

BEST APPROXIMATION OF BOUNDED FUNCTIONS  
BY CONTINUOUS FUNCTIONS

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**ABSTRACT:** Let  $X$  be a topological space and  $E$  be a Banach space. We study conditions under which a bounded  $E$ -valued function on  $X$  has a best approximant from some subspace of continuous  $E$ -valued functions. Applications to approximation of linear operators by compact operators are given.

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0. INTRODUCTION

It is a classical result that every closed subalgebra of  $C(X; \mathbb{R})$ ,  $X$  compact, is a proximal subspace. Proofs of this result were published by Pelczynski and Semadeni, but the result itself was known to Mazur. Several generalizations and extensions of Mazur's Theorem have appeared in the literature. For example, one may consider the problem of existence of Chebyshev centers for bounded subsets of  $C(X; \mathbb{R})$ , i.e. the problem of deciding when  $C(X; \mathbb{R})$  admits centers. This was settled by Kadets and Zamyatin, for  $X = [a, b]$ , and by Garkavi, for  $X$  compact (see [13] and [12]).

The existence of relative Chebyshev centers (also called restricted centers) with respect to a closed subalgebra  $A \subset C(X; \mathbb{R})$  was established in 1975 by Smith and Ward (see [24]), for any compact space  $X$ . An extension of this result to bounded functions, i.e. to closed subalgebras of  $\ell_\infty(X; \mathbb{R})$  was obtained by J. Mach in 1979 (see [18]).

Another line of generalizations of Mazur's Theorem consists in the consideration of vector-valued functions. The problem of existence of Chebyshev centers for bounded subsets of  $C_b(X; E)$ , the space of continuous and bounded  $E$ -valued functions, was solved by Ward in 1974 (see [25]) in the following two cases:

a)  $E$  is a finite-dimensional strictly convex normed space and  $X$  is paracompact; b)  $E$  is a Hilbert space and  $X$  is normal. D. Amir in 1978 (see [1]) generalized both results by proving that  $C_b(X;E)$  admits centers when  $E$  is a uniformly convex Banach space and  $X$  is any topological space.

When  $W \subset C_b(X;E)$  is a closed vector subspace one asks whether it is proximal or, more generally, whether any bounded subset of  $C_b(X;E)$  has a relative Chebyshev center with respect to  $W$ . Along this line we have the study of proximality of Grothendieck subspaces (in particular Stone-Weierstrass and Kakutani subspaces)  $W \subset C_0(X;E)$  made by J. Blatter (see [4]). For example, when  $E$  is a real Lindenstrauss space, any Stone-Weierstrass subspace  $W \subset C_0(X;E)$  is proximal ([4], Corollary 3.19); see also Yost ([26], Theorem 2.1). The case in which  $E$  is a uniformly convex Banach space, and  $W$  is a Stone-Weierstrass subspace was studied by Olech (see [21]) who proved proximality of such subspaces in  $C(X;E)$ ,  $X$  compact. K.S. Lau (see [15]) extended this to  $\ell_\infty(X;E)$ . In 1979, J. Mach (see [18]) generalized Olech's result showing that any bounded subset  $B \subset C(X;E)$ ,  $X$  compact and  $E$  uniformly convex, has a restricted center with respect to any Stone-Weierstrass subspace  $W \subset C(X;E)$ .

These results of the literature are particular cases of our Theorems.

We apply our results to establish existence of Chebyshev centers with respect to  $K(E;C_0(X))$ , the space of compact linear operators from a uniformly smooth space  $E$  into  $C_0(X)$ ,  $X$  locally compact, for bounded subsets  $B$  of  $\mathcal{L}(E;C_0(X))$ .

## 1. DEFINITIONS AND NOTATIONS

Let  $(E, \| \cdot \|)$  be a normed space over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . If  $B$  is a bounded set in  $E$ , we denote  $r(x, B) = \sup\{\|x - b\|; b \in B\}$  where  $x \in E$ . For any non-empty subset  $M$  of  $E$ , we define the *relative Chebyshev radius of  $B$  with respect to  $M$*  to be

$$\text{rad}(B, M) = \inf\{r(x, B); x \in M\}$$

and the set of *Chebyshev centers of  $B$  in  $M$*  to be

$$\text{cent}(B, M) = \{x \in M; r(x, B) = \text{rad}(B, M)\}.$$

Elements of  $\text{cent}(B, M)$  are also called *best simultaneous approximations of  $B$  by elements of  $M$* . When  $B$  is a single point  $y$ ,

$$\text{rad}(B, M) = \text{dist}(y, M)$$

and

$$\text{cent}(B, M) = P_M(y).$$

We say that  $M$  has the *relative Chebyshev center property in  $E$*  if  $\text{cent}(B, M) \neq \emptyset$  for every bounded set  $B$  in  $E$ . When  $M = E$  has this property, we say that  $E$  admits *Chebyshev centers*. Notice that if  $M$  has the relative Chebyshev center property in  $E$  then  $M$  is proximal in  $E$ . Finally, for each  $\varepsilon > 0$  define

$$\varphi_\varepsilon(u, v) = v \quad \text{if} \quad \|u - v\| \leq \varepsilon \quad \text{and}$$

$$\varphi_\varepsilon(u, v) = (1 - \varepsilon \|u - v\|^{-1})u + \varepsilon \|u - v\|^{-1} v \quad \text{otherwise,}$$

for  $u$  and  $v$  in  $E$ . The mapping

$$(u, v) \in E \times E \mapsto \varphi_\varepsilon(u, v) \in E$$

is continuous and satisfies

$$\|\varphi_{\varepsilon}(u,v) - u\| \leq \varepsilon$$

for all  $u, v \in E$ . We say that  $(E, \|\cdot\|)$  has property (P) if for every  $r > 0$  and  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\bar{B}(u, r + \delta) \cap \bar{B}(v, r + \theta) \subset \bar{B}(\varphi_{\varepsilon}(u,v), r + \theta)$$

for all  $0 < \theta < \delta$  and  $u, v \in E$ , where  $\bar{B}(y, s)$  denotes the closed ball with center in  $y$  and radius  $s$ . The most important class of spaces having property (P) is that of all uniformly convex Banach spaces (see Mach [17]).

## 2. SPACES OF BOUNDED MAPPINGS

Let  $X$  be a non-empty set and  $(E, \|\cdot\|)$  be a Banach space.  $\ell_{\infty}(X; E)$  denotes the space of all bounded  $E$ -valued functions  $f$  on  $X$  with the sup-norm  $\|f\| = \sup \{\|f(x)\|; x \in X\}$ . When  $X = \mathbb{N}$ , we write  $\ell_{\infty}(\mathbb{N}; E) = \ell_{\infty}(E)$  and  $\ell_{\infty}(\mathbb{N}; \mathbb{K}) = \ell_{\infty}$ .

DEFINITION 2.1: Let  $\varepsilon > 0$  be given. For  $f, g \in \ell_{\infty}(X; E)$  we define a function  $h_{\varepsilon}(f, g)$  by

$$h_{\varepsilon}(f, g)(x) = \varphi_{\varepsilon}(f(x), g(x))$$

for all  $x \in X$ , where  $\varphi_{\varepsilon}$  is the mapping that we defined in the introduction.

It follows that  $h_{\varepsilon}(f, g) \in \ell_{\infty}(X; E)$  and  $\|h_{\varepsilon}(f, g) - f\| \leq \varepsilon$ , for all  $f, g \in \ell_{\infty}(X; E)$ .

THEOREM 2.2: Let  $E$  be a Banach space with property (P). Let  $W \subset \ell_{\infty}(X; E)$  be a closed subset such that  $h_{\varepsilon}(f, g) \in W$  for all  $f, g \in W$  and  $\varepsilon > 0$ . Then  $W$  has the relative Chebyshev center property in  $\ell_{\infty}(X; E)$ . In particular,  $\ell_{\infty}(X; E)$  admits Chebyshev centers.



PROOF: Let  $r > 0$  and  $\varepsilon > 0$  be given. By the hypothesis,  $h_\varepsilon$  maps  $W \times W$  into  $W$ . Take  $\delta > 0$  as in Property (P) and let  $f, g \in \ell_\infty(X; E)$  and  $0 < \theta < \delta$  be given. If

$$w \in \bar{B}(f, r + \delta) \cap \bar{B}(g, r + \theta),$$

then by Property (P),

$$w(x) \in \bar{B}(\varphi_\varepsilon(f(x), g(x)), r + \theta)$$

for every  $x \in X$ . Hence

$$w \in \bar{B}(h_\varepsilon(f, g), r + \theta).$$

By Theorem 2 of Mach [17],  $W$  has the relative Chebyshev center property in  $\ell_\infty(X; E)$ .

COROLLARY 2.3: Let  $E$  be a Banach space with property (P). Every closed convex  $\ell_\infty(X; [0, 1])$ -submodule  $W$  of  $\ell_\infty(X; E)$  has the relative Chebyshev center property in  $\ell_\infty(X; E)$ .

PROOF: Let  $f, g \in W$  and  $\varepsilon > 0$  be given. If  $\beta_\varepsilon$  is the real-valued function defined by  $\beta_\varepsilon(t) = 1$  when  $|t| \leq \varepsilon$  and  $\beta_\varepsilon(t) = \varepsilon t^{-1}$  otherwise, then

$$\varphi_\varepsilon(f(x), g(x)) = \varphi(x) + \beta_\varepsilon(\|f(x) - g(x)\|) [g(x) - f(x)]$$

for every  $x \in X$ . Defining  $\psi_{\varepsilon, f, g} \in \ell_\infty(X; [0, 1])$  by

$$\psi_{\varepsilon, f, g}(x) = \beta_\varepsilon(\|f(x) - g(x)\|)$$

for all  $x \in X$ , we have

$$h_\varepsilon(f, g) = (1 - \psi_{\varepsilon, f, g})f + \psi_{\varepsilon, f, g} \cdot g$$

and thus  $h_\varepsilon(f, g) \in W$ .

Let  $E = \mathbb{R}$  normed by its usual absolute value. For  $\varepsilon > 0$  and  $f, g \in \ell_\infty(X; E)$ , we can write  $h_\varepsilon(f, g)$  in the form

$$h_\varepsilon(f, g) = \sup (\inf(g, f + \varepsilon), f - \varepsilon).$$

This and Theorem 2.2 give the following result.

**PROPOSITION 2.4:** *Let  $L$  be a closed sublattice of  $\ell_\infty(X; \mathbb{R})$  such that  $f \pm \varepsilon$  belong to  $L$ , for all  $f \in L$  and  $\varepsilon > 0$ . Then  $L$  has the relative Chebyshev center property in  $\ell_\infty(X; \mathbb{R})$ .*

**EXAMPLE 2.5:** Let  $(X, \leq)$  be a pre-ordered set and  $J$  be the set of all non-decreasing elements of  $\ell_\infty(X; \mathbb{R})$ . It is easy to see that  $J$  is a closed sublattice of  $\ell_\infty(X; \mathbb{R})$  and  $f \pm \varepsilon$  belong to  $J$ , for all  $f \in J$  and  $\varepsilon > 0$ . Thus  $J$  has the relative Chebyshev center property in  $\ell_\infty(X; \mathbb{R})$ .

For  $a, b \in \ell_\infty(X; \mathbb{R})$ , let  $[a, b]$  denote the order interval  $\{h \in \ell_\infty(X; \mathbb{R}); a(x) \leq h(x) \leq b(x), x \in X\}$ . It is easy to see that  $h_\varepsilon(f, g) \in [a, b]$  for all  $f, g \in [a, b]$  and  $\varepsilon > 0$ . Thus, by Theorem 2.2 we have

**PROPOSITION 2.6:** *Each order interval  $[a, b] \subset \ell_\infty(X; \mathbb{R})$  has the relative Chebyshev center property in  $\ell_\infty(X; \mathbb{R})$ .*

Franchetti and Cheney ([11], Lemma 3.5), proved the proximality of any order interval  $[a, b] \subset E$ , when  $E$  is a Banach lattice. Hence, our Proposition 2.6 extends Lemma 3.5 of [11] when  $E$  is the Banach lattice  $\ell_\infty(X; \mathbb{R})$ .

### 3. SPACES OF CONTINUOUS BOUNDED MAPPINGS

Let  $X$  be a topological space,  $(E, \| \cdot \|)$  be a Banach space over  $\mathbb{K}$  ( $\mathbb{R}$  or  $\mathbb{C}$ ) and  $C(X;E)$  be the space of all continuous  $E$ -valued functions on  $X$ . Let

$$C_b(X;E) = C(X;E) \cap \ell_\infty(X;E)$$

with the sup-norm induced by  $\ell_\infty(X;E)$ .

**THEOREM 3.1:** Suppose that  $E$  has Property (P). Then every closed  $C_b(X;\mathbb{K})$ -submodule of  $C_b(X;E)$  has the relative Chebyshev center property in  $\ell_\infty(X;E)$ .

**PROOF:** Similar to Corollary 2.3.

**COROLLARY 3.2:** If  $E$  has property (P) then  $C_b(X;E)$  has the relative Chebyshev center property in  $\ell_\infty(X;E)$ . In particular,  $C_b(X;E)$  admits Chebyshev centers.

**REMARK:** Kadets-Zamyatin [13] proved that  $C_b(X;E)$  admits Chebyshev centers when  $X = [a,b] \subset \mathbb{R}$  and  $E = \mathbb{R}$ . Ward [25] generalized this result in the following two cases: (a)  $E$  is a finite dimensional strictly convex normed space and  $X$  is paracompact. (b)  $E$  is a Hilbert space and  $X$  is normal. Amir [1] generalized all these results proving that  $C_b(X;E)$  admits Chebyshev centers when  $E$  is a uniformly convex Banach space. Notice that in all three cases  $E$  has property (P).

**COROLLARY 3.3:** Suppose that  $E$  has property (P) and  $Z \subset X$  is a closed subset. Then  $W = \{f \in C_b(X;E); f(x) = 0, x \in Z\}$  has the



relative Chebyshev center property in  $\ell_\infty(X;E)$ .

PROOF: Clearly,  $W$  is closed  $C_b(X;K)$ -submodule of  $C_b(X;E)$ .

PROPOSITION 3.4: Let  $Y$  be a topological space and suppose that  $E$  has property (P). Let  $M \subset C_b(Y;E)$  be a closed  $C_b(Y;K)$ -submodule and  $\pi$  a surjection from a set  $Z$  onto  $Y$ . Then  $\pi^O(M)$  has the relative Chebyshev center property in  $\ell_\infty(Z;E)$ , where  $\pi^O(f) = f \circ \pi$  for all  $f \in M$ .

PROOF: Define a topology on  $Z$  as follows:  $A \subseteq Z$  is open if and only if  $A = \pi^{-1}(B)$  with  $B$  open in  $Y$ . Thus  $C_b(Y;E)$  is isometrically isomorphic to  $C_b(Z;E)$  under  $\pi^O$ , and  $\pi^O(M)$  is a closed  $C_b(Z;K)$ -submodule of  $C_b(Z;E)$ . The result follows from Theorem 3.1.

REMARK: When  $E$  is uniformly convex and  $Y$ ,  $M$  and  $\pi$  are as in Proposition 3.4 Ka-Sing Lau ([15], Theorem 4.3) proved that  $\pi^O(M)$  is proximal.

THEOREM 3.5: Let  $E$  be a Banach space with property (P). Every Stone-Weierstrass subspace  $W$  of  $C_b(X;E)$  has the relative Chebyshev center property in  $\ell_\infty(X;E)$ .

PROOF: By definition, a Stone-Weierstrass subspace  $W$  has the following form

$$W = \{g \circ \pi; g \in C_b(Y;E)\}$$

where  $Y$  is a topological space and  $\pi$  is a closed continuous surjection of  $X$  onto  $Y$ . Thus  $W = \pi^O(C_b(Y;E))$  and the result follows from Proposition 3.4.

COROLLARY 3.6: Let  $X$  be a compact space and suppose that  $E$  has

property (P). Then every Stone-Weierstrass subspace of  $C(X;E)$  has the relative Chebyshev center property in  $\ell_\infty(X;E)$ .

When  $E$  is uniformly convex and  $X$  is compact, Olech [21] proved the proximality of the Stone-Weierstrass subspaces in  $C(X;E)$ . Mach [18] generalized Olech's result by proving that such subspaces have the Chebyshev center property in  $C(X;E)$ , under the hypothesis that  $E$  is uniformly convex and  $X$  is compact.

**PROPOSITION 3.7:** Let  $E$  be a Banach space with property (P). Every closed subset  $W \subset C_b(X;E)$  which is a convex  $C(X; [0,1])$ -submodule has the relative Chebyshev center property in  $\ell_\infty(X;E)$ , and a fortiori, in  $C_b(X;E)$ .

**PROOF:** Write  $h_\varepsilon(f,g)$  as in Corollary 2.3 and notice that the mappings  $\psi_{\varepsilon,f,g}$  belong to  $C(X; [0,1])$ .

**EXAMPLES 3.8:**

(a) If  $M$  is a closed convex subset of a Banach space  $E$ , let

$$C_b(X;M) = \{f \in C_b(X;E); f(X) \subset M\}.$$

It is easy to see that  $C_b(X;M)$  is a closed convex  $C(X; [0,1])$ -submodule, contained in  $\ell_\infty(X;E)$ .

(b) The space  $C^*(X;E) = \{f \in C_b(X;E); f(X) \text{ is compact in } E\}$  is a closed convex  $C(X; [0,1])$ -submodule contained in  $\ell_\infty(X;E)$ .

**THEOREM 3.9:** Let  $X$  be a compact space and  $E$  a Banach space with property (P). Every closed and self-adjoint polynomial algebra  $W \subset C(X;E)$  has the relative Chebyshev center property in  $\ell_\infty(X;E)$ , and, a fortiori, in  $C(X;E)$ .

PROOF: For every  $f, g \in W$  and  $\varepsilon > 0$ , one has  $h_\varepsilon(f, g) \in W$ , by Theorem 4.17 of Prolla [22].

COROLLARY 3.10: Let  $X$  be a non-empty set. Every closed and self-adjoint subalgebra  $A$  of  $\ell_\infty(X; \mathbb{K})$  has the relative Chebyshev center property in  $\ell_\infty(X; \mathbb{K})$ . If  $X$  is a topological space then every closed and self-adjoint subalgebra  $A$  of  $C_b(X; \mathbb{K})$  has the relative Chebyshev center property in  $\ell_\infty(X; \mathbb{K})$  and, a fortiori, in  $C_b(X; \mathbb{K})$ .

PROOF: Denote by  $X_d$  the topological space obtained endowing  $X$  with the discrete topology. Then  $\ell_\infty(X; \mathbb{K}) = C_b(X_d; \mathbb{K})$ . Since  $X_d$  is completely regular and Hausdorff,  $C_d(X_d; \mathbb{K})$  is isometrically isomorphic as a  $C^*$ -algebra to  $C(\beta X_d; \mathbb{K})$ . Since the definitions of subalgebra and polynomial subalgebra coincide in  $C(\beta X_d; \mathbb{K})$ , the result follows from Theorem 3.9. When  $X$  is a topological space, every closed subalgebra of  $C_b(X; \mathbb{K})$  is closed in  $\ell_\infty(X; \mathbb{K})$ , and the result follows from the first part.

EXAMPLE 3.11: Let  $X$  be a non-empty set and let  $L$  be a sublattice of the power set  $2^X$  (the lattice operations being  $\cup$  and  $\cap$ ), containing  $\emptyset$  and  $X$ . If  $f: X \rightarrow \mathbb{R}$ , one says that  $f$  is  $L$ -continuous if  $f(Y) \in L$  for every closed subset  $Y \subset \mathbb{R}$ . We will denote by  $C(L)$  the vector space of all real-valued  $L$ -continuous functions, and by  $C_b(L)$  the vector subspace of  $C(L)$  of all bounded real-valued  $L$ -continuous functions. When  $L$  is a *delta lattice* (i.e. closed under countable intersections) then  $C_b(L)$  is a Banach algebra under the sup norm i.e.,  $C_b(L)$  is a closed subalgebra of  $\ell_\infty(X; \mathbb{R})$ . By Corollary 3.10,  $C_b(L)$  has the relative Chebyshev center property in  $\ell_\infty(X; \mathbb{R})$ , for any delta lattice  $L$ . For the importance of  $C_b(L)$  and  $C(L)$ , see Bachman and Sultan [3].

Smith and Ward [24] proved that every closed subalgebra of  $C(X; \mathbb{R})$  has the relative Chebyshev center property in  $C(X; \mathbb{R})$  when  $X$  is compact. Yost proved that such subalgebras have the

$1\frac{1}{2}$  - ball property and therefore is proximal in  $C(X; \mathbb{R})$  (see [26], Lemma 1.1).

PROPOSITION 3.12: Let  $u, v$  be two functions in  $\ell_\infty(X; \mathbb{R})$  such that  $u \leq v$ . If the set  $I = \{f \in C_b(X; \mathbb{R}); u \leq f \leq v\}$  is non-empty, then it has the relative Chebyshev center property in  $\ell_\infty(X; \mathbb{R})$ .

PROOF: For  $\varepsilon > 0$  and  $f, g \in I$ , it is easy to see that the mapping  $h_\varepsilon(f, g)$  belongs to  $I$ , since  $h_\varepsilon(f, g)$  is continuous when  $f$  and  $g$  are.

REMARK: Given any two real-valued functions  $u$  and  $v$  on a topological space  $X$  the problem arises of inserting a continuous function  $f$  between  $u$  and  $v$ , i.e., to find a continuous mapping  $f$  such that  $u \leq f \leq v$ . More generally, let  $P_1$  and  $P_2$  be two classes of real-valued functions on a topological space  $X$  such that they contain the constant functions and  $P_i + C(X; \mathbb{R}) \subset P_i$  ( $i = 1, 2$ ). A space  $X$  has the weak insertion property for  $(P_1, P_2)$  if, and only if, for any pair of functions  $(f_1, f_2)$  with  $f_1 \leq f_2$ ,  $f_i \in P_i$  ( $i = 1, 2$ ), there exists a continuous function  $f$  on  $X$  such that  $f_1 \leq f \leq f_2$  (Lane [14], p. 181).

In the following the abbreviations  $\ell.s.c.$  and  $u.s.c.$  are used for lower semicontinuous and upper semicontinuous. The well-known Tong-Katetov Theorem states that a space  $X$  has the weak insertion property for  $(u.s.c., \ell.s.c.)$  if, and only if,  $X$  is normal. If one reverses the rôles of upper and lower semicontinuity, the following is true (see Stone [23], Lane [14]): a space  $X$  has the weak insertion property for  $(\ell.s.c., u.s.c.)$  if, and only if,  $X$  is extremally disconnected (i.e., the closure of each open set is open).

If  $C$  is a collection of subsets of  $X$ , a function  $f: X \rightarrow \mathbb{R}$  is called  $C$ -lower (resp.  $C$ -upper) semicontinuous if, for any  $r \in \mathbb{R}$ ,



the set  $\{x \in X; f(x) > r\}$  (resp. the set  $\{x \in X; f(x) \geq r\}$ ) belongs to  $C$ . The abbreviations  $C$ -l.s.c. and  $C$ -u.s.c. are used for  $C$ -lower semicontinuous and  $C$ -upper semicontinuous. Notice that  $C$ -l.s.c. reduces to l.s.c. when  $C$  is the collection of all open subsets of  $X$ ; and that  $C$ -u.s.c. reduces to u.s.c. if  $C$  is the collection of all closed subsets of  $X$ .

Let  $\mathcal{Z}$  denote the class of all zero sets of  $X$ , that is

$$\mathcal{Z} = \{Z(f); f \in C(X; \mathbb{R})\},$$

where, for each  $f \in C(X; \mathbb{R})$ ,  $Z(f) = \{x \in X; f(x) = 0\}$ . The following result is due to Stone (see Blatter and Seeever [5] and Lane [14]): if  $X$  is *basically disconnected* (i.e., the closure of the complement of each zero set is open), then  $X$  has the *weak insertion property* for (l.s.c.,  $\mathcal{Z}$ -u.s.c.) and for ( $\mathcal{Z}^c$ -l.s.c., u.s.c.). Here  $\mathcal{Z}^c$  denotes the class of all *cozero sets*, i.e., the complements of the zero sets.

There is a case in which it is possible to insert a continuous function with no restriction on  $X$ . Namely the following is true (see [5], Proposition 6.1, p. 41): Any topological space  $X$  has the *weak insertion property* for ( $\mathcal{Z}$ -u.s.c.,  $\mathcal{Z}^c$ -l.s.c.). Reversing the roles of u.s.c. and l.s.c. one gets the following ([5], Proposition 6.3, p. 42): if  $X$  is a  $\mathcal{Z}$ -space i.e. every zero set is a cozero set, then  $X$  has the *weak insertion property* for ( $\mathcal{Z}^c$ -l.s.c.,  $\mathcal{Z}$ -u.s.c.).

Let  $f: X \rightarrow \mathbb{R}$  be given. The upper and lower semicontinuous regularizations of  $f$  are defined as follows:

$$f^*(x) = \limsup_{y \rightarrow x} f(y)$$

$$f_*(x) = \liminf_{y \rightarrow x} f(y).$$

A mapping  $f$  is said to be *normal-l.s.c.* if  $f = (f^*)_*$ ; and



normal-u.s.c. if  $f = (f_*)^*$ . The following results are due to Lane [14]. A space  $X$  has the weak insertion property for (normal u.s.c., normal l.s.c.) if, and only if,  $X$  is mildly normal. (A space is mildly normal in case disjoint regular closed subsets are separated by disjoint open sets. A subset is regular closed if it is equal to the closure of its interior). A space  $X$  has the weak insertion property for (normal-u.s.c., l.s.c.) (resp. (u.s.c., normal-l.s.c.)) if, and only if,  $X$  is almost normal. (A space is almost normal in case disjoint closed sets, at least one of which is regular closed, are separated by disjoint open sets).

Combining these results with Proposition 3.12 one gets the following:

**THEOREM 3.13:** Let  $u$  and  $v$  be functions in  $\ell_\infty(X; \mathbb{R})$  such that  $u \leq v$ . The set  $I = \{f \in C_b(X; \mathbb{R}); u \leq f \leq v\}$  has the relative Chebyshev center property in  $\ell_\infty(X; \mathbb{R})$  in the following cases:

- (a)  $X$  is any topological space,  $u$  is  $\mathfrak{I}$ -u.s.c. and  $v$  is  $\mathfrak{I}^C$ -l.s.c.
- (b)  $X$  is a  $P$ -space,  $u$  is  $\mathfrak{I}^C$ -l.s.c. and  $v$  is  $\mathfrak{I}$ -u.s.c.
- (c)  $X$  is normal,  $u$  is u.s.c. and  $v$  is l.s.c.
- (d)  $X$  is extremally disconnected,  $u$  is l.s.c. and  $v$  is u.s.c.
- (e)  $X$  is basically disconnected,  $u$  is l.s.c. (resp.  $\mathfrak{I}^C$ -l.s.c.), and  $v$  is  $\mathfrak{I}$ -u.s.c. (resp. u.s.c.)
- (f)  $X$  is mildly normal,  $u$  is normal-u.s.c. and  $v$  is normal-l.s.c.
- (g)  $X$  is almost normal,  $u$  is normal-u.s.c. (resp. u.s.c.) and  $v$  is l.s.c. (resp. normal-l.s.c.).

**REMARK:** Part (c) of Theorem 3.13 generalizes Corollary 3.7 of

Franchetti and Cheney [11].

In the remainder of this section  $X$  is a locally compact space and  $C_0(X;E)$  is the subspace of all continuous  $E$ -valued functions on  $X$  vanishing at infinity. When  $X = \mathbb{N}$  we write  $c_0(E) = C_0(\mathbb{N};E)$  and  $c_0 = C_0(\mathbb{N};\mathbb{K})$ .

**THEOREM 3.14:** Suppose that  $E$  has property (P). Let  $W \subset C_0(X;E)$  be a closed subset such that  $h_\epsilon(f,g) \in W$  for all  $f, g \in W$  and  $\epsilon > 0$ . Then  $W$  has the relative Chebyshev center property in  $\ell_\infty(X;E)$  and, a fortiori, in  $C_b(X;E)$  and  $C_0(X;E)$ .

**PROOF:** Since  $C_0(X;E)$  is closed in  $\ell_\infty(X;E)$ , the result follows from Theorem 2.2.

**COROLLARY 3.15:** If  $E$  has property (P) then  $C_0(X;E)$  has the relative Chebyshev center property in  $\ell_\infty(X;E)$ , and, a fortiori, in  $C_b(X;E)$ . In particular,  $C_0(X;E)$  admits Chebyshev centers.

**PROOF:** Let  $f, g \in C_0(X;E)$  and  $\epsilon > 0$  be given. There exists a compact set  $K \subset X$  such that for every  $x \notin K$

$$\|f(x) - g(x)\| < \epsilon \quad \text{and} \quad \|g(x)\| < \epsilon.$$

By Definition 2.1,

$$\|h_\epsilon(f,g)(x)\| < \epsilon$$

for every  $x \notin K$ . Therefore,

$$h_\epsilon(f,g) \in C_0(X;E).$$

**COROLLARY 3.16:** If  $E$  has property (P) then  $c_0(E)$  has the relative Chebyshev center property in  $\ell_\infty(E)$ . In particular,  $c_0(E)$

admits Chebyshev centers.

REMARK: Yost ([26], Lemma 2.6) proved that  $c_0(E)$  is an M-ideal in  $\ell_\infty(E)$  when  $E$  is any Banach space, and therefore,  $c_0(E)$  is proximinal in  $\ell_\infty(E)$  even without the hypothesis that  $E$  has property (P).

COROLLARY 3.17: The space  $c_0$  has the relative Chebyshev center property in  $\ell_\infty$ . In particular,  $c_0$  admits Chebyshev centers.

#### 4. COMPACT LINEAR MAPPINGS

When  $E$  and  $F$  are normed spaces, we denote by  $\mathcal{L}(E;F)$  the vector space of all bounded linear operators  $T$  from  $E$  into  $F$  with the norm

$$\|T\| = \sup \{ \|Tx\|; \|x\| \leq 1 \}.$$

The subspace of  $\mathcal{L}(E;F)$  of compact linear operators will be denoted by  $K(E;F)$ .

Let  $X$  be a locally compact Hausdorff space and  $(E, \|\cdot\|)$  be a Banach space with dual  $E^*$ . We denote by  $C_0(X; (E^*, \tau))$  the space of all  $\tau$ -continuous  $E^*$ -valued functions on  $X$  vanishing at infinity when  $E^*$  has a topology  $\tau$ . Let

$$C_0(X; E^*) = \ell_\infty(X; E^*) \cap C_0(X; (E^*, \sigma))$$

where  $\sigma = \sigma(E^*, E)$  denotes the weak\* topology on  $E^*$ .

THEOREM 4.1: The space  $\mathcal{L}(E, C_0(X))$  is isometrically isomorphic to  $C_0(X; E^*)$  via the mapping  $\Phi$  defined by

$$\Phi(T)x = \delta_x \circ T$$

for all  $T \in \mathcal{L}(E; C_0(X))$  and  $x \in X$ , where  $\delta_x$  denotes the evaluation map at  $x$ . Under this mapping,  $K(E; C_0(X))$  is isometrically isomorphic to  $C_0(X; E^*)$ .

REMARK: This result is well known when  $X$  is compact (see Dunford-Schwartz [9], p. 490). Since the proof in the case of a locally compact Hausdorff space  $X$  is a straightforward generalization of the proof in the case of a compact space, we shall omit the proof of Theorem 4.1.

THEOREM 4.2: Let  $E$  be a uniformly smooth Banach space. Then  $K(E; C_0(X))$  has the relative Chebyshev center property in  $\mathcal{L}(E; C_0(X))$ .

PROOF: Since  $E^*$  is uniformly convex (see Diestel [7]) by Corollary 3.15  $C_0(X; E^*)$  has the relative Chebyshev center property in  $\ell_\infty(X; E^*)$ , and, a fortiori, in  $C_0(X; E^*)$ . Now the result follows from Theorem 4.1.

COROLLARY 4.3: If  $E$  is a Hilbert space, or a space  $\ell_p$  or  $L_p$ ,  $1 < p < \infty$  then  $K(E; C_0(X))$  has the relative Chebyshev center property in  $\mathcal{L}(E; C_0(X))$ .

PROOF: A Hilbert space, a space  $\ell_p$  or  $L_p$  with  $1 < p < \infty$  are uniformly smooth (see Clarkson [6]).

REMARK: Mach ([16], Corollary 3) proved that  $K(E; C(X))$  is proximal in  $\mathcal{L}(E; C(X))$  when  $X$  is compact and  $E$  is a Hilbert space, a space  $\ell_p$ ,  $1 < p < \infty$  or the space  $c_0$ .

COROLLARY 4.4: If  $E$  is a uniformly smooth Banach space then  $K(E; c_0)$  has the relative Chebyshev center property in  $\mathcal{L}(E; c_0)$ .

REMARK: Mach and Ward ([20], Theorem 3.1) and Yost ([26],



Corollary 2.7) proved that, for any Banach space  $E$ ,  $K(E; c_0)$  is an M-ideal of  $\mathcal{L}(E; c_0)$ .

COROLLARY 4.5: For each  $1 < p < \infty$  the space  $K(\ell_p; c_0)$  has the relative Chebyshev center property in  $\mathcal{L}(\ell_p; c_0)$ .

COROLLARY 4.6: Let  $X$  be a completely regular space and  $(E, \|\cdot\|)$  be a uniformly smooth Banach space. Then  $K(E; C_b(X; \mathbb{K}))$  has the relative Chebyshev center property in  $\mathcal{L}(E; C_b(X; \mathbb{K}))$ .

PROOF:  $\mathcal{L}(E; C_b(X; \mathbb{K}))$  is isometrically isomorphic to  $\mathcal{L}(E; C(\beta X; \mathbb{K}))$  where  $\beta X$  denotes the Stone-Čech compactification of  $X$ . Under this isomorphism  $K(E; C_b(X; \mathbb{K}))$  can be identified with  $K(E; C(\beta X; \mathbb{K}))$ . Now the result follows from Theorem 4.2.

REMARK: If we take  $\mathbb{K} = \mathbb{R}$  in Corollary 4.6 then the result is true for any topological space  $X$ , since  $C_b(X; \mathbb{R})$  is an AM-space with unit and so it is isometrically isomorphic to a  $C(Y; \mathbb{R})$  for some compact Hausdorff space  $Y$ .

Ka-Sing Lau ([15], Theorem 4.5, part (ii)) proved that for any topological space  $X$ ,  $K(X; C_b(X; \mathbb{R}))$  is proximal in  $\mathcal{L}(E; C_b(X; \mathbb{R}))$ , if  $E$  is a uniformly smooth Banach space.

COROLLARY 4.7: If  $(E, \|\cdot\|)$  is a uniformly smooth Banach space then  $K(E; \ell_\infty)$  has the relative Chebyshev center property in  $\mathcal{L}(E; \ell_\infty)$ .

PROOF: If  $\mathbb{N}$  has the discrete topology then  $\ell_\infty = C_b(\mathbb{N}; \mathbb{K})$ .

COROLLARY 4.8:  $K(\ell_p; \ell_\infty)$  has the relative Chebyshev center property in  $\mathcal{L}(\ell_p; \ell_\infty)$  when  $1 < p < \infty$ .

REMARK: Feder ([10], Theorem 1) proved that  $K(\ell_\infty; \ell_\infty)$  is not proximal in  $\mathcal{L}(\ell_\infty; \ell_\infty)$ .



Let  $(S, \Sigma, \mu)$  be a  $\sigma$ -finite positive measure space and  $(F, \| \cdot \|)$  be a uniformly convex Banach space. We denote by  $L_\infty(S, \Sigma, \mu; F)$  the space of all essentially bounded  $\mu$ -Bochner integrable functions  $f: S \rightarrow F$  normed by

$$\|f\| = \operatorname{ess\,sup}_{s \in S} \|f(s)\|.$$

The subspace of all elements of  $L_\infty(S, \Sigma, \mu; F)$  whose ranges are  $\mu$ -essentially relatively compact is denoted by  $K_\infty(S, \Sigma, \mu; F)$ . For definitions see Diestel-Uhl [8].

**THEOREM 4.9:** Let  $W \subset L_\infty(S, \Sigma, \mu; F)$  be a closed subset such that  $h_\varepsilon(f, g) \in W$  for all  $f, g \in W$  and  $\varepsilon > 0$ . Then  $W$  has the relative Chebyshev center property in  $L_\infty(S, \Sigma, \mu; F)$ .

**PROOF:** We can follow the same reasoning as in Theorem 2.2. The sup-norm is replaced by

$$\|f\| = \operatorname{ess\,sup}_{s \in S} \|f(s)\|$$

for all  $f \in L_\infty(S, \Sigma, \mu; F)$ .

**THEOREM 4.10:**  $K(L_1(S, \Sigma, \mu); F)$  has the relative Chebyshev center property in  $\mathcal{L}(L_1(S, \Sigma, \mu); F)$ .

**PROOF:** Since  $F$  is uniformly convex, it has the Radon - Nikodym property (see Clarkson [6]), and so  $\mathcal{L}(L_1(S, \Sigma, \mu); F)$  is isometrically isomorphic to  $L_\infty(S, \Sigma, \mu; F)$ . Also  $K(L_1(S, \Sigma, \mu); F)$  can be identified with  $K_\infty(S, \Sigma, \mu; F)$ . Let  $f, g \in K_\infty(S, \Sigma, \mu; F)$  and  $\varepsilon > 0$  be given. As in the proof of Corollary 2.3 we can write

$$h_\varepsilon(f, g)(s) = f(s) + \beta_\varepsilon (\|f(s) - g(s)\|) [g(s) - f(s)]$$

for all  $s \in S$ . If  $N_f$  and  $N_g$  denote the  $\mu$ -null sets such that

$K_f = f(S - N_f)$  and  $K_g = g(S - N_g)$  are relatively compact, then  $N = N_f - N_g$  is a  $\mu$ -null set and

$$h_\epsilon(f, g)(s) \in K_f + [0, 1] (K_g - K_f)$$

for all  $s \in S \setminus N$ . Hence  $h_\epsilon(f, g)(S \setminus N)$  is relatively compact.

REMARK: Ka-Sing Lau ([15], Theorem 4.5 part (i)) proved the proximality of  $K(L_1(S, \Sigma, \mu); F)$  under the same hypothesis of Theorem 4.10.

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