

$$\gamma_0 = \begin{bmatrix} 1_2 & 0 \\ 0 & -1_2 \end{bmatrix} ; \quad \gamma_k = \begin{bmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{bmatrix} \tag{34}$$

and where Dirac spinors are represented by the four-dimensional column matrices called *standard Dirac spinors*

$$\psi = \begin{bmatrix} \phi \\ \lambda \end{bmatrix} \tag{35}$$

where $\phi = \frac{1}{\sqrt{2}}(\xi+\eta)$; $\lambda = \frac{1}{\sqrt{2}}(\xi-\eta)$, where ξ and $\eta = \dot{\beta}(\dot{\eta},\ )$ are the undotted and dotted 2-component spinors introduced in (iii) of §1.

Introducing for each $x \in M$ a spinor basis $u_{s1}, u_{s2}, u_{s3}, u_{s4}$ at the fiber over x $(\mathbb{C}(4))$ in the spinor bundle[4] where

$$u_{s1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} ; u_{s2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} ; u_{s3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} ; u_{s4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (36)$$

we can show that ψ_s has the following representation in $\mathbb{R}_{1,3}$.

$$\mathbb{C}(4) \quad \psi_s \mapsto \Psi = \psi u_1 \ , \ \psi \in \mathbb{R}^+_{1,3} \ , \quad u_1 = \frac{1}{2}(1+e_o) \ ; \ u_1^2 = u_1 \qquad (37)$$

where e_μ, $\mu = 0,1,2,3$ is an orthonormal basis of $\mathbb{R}^{1,3}$(*), and u_1 is the representation of u_{s1} in $\mathbb{R}_{1,3}$. Interpreting the γ^μ as representations of the vectors $e^\mu = g^{\mu\nu} e_\nu$, we get the following representation of eq.(33) in the Clifford bundle over space time

$$(\hbar \Box \psi e_2 e_1 - qA\psi) e_o u_1 = m\psi u_1 \ ; \ \Box = e^\mu \partial_\mu \quad ; \quad A = e^\mu A_\mu . \qquad (38)$$

We observe that in eq. (38) the $i = \sqrt{-1}$ has been eliminated! Now altough u_1 has no inverse, the coefficients of u_1 can be equated and we have

$$\hbar \Box \psi e_2 e_1 - qA\psi = m\psi e_o \quad . \qquad (39)$$

Finally considering

$$\phi = \psi u \ ; \ u = \frac{1}{4} \ (1+e_o)(1+e_3 e_o) \qquad (40)$$

eq.(39) can be written as

$$\hbar \Box \phi e_5 - qA\phi = m\phi \quad ; \quad e_5 = e_o e_1 e_2 e_3 \qquad (41)$$

which appears originaly in ref.(18). It is quite clear that ϕ is an element of a minimal ideal in $\mathbb{R}_{1,3}$ since u is a primitive idempotent.

In resume, the *standard Dirac spinors* are represented in $\mathbb{R}_{1,3}$ by the elements of the minimal ideal $I_{1D} = \mathbb{R}^+_{1,3} u_1$ of $\mathbb{R}_{1,3}$ and a simple

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(*) Observe that the Minkowski-space M is the affine space constructed with $\mathbb{R}^{1,3}$.

Dirac equation is written using the elements of the minimal ideal $I_D = \mathbb{R}^+_{1,3}u$ of $\mathbb{R}_{1,3}$. Obviously I_{1D} and I_{2D} are isomorphic.

We can show without difficult that I_{1D} (or I_{2D}) is the carrier space of the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $SL(2,\mathbb{C})$. In refs(1,2) Hestenes calls $\psi \in \mathbb{R}^+_{1,3}$ a spinor. This seems to be non-sequitur in the sense that as $\psi \in \mathbb{R}^+_{1,3}$ is a sum of scalar, pseudo-scalar and bivector parts and the space of the ψ's is the carrier space of the representation $D^{(0,0)} \oplus D^{(1/2,0)} \otimes D^{(0,1/2)} \oplus D^{(0,0)}$ of $SL(2,\mathbb{C})$ (19). Nevertheless it is important to emphasize that working directly with eq.(39) (ie, with ψ) gives the correct giromagnetic factor for the electron!

These results show that it is not certain that the Kähler-Dirac equation as presented in ref.(17) can correctly represent the Dirac equation in the Clifford bundle over Minkowski space-time. We will analyse this point in another paper.

As to (B) we think that the present paper gives the relation between spinors and Spinors in a clear way. Our method of identification of spinors with Spinors are based on the concept of Spinorial metric and the observation that for $p + q \leq 5$, $\mathrm{Spin}^+(p,q)$ is the invariance group of the Spinorial metrics.

Among the important results obtained we emphsize that here for the first time there appears the representation of undotted and dotted two-component and Dirac spinors in $\mathbb{R}_{1,3}$. In particular we gave a rigorous proof that the space-time Spinors, ie, the elements of $I_D = \mathbb{R}_{1,3}f(e_{30})$ carry the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $SL(2,\mathbb{C})$ and thus can be said to give a representation of Dirac spinors as introduced in (iii) of §1.

Also the *standard Dirac spinors* [eq.(35)] are represented in $\mathbb{R}_{1,3}$ by the elements of the ideal $I_{1D} = \mathbb{R}^+_{1,3}u_1$ and a simple Dirac equation written using the Clifford bundle uses as field variable the element of $I_{2D} = \mathbb{R}^+_{1,3}u$.

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