THE FUNDAMENTAL EQUATIONS OF MINIMAL SURFACES IN $\mathbb{C}P^2$

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ABSTRACT: In this paper we give a characterization of minimal surfaces in $\mathbb{C}P^2$ in terms of the Gaussian curvature K, the normal curvature K_N and the Kahler angle α. Also, we obtain global restrictions of K, K_N and α.

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THE FUNDAMENTAL EQUATIONS OF MINIMAL SURFACES IN $\mathbb{C}P^2$

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0. Introduction.

The investigation of minimal surfaces in complex projective n-space has gained interest by recent work of the physicists Din and Zakrzewski [3],[4] and Glaser and Stora [6] . Mayor contributions are done by Eells and Wood [5] and Wolfson [16] , which among other results, give a complete description of the branched minimal immersions of the 2-sphere S^2 in $\mathbb{C}P^n$, using the nonexistence of nonzero global holomorphic differentials on S^2.

Our intention was to give a characterization of minimal surfaces in terms of local invariants alone. This could be done successfully in the case of real codimension 2 where the geometry of the normal bundle is given by a single curvature function K_N . We were motivated by earlier work of Tribuzy and Guadalupe [15] which gave such a characterization for minimal surfaces in S^4 in terms of the Gaussian curvature K and the normal curvature K_N. Replacing S^4 by $\mathbb{C}P^2$, we need an additional invariant related to the complex structure: The Kähler angle α introduced by Chern and Wolfson [2] (see § 1 for definition). However, we restrict our attention to (non-branched) immersions. As a particular case, we characterize the induced metrics of (anti-) holomorphic and totally real minimal immersions by a condition on the Gaussian curvature alone. All these results are stated in § 2, while the proofs of the main theorems are given in § 6 and § 7. As a tool, we need an existence and congruence theorem for mappings into symmetric spaces which is derived in § 5. We wish to mention that all local results can easily be generalized to Kähler 4-manifolds of arbitrary constant holomorphic sectional curvature c. Since the projective plane is the most interesting space for global applications, we restrict our attention to c = 4.

Global applications are given in § 3 and § 4. We avoid refering to the Riemann-Roch theorem but instead use elementary properties of the Laplacian. We improve the results of Eells and Wood on surfaces of higher genus and get in particular constraints for compact minimal immersions in terms of the genus, the degree and the self-intersection number. Curvature constraints are given in § 4.

Finally, we wish to thank many people whose hints and discussion made this paper possible, in particular Prof. Chern, who introduced us to the subject, Prof. Schoen, Prof. Lawson, J. Wolfson, Th. Langer and M. Micallef.

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1. Generalities

Let M be an oriented surface and $(P, <,>)$ a 4-dimensional oriented Riemannian manifold, $f : M \to P$ an immersion. Let $f^{*}TP = TM \oplus NM$ be the usual orthogonal decomposition into the tangent and the normal bundle, $\nabla = \nabla^{T} + \nabla^{N}$ be the corresponding decomposition of the Levi-Civita connection of P, pulled back to M. ∇^{T} is the Levi-Civita connection of the induced metric ds^2 on M, ∇^{N} the normal connection. Let R, R^T and R^N be the curvature tensors of ∇, ∇^T and ∇^N, respectively. Choose oriented orthonormal local frames e_1, e_2 of TM and e_3, e_4 of NM. The real valued functions

$$K = < R^T(e_1, e_2)e_2, e_1 > \quad , \quad K_N = < R^N(e_1, e_2)e_4, e_3 >$$

called Gaussian curvature and normal curvature, are independent of the choice of the frames. The Gauss - Bonnet - Chern theorem relate these to the Euler numbers χ of TM and χ_N of NM:

$$\chi = 2\pi \int_M K \, dM \quad , \quad \chi_N = 2\pi \int_M K_N \, dM$$

where dM is the volume element of M.

The second fundamental form $A : TM \otimes TM \to NM$ is defined by $A(X,Y) = (\nabla_X Y)^{\perp}$. Let $A^3 = < A, e_3 >$, $A^4 = < A, e_4 >$. We consider those as symmetric 2×2 matrices with coefficients $A^{\rho}_{ij} = < \nabla_{e_i} e_j, e_{\rho} >$, $i,j = 1,2$; $\rho = 3, 4$. These matrices are related to the curvatures by Gauss and Ricci equations

$$K = \hat{K} + \det A^3 + \det A^4 \ , \quad K_N = \hat{K}_N + < [A^3, A^4] e_2, e_1 >$$

where $\hat{K} = < R(e_1, e_2) e_2, e_1 >$ and $\hat{K}_N = < R(e_1, e_2) e_4, e_3 >$

The immersion f is called minimal if trace $A = A_{11} + A_{22} = 0$. So the matrices A^3 and A^4 have the form

$$A^3 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad \text{and} \quad A^4 = \begin{pmatrix} c & d \\ d & -c \end{pmatrix}$$

and the Gauss and Ricci equations become

(1.1) $\quad K = \hat{K} - (a^2 + b^2 + c^2 + d^2)$

(1.2) $\quad K_N = \hat{K}_N + 2 (ad - bc)$

Define the bundle map $\bar{A} : TM \to NM$ by $\bar{A}(X) = A(X, X)$. Let T^1_pM be the unit circle in T_pM. Then $\bar{A}(T^1_pM)$ is a (possibly degenerated) ellipse in N_pM, which is double covered by T^1_pM and in the minimal case centered at the origine. This is called the ellipse of curvature. Namely, if $X = (\cos\Theta) e_1 + (\sin\Theta) e_2$, then in the minimal case $A_{11} = - A_{22}$ we have $\bar{A} X = (\cos 2\Theta) A_{11} + (\sin 2\Theta) A_{12}$. The oriented area of this ellipse is given by $\pi \det (A_{11}, A_{12}) = \pi (ad - cb) = \frac{1}{2} \pi (K_N - \hat{K}_N)$. Thus outside the zero section, $\bar{A}$ has degree 2 if $K_N > \hat{K}_N$ and -2 if $K_N < \hat{K}_N$. The ellipse of curvature is a circle if and only if $A_{11} \perp A_{12}$, $\|A_{11}\| = \|A_{12}\|$, hence if and only if $a = \pm d$, $b = \mp d$.

Now let P be a Kähler manifold, i.e there is a tensor $J \in \mathrm{End}(TP)$ with $J^2 = -I$ and $\nabla J = 0$. J defines a $\mathbb{C}$ - scalar multiplication in any tangent space: If $c = a + ib \in \mathbb{C}$, $X \in T_pP$, then $cX := aX + bJX$. The Kähler form $\emptyset \in \Omega^2(P)$, defined by $\emptyset(X, Y) = \langle JX, Y\rangle$, represents a cohomology class in $H^2(P, \mathbb{R})$. Let $C_o > 0$ be the smallest number such that $C_o \emptyset$ represents an integral class. If $P = \mathbb{C}P^2$ with constant holomorphic curvature 4 (see below), then $C_o = \pi^{-1}$.

For the immersion $f : M \to P$ define $\alpha : M \to [0, \pi]$ by $\cos \alpha = \langle Je_1, e_2\rangle = f^*\emptyset(e_1, e_2)$, called the <u>Kähler angle</u>. $\cos\alpha$ is independent of the choice of the frame, in fact we have $f^*\emptyset = \cos\alpha \, dM$ where dM is the volume element of M. Thus $d := C_o \int_M \cos\alpha \, dM$ is an integral number, defined by the map $f^* : H^2P \to H^2M$. This is called the <u>degree</u> of f.

<u>Remark</u>. It is not clear whether the angle α itself can be defined globally as a smooth function. It certainly is smooth on the subset $\{\cos^2\alpha \neq 1\}$.

The immersion $f : M \to P$ is called <u>complex</u> if and only if $\sin\alpha \equiv 0$, hence if and only if $J(TM) = TM$. In this case either $\cos\alpha = 1$ and $Je_1 = e_2$ (holomorphic case) or $\cos\alpha = -1$ and $Je_1 = -e_2$ (antiholomorphic case). Sine $\nabla JX = J\nabla X$ and since A is symmetric, we have $A(JX, JX) = J^2A(X,X) = -A(X, X)$, thus f is minimal. Moreover, $A(X, JX) = JA(X,X) \perp A(X, X)$, hence the ellipse of curvature is a circle.

Now, we restrict our attention to $P = \mathbb{C}P^2 = \mathbb{C}^3 - \{0\} / \mathbb{C}^* = S^5/S^1 = \{[x] \,/\, x = (x_o, x_1, x_2) \in \mathbb{C}^3 - 0\}$. We choose the Riemannian metric on P so that $\pi : S^5 \to P$ becomes a Riemannian submersion. This is the Fubini-Study metric with holomorphic sectional curvature 4. For this metric, we have

(1.3) $\hat{K} = 3\cos^2\alpha + 1$

(1.4) $\hat{K}_N = 3\cos^2\alpha - 1$

(see e.g. Kobayashi and Nomizu ([11] , p.166). The group $G = SU(3)$ acts on P by holomorphic isometries, more precisely, G/Z with $Z = \{ z \cdot I \; ; \; z \in \mathbb{C} \, , \, z^3 = 1 \}$ is the connected component of the isometry group I (P).

It is well known that on P, beneath the complex immersions, there is another distinguished family of minimal immersions which we can construct as follows (see [5]). Let $p \in P$ and $x \in \pi^{-1}(p) \subset S^5 \subset \mathbb{C}^3$. Then $d\pi(x)$ maps the horizontal subspace $H_x = \{ v \in T S^5 , v \perp x \} = (\mathbb{C} \cdot x)^{\perp} \subset \mathbb{C}^3$ isometrically onto $T_p P$. At any other point $x' \in \pi^{-1}(p)$ we have $x' = e^{i\theta} x$ for some $\theta \in \mathbb{R}$, hence $d\pi(x')|_{H_{x'}} = e^{i\theta} d\pi(x)|_{H_x}$. Therefore, any complex line $\ell \subset T_p P$ (considered as a $\mathbb{C}$ - vector space) has a horizontal lift $\hat{\ell} \subset \mathbb{C}^3$ independent of the preimage $x \in \pi^{-1}(p)$. This defines an element $[\hat{\ell}] \in \mathbb{C}P^2 = P$.

Now if $h : M \to P$ is a complex immersion, then $dh(T_m M)$ defines a complex line $\ell(m)$ for any $m \in M$. Thus we get a mapping $f : M \to P$, $f(m) = [\hat{\ell}(m)]$, called the <u>associated map</u> to h (following [5]). This is known to be a branched minimal immersion. In fact we will show that f is unbranched if so is h (Thm. B).

Following the notation of [5] , we will call an immersion $f : M \to P$ <u>isotropic</u> if f is either complex or associated to a complex immersion(called " associated " for short).

Another interesting family of minimal surfaces are the real ones; these are minimal immersions which satisfy $\cos\alpha \equiv 0$, in other words: $J(TM) = NM$.

2. Local results.

The results in this chapter are local in the sense that we do not need assumptions about completeness or compactness. The proofs of the followings theorems A, B and C will be given in chapter 7.

Theorem A. Let (M, ds^2) be a surface with Gaussian curvature function K and Laplacian Δ.

(i) Let $f : M \to P = \mathbb{C}P^2$ be a minimal isometric immersion with normal curvature K_N and Kähler angle α. Let

$$k = K_N - K + 2 \qquad , \qquad \ell = k \sin^2\alpha$$

$$c = \cos(\alpha/2) \, , \qquad s = \sin(\alpha/2)$$

(a) If f is isotropic , then $\ell \equiv 0$

If f is not isotropic , then $\ell \geq 0$ with only isolated zeros and

(2.1) $\Delta \log \ell = 6K$ on $\{\ell \neq 0\}$

(b) In either case the following equations hold :

(2.2) $\|\nabla \cos\alpha\|^2 = \sin^2\alpha\,(6\cos^2\alpha - K - K_N)$ on M

(2.3) $\Delta \log \tan(\alpha/2) = 6\cos\alpha$ on $\{\sin\alpha \neq 0\}$

(2.4) $\Delta \log k = 2(2K - K_N)$ on $\{k \neq 0\}$

(2.5) $\Delta \log s^2 = K + K_N + 6\cos\alpha$ on $\{s \neq 0\}$

(2.6) $\Delta \log c^2 = K + K_N - 6\cos\alpha$ on $\{c \neq 0\}$

(ii) Let smooth functions $K_N : M \to \mathbb{R}$ and $\cos\alpha : M \to [-1, 1]$ be given which satisfy (2.2) , (2.3) , (2.4), (2.5) and (2.6) . Let U be a simply connected open subset of $M - (\partial \{ \sin \alpha = 0 \}) \cup \partial \{ k = 0 \}$. Then there exists an isometric minimal immersion $f : U \to P$ with normal curvature K_N and Kähler angle α . This is uniquely determined up to motion of P.

Remark . (2.5) + (2.6) yields

(2.7) $\Delta \log \sin^2 \alpha = 2 (K + K_N)$ on $\{ \sin \alpha \neq 0 \}$

and (2.5) - (2.6) yields (2.3) . Note further that (2.7) + (2.4) imply (2.1).

Equations (2.1),....,(2.7) will be called the fundamental equations. On $\{\rho \neq 0\}$, they can be derived from (2.4) , (2.5) and (2.6) alone . Note that (2.3) is equivalent to equation (2.58) in ([16] , p.40) .

Theorem B . Let $f : M \to P$ be a minimal immersion of a surface M. Then the ellipse of curvature is a circle everywhere if and only if f is either isotropic or real.

Theorem C . Let M be a surface.

(i) If $f : M \to P$ is a minimal immersion with Gaussian curvature K and normal curvature K_N , then $K - K_N \leq 2$, and equality holds everywhere if and only if f is associated to a holomorphic map $h : M \to P$.

(ii) Let $h : M \to P$ be a holomorphic immersion. Then the associated map $f : M \to P$ is an immersion, too, and the metrics ds_h^2 and ds_f^2 induced by h and f on M are conformal with

$$ds_h^2 = \sin^2 (\alpha / 2) \, ds_f^2$$

where α is the Kähler angle of f.

Theorem 2.1 . Let $f : M \to P$ be a minimal immersion of a surface M with K, K_N as above. Then $K + K_N \leq 6$, and equality holds everywhere if and only if f is complex. If f is real, then $K + K_N \equiv 0$.

Remark. We will see in proposition 4.2 (i) that the converse of the last statement also holds, provided that M is compact.

Proof. By (2.2), $\sin^2\alpha\,(K + K_N - 6) = -6\sin^4\alpha - \|\nabla\cos\alpha\|^2$. If $K + K_N = 6$, this implies $\sin\alpha \equiv 0$. If $\sin\alpha \not\equiv 0$, we have in turn $K + K_N < 6$. If $\sin\alpha = 0$ on an open set U, (2.5) or (2.6) imply $K + K_N = 6$ on U .

Theorem 2.2 . Let (M , ds^2) be a surface with Gaussian curvature K.

(i) Let $f : M \to P$ be a complex isometric immersion. Then

(2.8) $\Delta \log (4 - K) = 6 (K - 2)$ on $\{ K < 4 \}$

(ii) Suppose that M is simply connected and either $K \equiv 4$ or $K < 4$ everywhere and (2.8) holds. Then, there exists a complex isometric immersion $f: M \to P$, unique up to isometries of P.

Proof. If we let $K_N = 6 - K$, $\sin\alpha = 0$, then (2.8) is equivalent to (2.1),...., (2.6) (remember $K \leq \tilde{K} = 4$ by (1.1), (1.3) . Thus the result follows from theorem A.

Corollary 2.3 . Any complex immersion $f : M \to P$ of constant curvature K is (up to motions) a parametrization of the Veronese surface

$$V = \{ [x] \in P \ / \ x_0^2 + x_1^2 + x_2^2 = 0 \}$$

with $K = 2$, $K_N = 4$, or of the totally geodesic projective line

$$\mathbb{C}P^1 = \{ [x] \in P \ / \ x_2 = 0 \}$$

with $K = 4$, $K_N = 2$

Proof . By the preceding theorem, we have $K \equiv 4$ if and only if $f(M) \subset \mathbb{C}P^1$. If $K = \text{constant} \neq 4$, then $K = 2$ by (2.8). So there exists a complex surface with $K = 2$, $K_N = 4$. On the other hand, V is a complex surface and an orbit of the subgroup $H = SO(3)$ of the holomorphic isometry group $G = PSU(3)$ of P, namely $V = H[1, i, 0]$. So it has constant curvature K. But as an algebraic surface of degree 2, it is not congruent to $\mathbb{C}P^1$. Thus, it must be the unique complex surface of curvature 2.

Remark. Instead of V, one can consider

$$V' = \{ [x] \in P \ / \ x_1^2 = 2 x_o x_2 \}$$

which is congruent to V. This has the nice parametrization $f : \hat{\mathbb{C}} \to V' \subset P$, $f(z) = [1, z, \frac{1}{2} z^2]$.

Theorem 2.4 . Let (M, ds^2) be a surface with Gaussian curvature K.

(i) Let $f : M \to P$ be a real minimal isometric immersion. Then

(2.9) $\Delta \log (1 - K) = 6K$ on $\{K < 1\}$

(ii) Suppose that M is simply connected and either $K \equiv 1$ or $K < 1$ everywhere and (2.9) holds. Then, there exists a real minimal isometric immersion $f: M \to P$, unique up to isometries of P.

Proof. If we let $K_N = -K$, $\cos \alpha = 0$, then this is a special case of theorem A. Remember $K \leq \hat{K} = 1$ by (1.1), (1.3)

Corollary 2.5 . Any real minimal immersion of constant curvature is (up to motions) a parametrization of either the " Clifford torus "

$$CT = \{ [x] \ / \ x_o \bar{x}_o = x_1 \bar{x}_1 = x_2 \bar{x}_2 \}$$

with $K = 0$ or the totally geodesic real projective plane

$$\mathbb{R}P^2 = \{ [x] \ / \ \bar{x}_a = x_a , \ a = 0, 1, 2 \}$$

<u>Remark.</u> If a real minimal immersion $f : M \to P$ is also associated, then $K = -K_N$ and $K - K_N = 2$, hence $K = 1$ and $f(M) \subset \mathbb{R}P^2$, by this corollary. We will see in theorem 3.3 (i) and proposition 4.1, that this implies theorem 7 of Yau [17].

<u>Proof.</u> The mapping $\varphi : S^5 \to S^5$, $X \to \bar{X}$ (consider $S^5 \subset \mathbb{C}^3$) is an isometry which preserves the complex lines $\varphi(e^{i\theta} X) = e^{-i\theta}\varphi(X)$. So it induces an isometry $\bar{\varphi} : P \to P$ whose fixed point set is $\mathbb{R}P^2$. So $\mathbb{R}P^2$ is totally geodesic and real, hence $K = 1$. If $K = \text{const.} \neq 1$, then $K = 0$ by (2.9). On the other hand, $C\,T$ is an orbit of the subgroup

$$T = \left\{ \begin{pmatrix} e^{ia} & & \\ & e^{ib} & \\ & & e^{ic} \end{pmatrix} \Big/ \; a + b + c = 0 \right\}$$

of $SU(3)$, namely $C\,T = T[1,1,1]$ (with isotropy subgroup Z) and $SU(3)$ acts on P by isometries. So all the functions K, K_N and α are constant, in particular $K = 0$, since $C\,T$ is a torus and $K_N = 0$, since the normal bundle is trivial because the subgroup $T/Z \subset I(P)$ acts freely on $C\,T$.

An easy computation shows that $J(T(C\,T)) \perp T(C\,T)$, therefore $C\,T$ is real. Moreover, $C\,T$ is the projection of the 3 - dimensional Clifford torus $C\hat{T} = \{ (x_0, x_1, x_2) \, / \, x_0\bar{x}_0 = x_1\bar{x}_1 = x_2\bar{x}_2 = 1/3 \}$ in S^5, so $C\,T$ is minimal by the following lemma which finishes the proof.

<u>Lemma 2.6</u> . <u>Let $\rho : \hat{M} \to M$ and $\pi : \hat{P} \to P$ be Riemannian submersions. Suppose $\hat{f} : \hat{M} \to \hat{P}$ and $f : M \to P$ are isometric immersions, moreover suppose that</u>

$\hat{f}(\rho^{-1}(x)) = \pi^{-1}(f(x)) =: \hat{P}_x$ and $\hat{P}_x$ is a minimal submanifold of $\hat{P}$, for all $x \in M$. Then $\hat{f}$ is minimal if and only if f is minimal.

Proof. Let $e_1, \ldots, e_m$ be an orthonormal frame on M. Let $\hat{e}_1, \ldots, \hat{e}_m$ be the horizontal lift to $\hat{M}$; choose orthonormal fields $\hat{e}_{m+1}, \ldots, \hat{e}_n$ on $\hat{M}$ such that $\hat{e}_1, \ldots, \hat{e}_n$ is an orthonormal frame. Call $A_{ij} = (\nabla_{e_i} f_* e_j)^{\perp}$ and $\hat{A}_{ab} = (\hat{\nabla}_{\hat{e}_a} \hat{f}_* \hat{e}_b)^{\perp}$ the second fundamental forms of M and $\hat{M}$ (choose $i,j = 1,\ldots,m$; $a,b = 1,\ldots,n$; $\mu = m+1,\ldots,n$). Then $\hat{f}_* \hat{e}_i$ are horizontal in $\hat{P}$, hence $\pi_* (\hat{\nabla}_{\hat{e}_i} \hat{f}_* e_i) = \nabla_{e_i} f_* e_i$. On the other hand, $\hat{f}_* \hat{e}_{m+1}, \ldots, \hat{f}_* \hat{e}_n$ form an orthonormal frame of the fibres of $\pi : \hat{P} \to P$ in the range of f . These are minimal, hence $\sum_{\mu=m+1}^{n} \hat{A}_{\mu\mu} = 0$. So trace $\hat{A} = \sum_{i=1}^{n} \hat{A}_{ii}$, hence π_*(trace $\hat{A}$) = trace A . But trace $\hat{A}$ is a normal vector along $\hat{f}$, and $\hat{f}(\hat{M})$ contains the fibres, so trace $\hat{A}$ is horizontal. Therefore, trace $\hat{A} = 0$ if and only if trace $A = 0$.

Remark. Note that by the same argument as in the proof of corollary 2.5, any principal orbit of T is a torus with $K = K_N = \cos\alpha = 0$. But an orbit Tx is congruent to T [1, 1, 1] if and only if x = n [1, 1, 1] with n in the normalizer of T. All other orbits are not congruent, hence, by theorem 2.4 (ii), not minimal. This shows that only minimal surfaces are uniquely determined by the three functions K , K_N and α .

A submanifold $M \subset P$ is called homogeneous if a subgroup of $I(P)$ acts transitively on M. So, for a homogeneous surface in P, the functions K, K_N and α are constant . Thus we have proved

Proposition 2.7 . The only homogeneous minimal surfaces in $P = \mathbb{C}P^2$ are $\mathbb{C}P^1$, V, $\mathbb{R}P^2$ and CT.

Remark 2.8 . Let $h : M \to P$ be a holomorphic immersion and $f : M \to P$ the associated immersion. We have $K_{N_f} = K_f - 2$ and hence from (2.5) for the Kähler angle α of f and $s = \sin \alpha/2$

$$\Delta_f \log |s| = K_f - 1 + 3 \cos \alpha = K_f + 2 - 6 s^2$$

Theorem C (ii) yields $s^2 K_h = K_f - \Delta_f \log |s|$, so we get $s^2 = 2 / (6 - K_h)$ and therefore from theorem C (ii) again

(2.10) $$d s_f^2 = 1/2 \, (6 - K_h) \, d s_h^2$$

From (2.7) we get $4 (K_f - 1) = \Delta_f \log (s^2 (1 - s^2)) = \Delta_f \log (4 - K_h) - 2 \Delta_f \log (6 - K_h)$, and further, using $\Delta_f = 2 (6 - K_h)^{-1} \Delta_h$ and (2.8),

$$K_f = 1 + (6 - K_h)^{-1} [\, 3 (K_h - 2) - 4 \Delta_h \log (6 - K_h) \,]$$

So the geometry of $d s_h^2$ determines the geometry of $d s_f^2$.

If $f : M \to P$ is a noncomplex minimal immersion, we can rewrite the fundamental equations locally as follows

Case a) f not associated, i.e $k \neq 0$. Then $\{\ell = 0\}$ contains only isolated points. Let $M_o = \{\ell \neq 0\}$ and $\lambda = (1/4t\,\ell)^{1/6}$ on M for an arbitrary constant $t > 0$. Since $K = -\Delta \log \lambda$ by (2.1), the metric $d s_o^2 = \lambda^{-2} d s^2$ is flat. Hence $(M_o, d s_o^2)$ is isometrically covered by an open subset $\mathcal{U}$ of $\mathbb{C}$. We pull back all function to $\mathcal{U}$. Introducing $p = \lambda c$, $q = \lambda s$ we get from (2.3) and (2.7)

$$\Delta^o \log |p q| = t(\bar{p} q)^{-2} - 2(p^2 + q^2)$$

$$\Delta^o \log |q/p| = 6(p^2 - q^2)$$

where $\Delta^o = \lambda^2 \Delta$ is the euclidean Laplacian. Now setting

$$u = \log |p q| = \log (1/2\ \lambda^2\ |\sin \alpha|)$$

$$v = \log |q/p| = \log |\tan(\alpha/2)|$$

we end up with

$$\Delta^o u = t e^{-2u} - 4 e^u \cosh v \tag{2.11}$$

$$\Delta^o v = -12 e^u \sinh v \tag{2.12}$$

Since (2.1) is automatically true, by the choice of λ, these equations are equivalent to the fundamental equations. Therefore, by theorem A, any solution of these gives a unique local minimal immersion into P. A trivial solution of (2.12) is $v = 0$ which corresponds to a real minimal immersion.

Case b) f is associated. Here we have $K_N = K - 2$ instead of (2.4) and from (2.7) $\Delta \log |\sin \alpha| = 2K - 2$ on $\{\sin \alpha \neq 0\}$. Let z be any conformal coordinate system on $\{\sin\alpha \neq 0\}$ and λ its conformal factor. Using the equation $K = -\lambda^{-2}\Delta^{o}\log\lambda$ we get $\Delta^{o}\log\lambda^{2}\,|\sin\alpha| = 2\lambda^{2}$. From this and (2.3) we derive

$$\Delta^{o} u = -4 e^{u}\cosh v \tag{2.13}$$

$$\Delta^{o} v = \mp 12 e^{u}\sinh v \tag{2.14}$$

defining u and v as in case a . Any solution of these determines a unique associated local minimal immersion into P. In the special case v = 0, (2.13) leads to the equation $-\Delta^{o}\log\lambda = \lambda^{2}$ whose solutions are the conformal factors on the sphere of curvature 1.

Note that the differential equations (2.11), (2.13) and (2.14) have analytic coefficientes so that the Cauchy- Kowalewski theorem applies (see John [10] , Chap. II, 4).

3. Restrictions on χ, χ_N and d

The following well known lemma is basic. If M is compact and $f : M \to R$ has no zeros, then by the Stokes theorem, $\int_M \Delta\log|f| = 0$. R. Schoen indicated to us the following generalization:

Lemma 3.1 . Let M be a compact Riemannian manifold, $f : M \to R$ a C^{∞} - function whose zero set Z has measure zero, $g : M \to R$ a continuous function with

$$\Delta\log|f| = g \quad \text{outside } Z$$

Then $\int_M g \leq 0$.

If dim M = 2 and Z is isolated, then $\int_M g = 0$ only if $Z = \emptyset$

<u>Proof</u>. Let δ be a small regular value of $h = |f|$ and M^{δ} the set where $h < \delta$; choosing δ small we obtain $\left| \int_{M^{\delta}} g \right| \leq \varepsilon$. Let $M_{\delta} = M - M^{\delta}$. Then

$\int_{M_\delta} g = \int_{M_\delta} \Delta \log h = \int_{\partial M_\delta} - < \nabla h , \nabla \log h > / \| \nabla h \|$. Since

$- \nabla h / \| \nabla h \|$ is the normal vector pointing outward M_δ.

Now $\int_{M_\delta} g = - \int_{\partial M_\delta} \| \nabla h \| / h < 0$, hence $\int_M g < \varepsilon$ for any ε, hence $\int_M g \leq 0$.

We want to show that $\lim_{\delta \to 0} \int_{M_\delta} g < 0$ unless $Z = \emptyset$. Let C_δ be a connected component of ∂M_δ ; then C_δ is a curve surrounding a zero $p \in Z$. We may assume $f > 0$ on C_δ. Let $z : M \to \mathbb{C}$ be a Riemann normal chart at p with $z(p)=0$. If δ is small enough, the error $\left| \int_{C_\delta} \| \nabla f \| / f - \int_{C_\delta \circ z^{-1}} (\| \nabla f \| / f) \circ z^{-1} \right|$ gets small, so we may assume $M = \mathbb{C}$ and $p = 0$. Moreover, f may be replaced by its lowest order nonvashing Taylor polynomial f_k, since $\int_{C_\delta} \| \nabla f \| / f \sim$ $\int_{\{f_k = \delta\}} \| \nabla f_k \| / | f_k |$ as $\delta \to 0$. Now f_k is homogeneous of degree $k \geq 1$, while $b = \| \nabla f_k \| / | f_k |$ is of degree -1. Making the substitution $y = \delta^{-1/k} x$ we have $f(y) = 1$ if and only if $f(x) = \delta$ and hence

$$\int_{\{f_k = \delta\}} b(x)\, ds(x) = \int_{\{f_k = 1\}} b(\delta^{1/k} y)\, \delta^{1/k}\, ds(y) = \int_{\{f_k = 1\}} b(y)\, ds(y) > 0 .$$

Therefore, $\int_{C_\delta} \| \nabla f \| / f$ has a positive limit for $\delta \to 0$ and the result follows.

Let M be a compact connected surface of Euler characteristic χ and $f : M \to P$ a minimal immersion with degree d and normal Euler number χ_N (see § 1). Without loss of generality, we can assume that M is orientable. From equations (2.1), (2.4), (2.5), (2.6), (2.7) and lemma 3.1 , we get immediately the following global statements, if f is not isotropic:

(3.1) $\chi \leq 0$ and equality holds if and only if φ has no zeros.

(3.2) $2\chi - \chi_N \leq 0$ and equality holds if and only if $2 + K_N - K$ has no zeros.

(3.3) $\chi + \chi_N + 3 |d| \leq 0$ and equality holds if and only if $\sin \alpha$ has no zeros.

(3.4) $\chi + \chi_N \leq 0$ and equality holds if and only if $\sin \alpha$ has no zeros.

Combining (3.2) and (3.4), we get

(3.5) $\chi + |d| \leq 0$ and equality holds if and only if φ has no zeros.

Remark. From (3.1), (3.4) and (3.5) we see that φ has to have zeros unless $\chi = \chi_N = d = 0$. The torus C T introduced in § 2 is an example of this situation.

Theorem 3.2 . Let M be a compact orientable surface of genus g and $f : M \to P$ a minimal immersion of degree d.

(i) If g = 0, then f is isotropic

(ii) If g = 1 , d = 0 , then f is isotropic.

If d = 0 and f is not isotropic, then $\chi_N = 0$ and φ has no zero.

(iii) If $g > 1$ and $|d| \geq 2g - 2$, then f is istropic.

Remark. (i) is well known in [3], [4], [5], [6] and [16]. (ii) has been proved in [5]. (iii) is an improvement of proposition 7.8 in [5].

Proof. From (2.7) we see that f is isotropic if $|d| > 2g-2$. If $|d| = 2g-2$, then by (3.1) ρ has no zeros, hence by the preceding remark, M is a torus with $\chi_N = 0$. This proves the theorem.

Let M be a compact oriented surface and $f : M \to P$ an isometric minimal immersion. Integrating the inequalities in theorem C (i) and theorem 2.1, we get

(3.6) Area (M) $\geq \pi(\chi - \chi_N)$ and equality holds if and only if f is associated.

(3.7) Area (M) $\geq \pi/3\,(\chi - \chi_N)$ and equality holds if and only if f is complex.

In particular, if f is isotropic, then there is no minimal surface of smaller area in the same isotopy class.

Proposition 3.3 . Let M be a compact orientable surface and $f : M \to P$ an isotropic minimal immersion. Then one of the following cases hold:

a) $f(M) = \mathbb{C}P^1$, $|d| = 1$, $\chi = 2$, $\chi_N = 1$, area (M) = π

b) f is complex, area (M) = $\pi|d| = \pi/3(\chi + \chi_N) > \pi(\chi - \chi_N)$,

$2\chi - \chi_N \leq 0$

c) f is associated, area (M) = $\pi(\chi - \chi_N) > \pi/3(\chi - \chi_N)$,

$\chi + \chi_N + 3|d| \leq 0$.

Proof. If f is both complex and associated, then $K - K_N = 2$, $K + K_N = 6$ imply $K = 4$, $K_N = 2$, $\sin\alpha = 0$, hence $f(M) = \mathbb{C}P^1$, by the uniqueness part of theorem A (ii). If f is complex and not associated, then the zero set of k is a real analytic subvariety od M of dim ≤ 1. Hence, by lemma 3.1, the inequality (3.2) still holds. Also, we have equality in (3.7) and strict inequality in (3.6). Moreover, $\pi | d | = \text{Area}(M)$ since $|\cos\alpha| = 1$. This proves (b). Similarly, (c) follows from (3.3), (3.6) and (3.7), since $\sin\alpha \neq 0$

Remarks. 1) The equations in (3.6) and (3.7) should be compared to the equalities in proposition 7.1 (ii) of [5]. Recall that we choosed the holomorphic sectional curvature $c = 4$.

2) If $f : M \to P$ is associated to the holomorphic immersion $h : M \to P$, we have by (2.10) the relation

$$\text{Area } f(M) = 3 \text{ Area } h(M) - \pi \chi$$

Using (3.6) and (3.7) we get a relation between the normal Euler numbers χ_{N_f} and χ_{N_h} of f and h, respectively

$$\chi_{N_f} + \chi_{N_h} = \chi$$

The normal characteristic χ_N on an immersion $f : M \to P$ can be computed from the degree d and the self-intersection number I_f as follows:

Proposition 3.4 Let M be a compact oriented surface and $f : M \to P = \mathbb{C}P^2$ an immersion of degree d which has only regular self-intersections and no multiple points of multiplicity $k \geq 3$. Then

$$\chi_N = d^2 - 2 I_f$$

Proof. Let $S \subset f(M)$ be the set of points which have two preimages under f. S is a finite subset of P. Let $s \in S$, $s = f(x_1) = f(x_2)$. We define s to have the weight $w(s) = +1$ if $f_*(T_{x_1}M)$ and $f_*(T_{x_2}M)$ together define the positive orientation on $T_s P$, otherwise $w(s) = -1$. The self-intersection number I_f is defined as $I_f = \sum_{s \in S} w(s)$. Let $X = f^{-1}(S) \subset M$ and for $x \in X$ define $w(x) = w(f(x))$ and the zero-cycle $[X] = \sum_{x \in X} w(x)x \sim 2 I_f \cdot g \in H^0(M)$ where g denotes the generator of $H_0(M)$ dual to $1 \in H^0(M)$. Call $D : H^* \to H_*$ the Poicare duality map, $[M] \in H_2(M)$ the fundamental class of M. Let $e \in H^2(M)$ denote the Euler class of the normal bundle NM of f. By Herbert ([7], p. ix,x) and Lashof and Smale [11] we have

$$[X] = D_M (f^* D_P^{-1} f_* [M] - e)$$

Now, $< 1, D_M f^* D_P^{-1} f_* [M] > = < f^* D_P^{-1} f_* [M], [M] > = < D_P^{-1} f_* [M], f_* [M] >$ where $< , >$ denotes the pairing $H^* \otimes H_* \to \mathbb{Z}$. Let w be a generator of $H_2(M)$, then $f_* [M] = \pm dw$ and $< D_P^{-1} w, w > = 1$. Moreover, $D_M e = \chi_N g$. Therefore we get $2 I_f = d^2 - \chi_N$ which finishes the proof.

Using the last result, we can give necessary conditions for a minimal embedding where we have $\chi_N = d^2$.

Proposition 3.5 Let M be a compact orientable surface of genus g and $f : M \to P$ be a minimal embedding of degree d.

a) If f is not isotropic, then $2g \geq (d^2 + 3|d| + 2)$

b) If f is complex, then $2g = d^2 - 3|d| + 2$ (compare [7], p. 280)

c) If f is associated, then $f(M) = \mathbb{C}P^1$

Proof. a) and b) are immediate from (3.3) and proposition 3.3 b). If f is associated but $f(M) \not\subset \mathbb{C}P^1$, then by proposition 3.3 c) $\chi - \chi_N > 0$ and $\chi + \chi_N \leq 0$ which is impossible if $\chi_N = d^2 \geqslant 0$.

Remark. The last result can also be expressed by saying : Any full (i.e $\not\subset \mathbb{C}P^1$) associated map has topologically nontrivial self-intersections.

4. Global Restrictions of K, K_N and α.

Proposition 4.1 Let M be a complete Riemannian surface and $f : M \to P$ an isometric minimal immersion. Assume (a) $K \geqslant 0$ or (b) $K \leq 0$ and $\ell \geqslant a > 0$. Then either f is isotropic or M is flat and ℓ = constant.

Proof. Assume that f is not isotropic. Then from (2.1) we have $\Delta \log \ell = 6K$. If $K \geqslant 0$ (case (a)), this implies $\Delta \ell = \|\nabla \ell\|^2 / \ell + 6K\ell \geqslant 0$, hence ℓ is subharmonic. Moreover, recall that $K_N \leq 6 - K$ (Theorem 2.1), so $k = 2 + K_N - K \leq 8 - 2K \leq 8$ and $\ell = k \sin^2 \alpha \leq 8$. So ℓ is bounded from above and hence by Huber [8] $K \equiv 0$ and ℓ = constant.

In case b), we may proceed as in Yau [17] and Klotz and Osserman [12]. We may assume that M is simply connected. The metric $ds_o^2 = \ell^{1/3} ds^2$ on M is flat and complete since ℓ is bounded away from 0. If $\overline{\Delta}$ is the Laplacian of ds_o^2, we have $\Delta^o = \ell^{-1/3} \Delta$, so we get from (2.1) $\Delta^o \log \ell = 6 \ell^{-1/3} K \leq$ Therefore, $- \log \ell$ is a bounded subharmonic function on the euclidean plane which must be a constant. This proves the statement b).

Remark. Proposition 4.1 is a generalization of Theorem 7 in Yau [17]. If f is real, then $\ell = k = 2(1 - K)$. In this case, the assumption $K \leq 0$ in (b) implies $\ell \geqslant 2$.

<u>Theorem</u> 4.2 <u>Let</u> M <u>be a compact oriented surface with curvature</u> K <u>and</u> f: $M \to P$ <u>an isometric minimal immersion.</u>

(i) <u>If</u> $K_N \geqslant -K$, <u>then</u> f <u>is complex or real</u> .

(ii) <u>If</u> $K_N < -K$, <u>then</u> M <u>is a sphere of area</u> 6π <u>and</u> f <u>is associated with degree</u> d=0.

(iii) <u>If</u> $K_N \leqslant 2K$, <u>then</u> f <u>is associated with</u> $K \geqslant -2$ <u>or</u> f(M) = V <u>or</u> f(M) = CT .

<u>Proof.</u> (i) Assume that f is not complex. Then $\sin\alpha$ is nonzero on a dense open subset M_o . On M_o we have by (2.7)

$$\Delta \sin^2\alpha = \|\nabla \sin^2\alpha\|^2 / \sin^2\alpha + 2\sin^2\alpha\,(K + K_N) \geqslant 0$$

So $\Delta \sin^2\alpha \geqslant 0$ on the whole of M, hence $\sin\alpha =$ constant $\neq 0$. By (2.3) we get $\cos\alpha = 0$ which proves the result.

(ii) From (2.2) we can see that the only critical points of $\cos\alpha$ are those where $\sin\alpha = 0$, hence maxima or minima. By a standard Morse theory argument we see that there is exactly one maximum and one minimum and M is a sphere. In particular, f is isotropic(Theorem 3.2(i)) . So f is associated since in the complex case we would have $K_N = 6 - K > -K$ (Theorem 2.1). So $K - K_N = 2$, and $K < -K_N$ implies that $K_N < -1$. Since $\hat{K}_N = 3\cos^2\alpha - 1 \geqslant -1$, the oriented area of the ellipse of curvature(in this case a circle) is everywhere strictly negative (§ 1). Therefore the bundle map $\bar{A} : TM \to TN$ introduced in § 1 has degree deg ($\bar{A}$) = - 2 everywhere outside the zero section. We now use an argument of Asperti-Ferus-Rodriguez ([1] , Thm.1): The index formula for the Euler number, applied to generic tangent vector field $X \in \Gamma TM$ and to the normal field $\bar{A} \circ X \in \Gamma NM$ yields $\chi_N = \deg(\bar{A})\chi = -2\chi$. Hence from proposition 3.3 b), area(M) = $3\chi\pi = 6\pi$ and from part c) , $3\,|\,d\,| \leq -\chi - \chi_N = 2$, so d = 0.

(iii) If f is associated, then $K_N = K - 2$. Hence $K_N \leqslant 2K$ if and only if $K \geqslant -2$.

Now assume f not to be associated. Then, by a similar argument as in (i), we get $k = K_N - K + 2 = \text{constant} > 0$ and $K_N = 2K$, thus both K and K_N are constant. If f is complex, then by corollary 2.3, $f(M) = V$. So assume f to be not isotropic. Then by (3.5) ($\mathcal{V} + |d| \leq 0$), we have $K \leq 0$. However, the case $K < 0$ is impossible since then $K + K_N < 0$ which is excluded by (ii). Thus $K = 0$ and M is a flat torus. By (2.7), $\sin^2\alpha$ is a subharmonic function, hence a nonzero constant. By (2.3) then f is real and corollary 2.5 gives $f(M) = CT$.

Corollary 4.3 Let (M, ds^2) be a compact oriented surface of constant curvature K. If $f: M \to P$ is an associated isometric immersion, then M is a sphere, and $f(M) = \mathbb{C}P^1$ or $f(M) = \mathbb{R}P^2$ or $K = 2/3$, $d = 0$.

Proof. We have $K_N = K - 2$. If $K \geq 1$, then $K_N \geq -1 \geq -K$, so f is either complex or real by (i), with $K = 4$ or $K = 1$, by theorem 2.2 and theorem 2.4, respectively. Now corollary 2.3 and corollary 2.5 imply $f(M) = \mathbb{C}P^1$ or $f(M) = \mathbb{R}P^2$. If $K < 1$, then $K_N < -1 < -K$, so by (i) we have $d = 0$ and $\text{area}(M) = 6\pi$, so $K = 2/3$.

Remark. The question remains open whether or not there exists an associated immersion of constant curvature $K = 2/3$. Clearly, this could not be a homogeneous immersion. More general, the existence of an associated sphere of area 6π (see Prop. 4.2 (ii)) is in doubt.

5. Structure equations and an embedding theorem

Let $(P , < , > , J)$ be a 4-dimensional Kähler manifold. On any tangent space $T_p P$ we define the hermitian product $(X , Y) = < X , Y > + i < X , J Y >$. A unitary basis at p is formed by vectors $E_1 , E_2 \in T_p P$ such that $(E_i , E_j) = \delta_{ij}$.

Now let M be a surface and $f : M \to P$ a smooth mapping. Consider the bundle $f^* TP$ on M with pulled back hermitian metric (,) and connection ∇. Let (E_1 , E_2) a unitary frame along f, i.e a uniatry frame in $f^* TP$. We define complex valued forms $w_i , w_{ij} \in \Omega^1_C (M)$ and $\Omega_{ij} \in \Omega^2_C (M)$ by $w_i (X) = (d f(X) , E_i)$, $w_{ij} (X) = (\nabla_X E_i , E_j)$, $\Omega_{ij} (X , Y) = (R (X , Y) E_i , E_j)$ for $X, Y \in TM$. Then we have $w_{ij} = - \bar{w}_{ji}$, $\Omega_{ij} = - \bar{\Omega}_{ji}$ and the structure equations

(5.1) $\quad d w_i = \sum_k w_k \wedge w_{ki} \; , \qquad d w_{ij} = \Omega_{ij} + \sum_k w_{ik} \wedge w_{kj}$

Now let $P = \mathbb{C} P^2$ and $UP = \{ (p ; E_1 , E_2) \}$ be the bundle of unitary frames on P. Since G = SU(3) is a group of holomorphic isometries on P, it acts on UP, and ~~this action is transitive.~~ In fact, let $\{e_0 , e_1 , e_2\}$ be the standard basis of $\mathbb{C}^3$. Recall that G acts transitively on $S^5 \subset \mathbb{C}^3$ and $\pi : S^5 \to P$ is a G-equivariant Riemannian submersion. Then $\{E^o_1 , E^o_2\}$ with $E^o_i = d\, \pi_{e_o} (e_i)$ forms a unitary basis at $\theta = \pi e_o \in P$. Let $Z = \{ e^{2 \pi ia/3} I \; ; \; a = 0,1,2\}$ denote the center of G. Then the mapping $G/Z \to P$ defined by

$$g Z \to (g\theta \; ; g . E^o_1 , g . E^o_2) = (\pi g e_o ; d\, \pi_{g e_o} g(e_1) , d\, \pi_{g e_o} g(e_2))$$

is bijective.

A smooth $f : M \to P$ with unitary frame $\{E_1, E_2\}$ along f yields a mapping $(f ; E_1, E_2) : M \to P = G/Z$. Let $g : M \to G$ be a (local) lift of this mapping, i.e $f = \pi g e_o$, $E_i = d\pi_{g e_o}(g e_i)$. For any $X \in TM$, we have

(5.2) $w_i(X) = (df(X), E_i) = (d\pi_{ge_o}(dg(X)e_o), d\pi_{ge_o}(ge_i))$

$= (dg(X)e_o, ge_i) = (g^{-1}dg(X)e_o, e_i)$

since π is a Riemannian submersion and ge_i is a horizontal vector at ge_o .

Let $\underline{g} : G \to G$ denote the identity and $\psi = \underline{g}^{-1} d\underline{g} : TG \to T_1 G = \mathfrak{g}$ the Maurer-Cartan form. Put $\psi_{ab} = (\psi e_a, e_b) \in \Omega^1_{\mathbb{C}}(G)$. By (5.2) we have

(5.3) $w_i = g^* \psi_{oi}$

On the other hand we have $\psi_{ba} = -\bar{\psi}_{ab}$ and

(5.4) $d\psi_{ab} = \sum_c \psi_{ac} \wedge \psi_{cb}$

(Maurer-Cartan equations). It follows that

$$dw_i = \sum_k w_k \wedge g^* \varphi_{ki}$$

with $\varphi_{ki} = \psi_{ki} - \delta_{ki}\psi_{oo}$. So, by the structure equations and the Cartan lemma, we get

(5.5) $w_{ki} = g^* \varphi_{ki}$

If we apply (5.4) to φ_{ki} , (5.1) yields

$$\Omega_{ij} = -\bar{w}_i \wedge w_j + \delta_{ij} \sum_k w_k \wedge \bar{w}_k$$

<u>Theorem 5.1</u> Let M <u>be a simply connected surface</u> and $w_i, w_{ij} \in \Omega^1_C(M)$, i,j=1,2, <u>which satisfy</u> $w_{ij} = -\bar{w}_{ji}$ <u>and the structure equations of</u> P

$$(5.6) \qquad d w_i = \sum_k w_k \wedge w_{ki} \,,$$

$$d w_{ij} = \sum_k w_{ik} \wedge w_{kj} + \Omega_{ij} \,, \quad \Omega_{ij} = -\bar{w}_i \wedge w_j + \delta_{ij} \sum_k w_k \wedge \bar{w}_k$$

<u>Then there exists a smooth map</u> $f : M \to P$ <u>and a unitary basis</u> $\{E_1, E_2\}$ <u>along</u> f <u>such that</u> $w_i = (df, E_i)$, $w_{ij} = (\nabla E_i, E_j)$. <u>Moreover,</u> $(f; E_1, E_2)$ <u>is uniquely determined up to motions of</u> P.

<u>Remark.</u> Consider the quadratic form $ds^2 = w_1 \bar{w}_1 + w_2 \bar{w}_2$ on M. The theorem implies that $ds^2 = f^* d\hat{s}^2$, where $d\hat{s}^2$ denotes the Riemannian metric on P. Thus f becomes an immersion if and only if ds^2 is everywhere positive, and then f is isometric with respect to ds^2.

<u>Proof.</u> We use the same idea as in Spivak ([14] , p.67). It is sufficient to construct a map $g : U \subset M \to G$ such that $w_i = g^* \psi_{oi}$, $w_{ij} = g^* \varphi_{ij}$ (see (5.3) and (5.5)). Them we put $f(m) = g(m)\sigma$ and $E_i(m) = g(m)_* E^o_i$.

~~We will first construct $\Gamma = \text{graph}(g) \subset M \times G$.~~ We consider w_i, w_{ij}, ψ_{oi}, φ_{ij} as being forms on M×G, by pulling back via the projections π_M, π_G on $M \times G$ onto M and G. Put $\chi_i = w_i - \psi_{oi}$, $\chi_{ij} = w_{ij} - \varphi_{ij}$, i,j = 1,2. Thus we have defined 8 real valued 1-forms on $M \times G$, namely real and imaginary part of χ_1, χ_2, χ_{12} and $i^{-1}\chi_{11}$, $i^{-1}\chi_{22}$. These are linear independent since real and imaginary part of ψ_{o1}, ψ_{o2}, φ_{12} and $i^{-1}\varphi_{11}$,

$i^{-1}\varphi_{22}$ form a basis of $\mathcal{G}^*$. So $\{\chi_i = 0\ ,\ \chi_{ij} = 0\}$ is a 2-dimensional distribution (codimension 8) on $M\times G$. Using (5.4) and (5.1) we get

$$d\chi_i = \sum_k w_k \wedge w_{ki} - \psi_{ok} \wedge \psi_{ki}$$

$$d\chi_{ij} = -(\bar{w}_i \wedge w_j - \bar{\psi}_{oi} \wedge \psi_{oj}) + \sum_k (w_{ik} \wedge w_{kj} - \varphi_{ik} \wedge \varphi_{kj})$$

$$-\delta_{ij} \sum_k (w_k \wedge \bar{w}_k - \psi_{ok} \wedge \bar{\psi}_{ok})$$

(observe that $\sum_k \psi_{ik} \wedge \psi_{kj} = \sum_k \varphi_{ik} \wedge \varphi_{kj}$). These are in the ideal generated by χ_i and χ_{ij} , since in any ring we have 2 (ab - cd) = (a + c) (b - d) + (a - c) (b + d). Thus, by Frobenius theorem, the distribution is integrable.

Let Γ be an integral leaf through (m,g_o) and $X \in T_{(m,g_o)}\Gamma$. If $d\pi_M(X) = 0$, we have $0 = \chi_i(X) = \psi_{oi}(X)$, $0 = \chi_{ij}(X) = \varphi_{ij}(X)$. Therefore $\psi(X) = 0$, so $d\pi_G(X) = 0$, hence $X = 0$. It follows that Γ is locally a graph over M: There exists a neighborhood U of m in M and a map $g : U \to G$ such that graph(g) is an open subset of Γ.

If $g_1 \in G$ is arbitrary, we can define an integral leaf Γ_1 through (m,g_1) as follows: $\Gamma_1 = \{ (u, g_1 g_o^{-1} g) ; (u, g) \in \Gamma \}$. This is an integral leaf since the forms ψ_{ab} are left invariant. So, by the uniqueness part of the Frobenius theorem, any integral leaf over m arises that way. Since the graph of any admissible map $g : U \to P$ is an integral leaf, g is unique up to left translations in G. This uniqueness together with paracompactness and simple connectivity allow to extend g to all of M. This proves the theorem.

Remark. In this proof, we did not make use of the dimension of M. Also it has a straightforward generalization to any Riemannian symmetric space. Note that the forms ψ_{oi} and ψ_{ij} can be defined in terms of the Cartan decomposition $\mathfrak{g} = k \oplus p$. If $\psi = \psi_k + \psi_p$ is the corresponding decomposition of the Maurer-Cartan form and ^ the usual identification of p with C^2, we have $\psi_{oi} = (\psi^p, \hat{e}_i)$ and $\psi_{ij} = ([\psi^k, \hat{e}_i], \hat{e}_j)$. In general, one has to use an orthonormal basis $\{\hat{f}_i\}$ of p instead.

6. A special unitary frame

Let M be an orientad surface and $f : M \to P$ an immersion. Let $\{e_1, e_2\}$ be an orientad orthonormal tangent frame with respect to the induced metric ds on M. E.g., if $z = x + iy : M_{loc} \to C$ is an oriented conformal coordinate with conformal factor λ, i.e $ds^2 = \lambda^2 dz d\bar{z}$, we can put $e_1 = \lambda^{-1}\partial/\partial x$ and $e_2 = \lambda^{-1}\partial/\partial y$. Then $\{e_1, e_2\}$ will be called the frame of the coordinate z.

For given $\{e_1, e_2\}$ let $\theta_i = \langle df, e_i \rangle$ and $\varphi = \theta_1 + i\theta_2$. A 1-form w of ~~$\Omega^1_c(M)$ is called (1,0) form if $w = h\varphi$ locally for a function $h : M_{loc} \to C$.~~

On the subset $\{\sin\alpha \neq 0\}$ define

(6.1) $\quad E_1 = (2c)^{-1}(e_1 - Je_2), \quad E_2 = (2s)^{-1}(e_1 + Je_2)$

where $c = \cos(\alpha/2)$, $s = \sin(\alpha/2)$; these form a unitary frame along f. If $e_1' = e_1\cos\tau + e_2\sin\tau$, $e_2' = -e_1\sin\tau + e_2\cos\tau$, then $E_1' = e^{-i\tau}E_1$, $E_2' = e^{i\tau}E_2$ (recall that we consider T_pP as a complex vector space, cf. §1). If $\sin\alpha(m) = 0$, then either E_1 or E_2 is still defined at m by (6.1) and smooth in a neighborhood of m. Thus we get an orthonormal splitting $f^{*}TP = H_1 \oplus H_2$ which is smooth and independent of the choice of $\{e_1, e_2\}$ and $H_1 = \mathbb{C}E_1$, $H_2 = \mathbb{C}E_2$ whenever E_1 or E_2 are defined. If $\sin\alpha = 0$ on some open subset U, the choice of a section of one of the bundles H_1 or H_2 is arbitrary. Thus $\{E_1, E_2\}$ can be defined in a neighborhood of any point outside $\partial\{\sin\alpha = 0\}$.

<u>Remark</u>. If $T'P \subset TP \otimes \mathbb{C}$ denotes the set of (1,0)-vectors and $X' = \frac{1}{2}(X - \sqrt{-1}JX)$ the projection on $T'P$, then E_1' is proportional to $(\partial f/\partial z)$ and E_2' is proportional to $(\partial f/\partial\bar{z})$ for any conformal coordinate $z : M \to \mathbb{C}$. However, we only use the real tangent bundle T P.

Now assume that $\{E_1, E_2\}$ is defined on an open subset U of M. Let $w_i = (df, E_i)$, $w_{ij} = (\nabla E_i, E_j)$ as above. We have

(6.2) $e_1 = cE_1 + sE_2$, $e_2 = cJE_1 - sJE_2$,

hence

(6.3) $w_1 = c\varphi$, $w_2 = s\bar{\varphi}$.

Now put

(6.4) $e_3 = -sE_1 + cE_2$, $e_4 = sJE_1 + cJE_2$.

$\{e_3, e_4\}$ is an orthonormal frame of the normal bundle NM, and $\{e_1, e_2, e_3, e_4\}$ is an oriented frame on P.

Next we compute the second fundamental form A. Put $W = \frac{1}{2}d\alpha - sc(w_{11} + w_{22})$, we get from (6.2) and (6.4)

$<\nabla e_1, e_3> = \mathrm{Re}(W + w_{12})$, $<\nabla e_1, e_4> = \mathrm{Im}(-W + w_{12})$

$<\nabla e_2, e_3> = \mathrm{Im}(-W - w_{12})$, $<\nabla e_2, e_4> = \mathrm{Re}(-W + w_{12})$

Let h^3, h^4 be the mean curvatures: $\frac{1}{2}(A_{11}+A_{22}) = h^3 e_3 + h^4 e_4$. Then A has the form

$$A^3 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} + h^3 I \quad , \quad A^4 = \begin{pmatrix} c & d \\ d & -c \end{pmatrix} + h^4 I$$

which implies

(6.5) $\quad 2\,w_{12} = h_1 \varphi + h\bar{\varphi} \quad , 2\,W = h_2\varphi + \bar{h}\bar{\varphi}$

with $h = h^3 + i\,h^4$, $h_1 = (a+d) + i(c-b)$, $h_2 = (a-d) - i(c+b)$

In particular, f is minimal if and only if w_{12} and W are (1,0)-forms. In the minimal case, the structure equations for (f, E_1, E_2) are (see (5.6))

$$(6.6)\quad \begin{aligned} &\text{(a) } d\,w_1 = w_1 \wedge w_{11} \quad , \quad \text{(b) } d\,w_2 = w_2 \wedge w_{22} \\ &\text{(c) } d\,w_{12} = w_{12} \wedge (w_{22} - w_{11}) \\ &\text{(d) } d\,w_{11} = \bar{w}_{12} \wedge w_{12} - 2\bar{w}_1 \wedge w_1 + w_2 \wedge \bar{w}_2 \\ &\text{(e) } d\,w_{22} = -\bar{w}_{12} \wedge w_{12} - \bar{w}_1 \wedge w_1 + 2\,w_2 \wedge \bar{w}_2 \end{aligned}$$

Now assume f to be minimal. Note that the Gauss and Ricci equations yield

$$|h_1|^2 = K_N - K + \hat{K} - \hat{K}_N \quad , \quad |h_2|^2 = -K_N - K + \hat{K} + \hat{K}_N$$

For $P = \mathbb{C}P^2$ we have $K = 3\cos^2\alpha + 1$, $K_N = 3\cos^2\alpha - 1$ (e.g. compute $d\langle\nabla e_1, e_2\rangle$ and $d\langle\nabla e_3, e_4\rangle$ with (6.2), (6.4) or use ([|4] , p.166)).

So we get

(6.7) $|h_1|^2 = K_N - K + 2 = k \quad , \quad |h_2|^2 = -K_N - K + 6\cos^2\alpha$.

If $w \in \Omega^1_{\mathbb{C}}(M)$, let $\|w\|^2 = |w(e_1)|^2 + |w(e_2)|^2$. If $w = h\varphi$, then $\|\mathrm{Re}\,w\| = \|\mathrm{Im}\,w\| = |h|$. Applying this to the form $w = 2\,W$, we derive (2.2) in

Theorem A from (6.7). On the subset $\{\sin\alpha \neq 0\}$, we have

$$\|\nabla\cos\alpha\|^2/\sin^2\alpha = \|\nabla\alpha\|^2 = \|d\alpha\|^2 = \|\mathrm{Re}\,2W\|^2 = |h_2|^2 = 6\cos^2\alpha - K_N - K$$

Then (2.2.) follows by continuity on the whole of M.

For $X \in f^*TP$, let $X = X_1 + X_2$ denote the decomposition according to $f^*TP = H_1 \oplus H$ Following [14], we define a cubic form Λ on M by $\Lambda(X) = (\nabla_X X_1, X_2)$. This is tensorial since $(X_1, X_2) = 0$. On $M-\partial\{\sin\alpha = 0\}$, we have

$$(6.8) \qquad \Lambda = w_1\bar{w}_2 w_{12} = 4^{-1} h_1 \sin\alpha\, \varphi^3$$

and by continuity $\Lambda = 0$ exactly at the zero set of $\ell = k\sin^2\alpha$. It is well known that Λ is holomorphic (see [4], [14]):

Let $m \in M-\partial\{\sin\alpha = 0\}$ and $U \subset M$ be a neighborhood of m where $\{E_1, E_2\}$ is defined and let $z = x + iy : M \to C$ be a conformal oriented chart with conformal factor λ. By (6.3) and (6.5) we have $w_1 = p\,dz$, $\bar{w}_2 = q\,dz$, $w_{12} = r\,dz$ with $p = c\lambda$ $q = s\lambda$, $r = 2^{-1}h_1\lambda$, and $\Lambda = pqr\,dz^3$. Applying the structure equations (6.6) we see that

$$(6.9) \qquad \begin{aligned} &(a) \quad (dp + p\,w_{11}) \wedge dz = 0 \\ &(b) \quad (dq - q\,w_{22}) \wedge dz = 0 \\ &(c) \quad (dr - r(w_{11} - w_{22})) \wedge dz = 0 \end{aligned}$$

Thus $d(pqr) \wedge dz = 0$, so pqr is holomorphic. This proves that Λ is holomorphic everywhere since $M-\partial\{\sin\alpha = 0\}$ is a dense subset. Since $\ell = |h_1 \sin\alpha|^2$, we conclude in particular from (6.8) either $\ell \equiv 0$ or $\ell = 0$ only at isolated points.

Let Λ be a holomorphic symmetric p-form on (M, ds^2), $\Lambda(m) \neq 0$ at a point $m \in M$. Then, in a neighborhood U of m, $\Lambda = h(dz)^p$ where $z : U \to \mathbb{C}$ is a conformal chart and h holomorphic, $h(m) \neq 0$. If U is small enough, there exists a holomorphic p-th root g on U, so $\Lambda = (g\,dz)^p$. But the differential $g\,dz$ is closed, so we find a conformal coordinate $v : U \to \mathbb{C}$ with $dv = g\,dz$, if U is small enough, hence $\Lambda = (dv)^p$.

Define the invariant $|\Lambda| : M \to \mathbb{R}_+$ by $\Lambda\bar{\Lambda} = |\Lambda|^2 (ds^2)^p$. In the special coordinate v we have $|\Lambda| = \lambda^{-p}$ where λ is the (positive) conformal factor defined by $ds^2 = \lambda^2\, dv\, d\bar{v}$. Thus we see that $\Delta \log|\Lambda| = -p\, \Delta \log \lambda = p K$. In the case of the holomorphic 3-form Λ defined in (6.8) we have $|\Lambda|^2 = 16 \ell$, hence $\Delta \log \ell = 6K$ outside $\{\ell = 0\}$. This proves equation (2.1) in Theorem A.

Remark. One might ask what are the Euler numbers of the two complex line bundles H_1 and H_2 introduced before. Using the frames $\{E_1, JE_1\}$ and $\{E_2, JE_2\}$ of H_1 and H_2 the connection forms are $\langle E_1, JE_1\rangle = i^{-1} w_{11}$ and $\langle E_2, JE_2\rangle = i^{-1} w_{22}$. Now by the structure equations we get

(6.10) $\quad d w_{11} = 1/4\,(K_N - K - 6\cos\alpha)\,\bar{\varphi}\wedge\varphi, \quad d w_{22} = 1/4(K - K_N - 6\cos\alpha)\,\bar{\varphi}\wedge\varphi$

Integrating (6.10) and using the Gauss-Bonnet-Chern Theorem, we get

$$\chi(H_1) = \frac{1}{2}\,(3d + \chi - \chi_N)$$

$$\chi(H_2) = \frac{1}{2}\,(3d - \chi + \chi_N)$$

We already knew from proposition 3.4 that these numbers are integers. In the complex case either $H_1 = TM$ or $H_2 = -TM$, and we get again the formula $\chi = \chi_N - 3|d|$ which was already used in proposition 3.5 b) (see [7], p. 280).

7. Proof of Theorems A, B and C

Let $g : M \to \mathbb{R}$ be a smooth function and $\emptyset \in \Omega^1_C(M)$ a purely imaginary form, i.e $\emptyset = u\,dz - \bar{u}\,d\bar{z}$ for any conformal coordinate z. Further assume that $w = dg + \emptyset \in \Omega^{1,0}(M)$. Then $0 = w \wedge dz = (g_{\bar{z}} - \bar{u})\,d\bar{z} \wedge dz$. Thus $g_z = u$ and

(7.1) $\emptyset = 2\,i\,\mathrm{Im}(g_z\,dz)$

Moreover, $d\emptyset = 2\,g_{z\bar{z}}\,d\bar{z} \wedge dz = 1/2\,\Delta^{\circ} g\,\lambda^{-2}\,\bar{\varphi} \wedge \varphi = 1/2\,\Delta g\,\bar{\varphi} \wedge \varphi$, hence

(7.2) $\Delta g\,\bar{\varphi} \wedge \varphi = 2\,d\emptyset$

This is the key observation for the equivalence of the fundamental and the structure equations.

Let $f : M \to P$ be a minimal immersion, z a conformal coordinate on M with conformal factor λ and $\{e_1, e_2\}$ be the corresponding frame. By (6.6), we can apply the above computation to $w = 2\,W / \sin\alpha = d(\log(s/c)) - (w_{11} + w_{22})$ on the subset $\{\sin\alpha \neq 0\}$. Then (7.2) yields

(7.3) $w_{11} + w_{22} = -2\,i\,\mathrm{Im}((\log(s/c))dz)$ on $\{\sin\alpha \neq 0\}$

Similarly, by (6.9 c), we can apply the above argument to $w = d\log r - (w_{11} - w_{22})$ ~~provided that r is positive real~~ i.e $r = \frac{1}{2}\,k^{1/2}\,\lambda$. This can be achieved on an open subset U if either ℓ has no zeros on U and z is a coordinate with $\Lambda = dz^3$, i.e $pqr = 1$, or if $\sin\alpha = 0$, $k \neq 0$ and $\{E_1, E_2\}$ is chosen suitably. Therefore, (7.2) implies

(7.4) $w_{11} - w_{22} = -2\,i\,\mathrm{Im}((\log r)_z\,dz)$ on $\{k \neq 0\}$

By (6.9 a, b) and applying again the above argument to $w = d\log p + w_{11}$ and to $w = d\log q - w_{22}$, then (7.2) yields

(7.5) $w_{11} = i \, \mathrm{Im}\,((\log p^2)_z \, dz)$ on $s \neq 0$

(7.6) $w_{22} = - i \, \mathrm{Im}\,((\log q^2)_z \, dz)$ on $c \neq 0$

Now from (7.2), we see the following

Lemma 7.1 Suppose $w_1 = p \, dz$, $\bar{w}_2 = q \, dz$, $w_{12} = r \, dz$ with $p = c\lambda$, $q = s\lambda$, $r = \frac{1}{2} k^{1/2} \lambda$.

(i) If (7.3) holds, then (6.6 d) + (6.6 e) is equivalent to (2.3).

(ii) If (7.4) holds, then (6.6 c) is true, and (6.6 d)- (6.6 e) is equivalent to (2.4).

(iii) If (7.5) holds, then (6.6 a) is true, and (6.6 d) is equivalent to (2.5).

(iv) If (7.6) holds, then (6.6 b) is true, and (6.6 e) is equivalent to (2.6).

Therefore, if $f : M \to P$ is a minimal immersion we get (2.3), (2.4), (2.5) and (2.6) on their domains.

Conversely, let (M , ds^2) be a surface and $K_N : M \to P$ and $\cos\alpha : M \to [-1, 1]$ functions which satisfy (2.2), (2.3), (2.4). Let U be an open subset outside $\partial \{ \sin\alpha = 0 \}$ and $\partial\{ k = 0\}$. If $\ell > 0$ on U, let $z : U \to \mathbb{C}$ an isometry with respect to the metric $ds_o^2 = \lambda^{-2} ds^2$ on U, where $\lambda = \ell^{-1/6}$; by (2.1) this metric is flat. If $\ell \equiv 0$ on U, choose an arbitrary conformal chart $z : U \to \mathbb{C}$ with conformal factor λ. Let w_1, w_2, w_{12} as in lemma 7.1. This implies that the structure equations (6.6) are true, if we define w_{11} and w_{22} as follows:

(a) $\ell > 0$: w_{11} by (7.5) and w_{22} by (7.6)

(b) $k = 0$, $\sin^2\alpha > 0$: as in (a)

(c) $k > 0$, $c = 1$, $s = 0$: w_{11} by (7.5), $w_{11} - w_{22}$ by (7.4)

(d) $k > 0$, $c = 0$, $s = 1$: w_{22} by (7.6), $w_{11} - w_{22}$ by (7.4)

(e) $k = 0$, $c = 1$, $s = 0$: w_{11} by (7.5), w_{22} so that $dw_{22} = -\bar{\varphi}\wedge\varphi$

(f) $k = 0$, $c = 0$, $s = 1$: w_{22} by (7.6), w_{11} so that $dw_{11} = \varphi\wedge\varphi$

To finish the proof of Theorem A, we have to examine the case $\ell \equiv 0$. If $\sin\alpha \equiv 0$, the immersion f is complex. So assume that $\sin\alpha \neq 0$ somewhere. Then $k \equiv 0$ on an open subset $U \subset M$, hence $w_{12} = 0$ on U. Consider a lift $g : U_{loc} \to G = SU(3)$ of the map $(f, E_1, E_2) : U_{loc} \to UP$ (see §5). Claim: The map $h : U \to P$, $h(m) = \pi(g(m)e_2)$ is holomorphic. In fact, $g^{-1}\,dg\,e_2 = g^*\psi\,e_2 = \sum_a g^*\psi_{2a}\,e_a = \bar{w}_2\,e_0 + w'_{22}\,e_2$, by (5.2) and (5.3), where we put $w'_{ij} = g^*\psi_{ij}$ and used $w'_{12} = w_{12} = 0$. Therefore from (6.3), we have

(7.7) $$dh = d\pi_{ge_2}(dg\,e_2) = s\varphi\, d\pi_{ge_2}\, ge_0$$

So dh is a (1,0) form, hence h holomorphic. Also we see that $\mathbb{C}g(m)e_0$ is the horizontal lift of the complex line $dh(T_mM)$. So the associated surface $\tilde{f} : U \to P$ is given by $\tilde{f}(m) = [\mathbb{C}g(m)e_0] = \pi(g(m)e_0) = f(m)$ (see §1).

In particular, f is analytic on $\{\sin\alpha \neq 0\}$, and so is $\sin\alpha$. Consequently, $\{\sin\alpha = 0\}$ has no inner point which implies that $k \equiv 0$ on M. It follows that $f : M \to P$ is associated to a holomorphic map $h : M \to P$ since the definition of h only depends on the subbundle $\mathbb{C}E_2 = H_2 \in f^*TP$ which is globally defined (see §6). This finishes the proof of Theorem A.

Moreover, we see from (7.7) that for $X \in T_mM$, $\|dh(X)\| = s(m)\,|\varphi(X)| = s(m)\,\|df(X$ since $g(m)e_0$ is a horizontal unit vectors at $g(m)e_2$. In particular, $ds_h^2 \leq ds_f^2$ which implies that f is an immersion if so is h. This proves Theorem C (ii). Part (i is proved by equation (6.7) together with above argument.

The ellipse of curvature of f is everywhere a circle if and only if $a + d = 0$, $b - c = 0$ or $a - d = 0$, $b + c = 0$ (see § 1), hence if and only if either $h_1 \equiv 0$ or $h_2 \equiv 0$ on $M - \partial\{\sin\alpha = 0\}$. If $h_1 \equiv 0$, then $k \equiv 0$ on M, so M is associated. If $h_2 \equiv 0$, then $2\chi = d\alpha - \sin(w_{11} + w_{22})$ vanishes. In particular, α = constant which by (2.3) implies $\sin\alpha = 0$ or $\cos\alpha = 0$ and therefore M is real or complex. Conversely, if M is real, then $w_{11} + w_{22} = 0$ by (7.3), so $\chi = 0$. Thus we proved Theorem B.

Remark: We saw earlier for $\ell = k\sin^2\alpha$ that either $\ell = 0$ or ℓ has only isolated zeros. Due to an idea of J. Wolfson, the same can be shown for the factors k and $\sin^2\alpha$ even in the case $\ell \equiv 0$. To see this, let E_i be arbitrary sections of unit length of the bundles H_i $(i = 1, 2)$ introduced in Ch. 5, and $\{w_1, w_2\}$ the corresponding coframe. Then we still have $w_1 = p\,dz$, $\bar{w}_2 = q\,dz$, $w_{12} = r\,dz$ where p,q,r now take complex values, and $p = \lambda c$, $q = \lambda s$, $r^2 = \lambda^2 k / 4$. From the equations in (6.9) we see

$$\partial t / \partial\bar{z} = h\,t$$

where t is any of the functions p,q,r and h the $d\bar{z}$ - coefficient of $-w_{11}$, w_{22}, $w_{11} - w_{22}$ resp. This equation implies that t has isolated zeros unless $t \equiv 0$ [16].

In particular, for complex immersions f we have $K = 4$ only at isolated points unless $f(M) \subset \mathbb{C}P^1$.

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