To appear in Studia Logica

PROPER n-VALUED LUKASIEWICZ ALGEBRAS AS S-ALGEBRAS OF LUKASIEWICZ n-VALUED PROPOSITIONAL CALCULI

Roberto Cignoli

RELATÓRIO INTERNO Nº 184

ABSTRACT: Proper n-valued Lukasiewicz algebras are obtained by adding some binary operators, fulfilling some simple equations, to the fundamental operations of n-valued Lukasiewicz algebras. They are the S-algebras corresponding to an axiomatization of Lukasiewicz n-valued propositional calculus that is an extention of the intuitionistic calculus.

Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
IMECC – UNICAMP
Caixa Postal 1170
13.100 - Campinas - SP.
BRASIL

Novembro - 1980

PROPER n-VALUED LUKASIEWICZ ALGEBRAS AS S-ALGEBRAS OF LUKASIEWICZ n-VALUED PROPOSITIONAL CALCULI

Roberto Cignoli

Dedicated to the memory of Gregorious C. Moisil.

INTRODUCTION: Many-valued logics were introduced by J. Lukasiewicz in 1920. With the aim of developing an algebraic theory of the n-valued Lukasiewicz propositional calculi, G. Moisil introduced in 1940 the n-valued Lukasiewicz algebras. These algebras are bounded distributive lattices on which are defined a De Morgan negation and $n-1$ modal operators fulfilling some axioms that are given in § 1.

The three-and four-valued Lukasiewicz algebras are the algebraic counterpart of the three-and four-valued Lukasiewicz propositional calculi respectively, but it was shown by A. Rose that this is not the case if $n \geq 5$ (see the introduction of [4]). The adequate algebraic notion, corresponding to all $n \geq 2$, was introduced by R. Grigolia [7] following some ideas of C. C. Chang, under the same of MV_n-algebras. But these structures are not directly based on lattices, and so their comparision with algebraic structures corresponding to other logical calculi (for instance, classical, Post, intuitiomistic calculi) is not easy.

In this paper we show that adequate algebraic counterparts for the n-valued Lukasiewicz propositional calculi can be obtained by adding a set of $(n(n-5)+2)/2$ $(n \geq 5)$ binary operators, satisfying very simple equations, to the primitive operations of n-valued Lukasiewicz algebras. The structures so defined are called Proper n-valued Lukasiewicz algebras, and they were introduced by the author in [5]. In case $n = 2, 3$ or 4, Proper n-valued Lukasiewicz algebras coincide with n-valued Lukasiewics algebras. Since it was proved by L. Iturrioz [8] that n-valued Lukasiewicz algebras can be characterized as Heyting algebras with some unary operators, if follows that Proper n-valued Lukasiewicz algebras provide an axiomatization of the n-valued calculi of Lukasiewicz that is an extension of the classical intuitiomistic calculus by means of the modal operators introduced by Moisil, some new binary operators and a rule of inference. The details are given in § 5. Note that if in the axiomatization given in § 5, we sustitute axioms (A 17) and (A 18) for axioms corresponding to the constants operators e_i, we get an axiomatization of n-valued Post propositional calculi [11]. On the other hand, axioms A 1) - A 16) and the rules of inference modus ponens and (r_n) give an axiomatization of a propositional calculus that has the n-valued Lukasiewicz algebras as S-algebras. It is fair to call these calculi, for $n \geq 2$, Moisil n-valued propositional calculi.

Thus we have that Moisil calculi are extentions of the intuitiomistic calculus. The Post calculi are obtained by adding some constants connectives to the Moisil calculi, and the Lukasiewicz calculi are obtained by adding some binary connectives. In § 2, § 3 and § 4 we develop the part of the theory of Proper n-valued Lukasiewicz algebras that we need in § 5. An investigation on the lattice theory of these algebras and their topological representations will be published elsewere.

ACKNOWLEDGEMENT: The author mantained some interesting conversations on the subject-matter of this paper with Professor H. Rasiowa, who visited the Universidade Estudal de Campinas supported by the Grant 80/0165 from the Fundação de Amparo a Pesquisa do Estado de São Paulo (FAPESP).

1. LUKASIEWICZ ALGEBRAS

For the pourpose of this paper it is convinient to consider the characterization of Lukasiewicz algebras in terms of symmetric Heyting algebras given by L. Iturrioz in [8] . For some historical remarks on the relations between Heyting and Lukasiewicz algebras, see [5] .

Recall that a Heyting algebra is an algebra $\langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ of type (2,2,2,0,0) such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a lattice with zero 0 and unit 1 and, for each x, y in A, $x \Rightarrow y$ is the greatest element in A such that $x \wedge (x \Rightarrow y) \leq y$. A symmetric Heyting algebra is an algebra $\langle A, \vee, \wedge, \Rightarrow, \sim, 0, 1\rangle$ such that $\langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ is a Heyting algebra and $\sim$ is a unary operation that satisfies the following two axioms:

M 1) $\sim\sim x = x$ and M 2) $\sim(x \wedge y) = \sim x \vee \sim y$. Since the underlying lattice of a Heyting algebra is always distributive, the reduct $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ of a symmetric Heyting algebra is a De Morgan algebra. Heyting and De Morgan algebras are studied in [1] and [11]. (In the second book, Heyting algebras are called pseudo-Boolean algebras, and De Morgan algebras, quasi-Boolean algebras).

1.1. DEFINITION [8] : An n-valued Lukasiewicz algebra (n an integer ≥ 2) is an algebra $\langle A, \vee, \wedge, \Rightarrow, \sim, \sigma_1^n, \ldots, \sigma_{n-1}^n, 0, 1\rangle$ such that $\langle A, \vee, \wedge, \Rightarrow, \sim, 0, 1\rangle$ is a symmetric Heyting algebra and σ_i^n, $1 \leq i \leq n-1$, are unary operations that satisfy the following axioms:

L1) $\sigma_i^n(x \vee y) = \sigma_i^n x \vee \sigma_i^n y$

L2) $\sigma_i^n(x \Rightarrow y) = \bigwedge_{j=i}^{n-1} (\sigma_j^n x \Rightarrow \sigma_j^n y)$

L3) $\sigma_i^n \sigma_j^n x = \sigma_j^n x, \quad 1 \leq i, j \leq n-1$

L4) $\sigma_1^n x \vee x = x$

L5) $\sigma_i^n \sim x = \sim\sigma_{n-i}^n x$

L6) $\sigma_1^n x \vee \sim \sigma_1^n x = 1$

n-valued Lukasiewicz algebras are considered in [4] and [1] (Note that in this last reference, the operators Di correspond to our σ_{n-i}^n).

We are going to limit ourselves to recall some properties that we shall need in what follows. We start by given an important example of n-valued Lukasiewicz algebras. Let K_n $(n \geq 2)$ denote the chain formed by the fractions $j/(n-1)$, $j = 0,1,\ldots,n-1$. Then $L_n = \langle K_n, \vee, \wedge, \Rightarrow, \sim, \sigma_1^{n-1}, \ldots, \sigma_{n-1}^n, 0, 1\rangle$ where $x \vee y = \max(x,y)$, $x \wedge y = \min(x,y)$, $0 = 0/(n-1)$, $1 = (n-1)/(n-1)$; $\sim x = 1 - x$,

$$x \Rightarrow y = \begin{cases} 1 & \text{if} \quad x \leq y \\ y & \text{if} \quad x > y \end{cases}$$

and

$$\sigma_i^{n-1}(j/(n-1)) = \begin{cases} 1 & \text{if} \quad i + j \geq n \\ 0 & \text{if} \quad i + j < n \end{cases}$$

We shall denote by $\mathcal{L}_n$ the (equational) class of n-valued Lukasiewicz algebras, and by $\mathbf{L}_n$ the corresponding category. It is well known that $\mathcal{L}_n$ is the equational class generated by the algebra L_n. An important property is the following:

MOISIL'S DETEMINATION PRINCIPLE: Let $A \in \mathcal{L}_n$ and x, y be elements of A. Then $x = y$ if and only if $\sigma_i^n x = \sigma_i^n y$ for $i = 1,2,\ldots,n-1$.

It is convinient to set $\sigma_n^n x = 1$ and $\sigma_o^n x = 0$, and to define the following operators $J_i^n : A \to A$ for $i = 0,1,\ldots,n-1$:

$$(1.1) \qquad J_i^n x = \sigma_{n-i}^n x \wedge \sim \sigma_{n-i-1}^n x$$

Note that in the algebra L_n, we have that

$$J_i^n(j/(n-1)) = \begin{cases} 1 & \text{if} \quad j = i \\ 0 & \text{if} \quad j \neq i \end{cases}$$

Moreover it is not hard to see that in any $A \in L_n$ we have that:

$$(1.2) \qquad \sigma_i^n(x) = \cancel{\bigvee_{j=1}^{n-1} J^n_{n-j}(x)} \qquad \bigvee_{j=1}^{i} \overline{J}^n_{n-j}(x)$$

If A is a distributive lattice, we denote by B(A) the Boolean algebra of all complemented elements of A. If $A \in L_n$, then $x \in B(A)$ if and only if $x = \sigma_i(x)$ for some $1 \leq i \leq n-1$. Moreover, $\sigma_1^n x$ is the greatest element in B(A) such that $\sigma_1^n x \leq x$, and then it follows that:

(1.3) $\sigma_1^n(x \Rightarrow y)$ is the greatest element in B(A) such that

$$x \wedge \sigma_1^n(x \Rightarrow y) \leq y$$

The last property implies that (cp [6]):

$$(1.4) \qquad x \Rightarrow y = \sigma_1^n(x \Rightarrow y) \vee y.$$

The following results are usfull in § 3:

1.2. LEMMA: (i) If A is a Heyting algebra and $a \in B(A)$, then $a \Rightarrow x = -a \vee x$, where $-a$ denotes the (Boolean) completent of a,

(ii) If $A \in L_n$ and $b \in B(A)$, then $x \Rightarrow b = \sim \sigma_{n-1}^n x \vee b$.

PROOF: (i) Note first that $a \wedge (-a \vee x) = (a \wedge -a) \vee (-a \wedge x) = -a \wedge x \leq x$.

Now, if $a \wedge y \leq x$, then $y \leq y \vee -a = (y \wedge a) \vee -a \leq x \vee -a$. Consequently $a \Rightarrow x = -a \vee x$.

(ii) By (1.4) and (L2) we have that:

$$x \Rightarrow b = \sigma_i^n(x \Rightarrow b) \vee b = b \vee \bigwedge_{i=1}^{n-1} (\sigma_i^n x \Rightarrow \sigma_i^n b)$$

Since $b \in B(A)$, it follows that $\sigma_i^n b = b$ for $i = 1,2,\dots,n-1$, and since $\sigma_i^n x \in B(A)$ for $i = 1,2,\dots,n-1$, (i) implies that $\sigma_i^n x \Rightarrow \sigma_i^n b = -\sigma_i^n x \vee b$. Therefore $a \Rightarrow b = b \vee ((\bigwedge_{i=1}^{n-1} -\sigma_i x) \vee b) = -\sigma_i^n x \vee b$

Finally we are going to recall some facts about congruences of n-valued Lukasiewicz algebras (see [4] or [1]).

Let $A \in L_n$. A subset F of A is said to be a Stone filter of A if (1) F is a filter of the underlying lattice of A and (2) if $x \in F$, then $\sigma_1^n x \in F$.

If F is a Stone filter of $A \in L_n$, then the binary relation $\theta(F)$ defined by the condition: $(x,y) \in \theta(F)$ if and only if there is $f \in F$ such that $x \wedge f = y \wedge f$, is a congruence on the algebra A and, moreover, $F = \{x \in A: (x,1) \in \theta(F)\}$. Conversely, if θ is a congruence on A, we have that $F = \{x \in A: (x,1) \in \theta\}$ is a Stone filter of A and that $\theta = \theta(F)$.

2. PROPER LUKASIEWICZ ALGEBRAS

The following notations will used systematically in what follows: N will denote the set of non-negative integers, $S_n = \{(i,j) \in N \times N: 3 \leq i \leq n-2, 1 \leq j \leq n-4, j < i\}$ provided that $n \geq 5$ and $S_n = \emptyset$ if $n < 5$; $T_n = \{(i,j) \in N \times N: 2 \leq i \leq n-2, 1 \leq j \leq n-3, j < i\}$ if $n \geq 4$ and $T_n = \emptyset$ if $n < 4$. For any set X, $|X|$ will denote the number of elements of X. Note that for $n \geq 5$, $|S_n| = (n(n-5)+2)/2$, and $|T_n| = |S_n| + 2$, but $|T_4| = 1$.

2.1. DEFINITION: A proper n-valued Lukasiewicz algebra ($n \in N$, $n \geq 2$) is a system $\langle A, \vee, \wedge, \Rightarrow, \sim, \{\sigma_i^n\}\ 1 \leq i \leq n-1,\ \{F_{ij}^n\}(i,j) \in S_n,\ 0,\ 1\rangle$ such that $\langle A, \vee, \wedge, \Rightarrow, \sim\ \{\sigma_i^n\}\ 1 \leq i \leq n-1, 0, 1\rangle$ is an n-valued Lukasiewicz algebra and F_{ij}^n, $(i,j) \in S_n$ are binary operators defined on A satisfying the following equations:

$$(P) \qquad \sigma_k^n(F_{i,j}^n(x,y)) = \begin{cases} 0 & \text{if } k \leq i-j \\ J_i^n(x) \wedge J_j^n(y) & \text{if } k > i-j \end{cases} \qquad 1 \leq k \leq n-1$$

The equational classes of proper n-valued Lukasiewicz algebras will be denoted by $\mathcal{P}_n$ and the corresponding categories, by $\mathbf{P}_n$. Note that for $2 \leq n \leq 4$, $\mathcal{P}_n = \mathcal{L}_n$ (because in these cases, $S_n = \emptyset$).

2.2. REMARK: If $A \in \mathcal{L}_n$, $n \geq 4$, define $F_{21}^n(x,y) = J_2^n(x) \wedge J_1^n(y) \wedge \sim y$ and $F_{n-2\ n-3}^n(x,y) = J_{n-2}^n(x) \wedge J_{n-3}^n(y) \wedge x$. It is very to check that $\sigma_1^n F_{21}^n(x,y) = \sigma_1^n F_{n-2\ n-3}^n(x,y) = 0$ and that for $2 \leq k \leq n-1$, $\sigma_k^n F_{21}^n(x,y) = J_2^n(x) \wedge J_1^n y$ and $\sigma_k^n F_{n-2\ n-3}^n(x,y) = J_{n-2}^n(x) \wedge J_{n-3}^n(y)$. Thus in a proper n-valued Lukasiewicz algebra we have in fact a set F_{ij}^n of binary operators satysfing (P) for $(i,j) \in T_n$.

2.3. EXAMPLE: We are going to denote by P_n the proper n-valued Lukasiewicz algebras obtained by adding the operators:

$$F_{ij}^n\left(\frac{r}{n-1}, \frac{s}{n-1}\right) = \begin{cases} \dfrac{n-1-i+j}{n-1} & \text{if } (r,s) = (i,j) \\ 0 & \text{otherwise} \end{cases} \qquad (i,j) \in S_n$$

to the n-valued Lukasiewicz algebras L_n. Note that $L_n = P_n$ if $2 \leq n \leq 4$.

2.4. LEMMA: The following properties hold true in any $A \in P_n$, where $(i,j) \in T_n$, x, y are elements in A and a, b elements in B(A):

(i) $F^n_{ij}(\sim y, \sim x) = F^n_{(n-1)(n-1-i)}(x,y)$

(ii) $F^n_{ij}(x \wedge a, y \wedge b) = F^n_{ij}(x,y) \wedge a \wedge b$

(iii) $F^n_{ij}(x \vee a, y \vee b) = F^n_{ij}(x,y) \wedge -a \wedge -b$

(iv) $F^n_{ij}(x,b) = F^n_{ij}(a,y) = 0$

PROOF: In order to prove (i) and (ii) apply the operators σ^n_k $(1 \leq k \leq n-1)$ to both membres of the equations and then use the Moisil's determination principle. (iii) is a consequence of (i) and (ii). (iv) is a consequence of (ii) and (iii). For, $F^n_{ij}(x,b) = F^n_{ij}(x \vee 0, b \vee 0) = F^n_{ij}(x,0) \wedge -b$; and $F^n_{ij}(x,0) = F^n_{ij}(x \wedge 1, 0 \wedge 0) = F^n_{ij}(x,0) \wedge 1 \wedge 0 = 0$.

Let F be a Stone filter of $A \in P_n$, and suppose that (x,x') and (y,y') belong to $\theta(F)$ (see §1). Then there are elements a, b belonging to $F \cap B(A)$ such that $x \wedge a = x' \wedge a$ and $y \wedge b = y' \wedge b$, and it follows from (ii) of Lemma 2.4 that $F^n_{ij}(x,y) \wedge a \wedge b = F^n_{ij}(x \wedge a, y \wedge b) = F^n_{ij}(x' \wedge a, y' \wedge b) = F^n_{ij}(x',y') \wedge a \wedge b$. Since $a \wedge b \in F$, we have that $(F^n_{ij}(x',y'), F^n_{ij}(x,y)) \in \theta(F)$, for $(i,j) \in T_n$. Consequently, $\theta(F)$ is also a Proper n-valued Lukasiewicz algebra congruence, and it is very easy to complete the proof of the following:

2.5. THEOREM: Let $A \in P_n$. Then the correspondence $F \to \theta(F)$ establishes an order isomorphism between the lattice of Stone filters of A (ordered by inclusion) and the lattice of congruences of A.

2.6. COROLLARY: Let $A \in P_n$. Then the correspondence $F^* \to \theta((\sigma^n_1)^{-1}(F^*))$

establishes an order isomorphism between the lattice of filters F^* of the Boolean algebra $B(A)$ and the lattice of congruences on A.

Suppose A, A' are in P_n and $h : A \to A'$ is an n-valued Lukasiewicz algebra homomorphism. Since $\sigma_k^n(hF_{ij}^n(x,y) = h(\sigma_k^n F_{ij}^n(x,y)) = h(0) = 0 = \sigma_k^n F_{ij}^n(h(x),h(y))$ for $1 \leq k \leq i-j$ and $\sigma_k^n(h(F_{ij}^n(x,y))) = h(\sigma_k^n(F_{ij}^n(x,y)) = h(J_i^n(x) \wedge J_j^n(y)) = J_i^n(h(x)) \wedge J_j^n(h(y)) = \sigma_k^n F_{ij}^n(h(x),h(y))$ for $i-j < k \leq n-1$, it follows by Moisil determination principle that $h(F_{ij}^n(x,y)) = F_{ij}^n(h(x),h(y))$, for each $(i,j) \in T_n$. Consequently, h is also a Proper n-valued Lukasiewicz algebra homomorphism, and we have:

2.7. PROPOSITION: $\mathbf{P}_n$ is a full subategory of $\mathbf{L}_n$ for $n \leq 2$. (Note that $\mathbf{P}_n = \mathbf{L}_n$ for $2 \leq n \leq 4$).

The next theorem is an easy consequence of Th. 1 Chap. X 1 § 6 of [1] and of our previous results (see also [4]).

2.8. THEOREM: The following are equivalent conditions for each $A \in P_n$ (with $|A| \geq 2$):

(i) $B(A)$ is the two-element Boolean algebra $\{0,1\}$.

(ii) A is a chaim (i.e., A is totally ordered).

(iii) A is isomorphic to a subalgebra of P_n.

(iv) A is simple.

(v) A is subdirectly irreducible.

(vi) A is irreducible.

2.9. COROLLARY: Each $A \in P_n$ with $|A| \geq 2$, is a subdirect product of a family of subalgebras of P_n.

2.10. COROLLARY: P_n is the equational class generated by the algebra P_n, for each $n \in N$, $n \geq 2$.

3. THE LUKASIEWICZ IMPLICATION

Let $A \in P_n$. For each x, y in A, define the binary operator $x \to y$ as follows:

$$(3.1) \qquad x \to y = (x \Rightarrow y) \vee \sim x \vee \bigvee_{(i,j) \in T_n} F^n_{ij}(x,y)$$

3.1. REMARK: Since $T_3 = \emptyset$ and $T_4 = \{(2,1)\}$ we have that:

$$(3.2) \qquad \text{If } A \in P_3 = L_3, \text{ then } x \to y = (x \Rightarrow y) \vee \sim x$$

$$(3.3) \qquad \text{If } A \in P_4 = L_4, \text{ then } x \to y = (x \Rightarrow y) \vee \sim x \vee F_{21}(x,y) =$$
$$= (x \Rightarrow y) \vee \sim x \vee (J^n_2(x) \wedge J^n_1(y) \wedge \sim y)$$

3.2. PROPOSITION: The following properties hold in each $A \in P_n$:

(i) $\sigma^n_1(x \to y) = \sigma^n_1(x \Rightarrow y)$

(ii) $x \Rightarrow y = \sigma^n_1(x \to y) \vee y$

(iii) If $a \in B(A)$, then $x \to a = \sim x \vee a$

(iv) If $b \in B(A)$, then $b \to x = \sim b \vee x$

(v) If a, b are in $B(A)$, then $a \to b = \sim a \vee b$

(vi) $1 \to x = x$

(vii) $x \to y = 1$ if and only if $x \leq y$.

PROOF: (i) Since $1 \leq i - j$ for each $(i,j) \in T_n$, we have that $\sigma^n_1 F^n_{ij}(x,y) = 0$, and it follows at once from L2) that $\sigma^n_1 \sim x \leq \sigma^n_1(x \Rightarrow y)$. Hence $\sigma^n_1(x \to y) = \sigma^n_1(x \Rightarrow y) \vee \sigma^n_1 \sim x = \sigma^n_1(x \Rightarrow y)$.

(ii) follows at once from (i) and (1.4).

(iii) From (iv) of Lemma 2.4 it follows that $x \to a = (x \Rightarrow a) \vee \sim x$, and (ii) of Lemma 1.2 implies that $(x \Rightarrow a) \vee \sim x = \sigma_1 \sim x \vee a \vee \sim x = a \vee \sim x$.

(iv) Follows from (iv) of Lemma 2.4 and (i) of Lemma 1.2.

(v) Is an obvious consequence of (iii) and (iv).

(vi) Is a particular case of (iv).

(vii) It follows from (i) that the following are equivalent conditions: (1) $x \to y = 1$; (2) $\sigma_1^n(x \to y) = 1$, (3) $\sigma_1^n(x \Rightarrow y) = 1$, (4) $x \Rightarrow y = 1$ and (5) $x \leq y$.

In the algebras P_n the operator $\to$ coincides with the Lukasiewicz n-valued implication [9] : $x \to y = \min(1, 1 - x + y)$. For, if $x \leq y$, it follows from (vii) of the above Proposition that $x \to y = 1$, and $1 - x + y \geq 1$. If $y < x$, set $y = q/(n-1)$ and $x = p/(n-1)$. Since in this case $F_{ij}^n(x,y) = 0$ if $(i,j) \neq (p,q)$ and $F_{pq}^n(x,y) = 1 - x + y$ and $x \Rightarrow y = y$, we have that if $(p,q) \in T_n$, then

$$x \to y = y \vee \sim x \vee (1 - x + y) = \max(y, 1 - x, 1 - x + y) = 1 - x + y \leq 1.$$

If $(p,q) \notin T_n$, $x \to y = \max(y, 1 - x)$. If $q = 0$, then $y = 0$ and $\max(y, 1 - x) = 1 - x = 1 - x + y \leq 1$. If $q \neq 0$, then $p = n - 1$, i.e. $x = 1$, and $\max(y, 1 - x) = y = 1 - x + y \leq 1$.

On the other hand, Lukasiewicz remarked that if $x, y \in P_n$, Then:

$$x \vee y = \max(x,y) = (x \to y) \to y \tag{3.4}$$

and

$$x \wedge y = \min(x,y) = \sim(\sim x \vee \sim y). \tag{3.5}$$

Moreover, Rosser and Turquette ([12], pp. 18 - 22) showed that the operators J_i^n, $0 \leq i \leq n-1$ of the algebras P_n can also be defined in terms of $\sim$ and $\to$. They defined first unary operators $H_k^n : P_n \to P_n$ as follows:

(3.6) $\quad H_1^n(x) = \sim x \quad$ and $\quad H_{k+1}^n(x) = x \rightarrow H_k^n(x)$,

and they showed that for each x in P_n:

(3.7) $\quad J_{n-1}^n(x) = \sim H_{n-1}^n(x) \quad$ and $\quad J_o^n(x) = J_{n-1}^n(\sim x)$.

The operators J_{n-i}^n, $i = 2,3,\dots,n-1$ were defined inductively as follows. For each $1 \leq k \leq n-2$, denote by $i(k)$ the greatest integer $<(n-1)/(n-1-k)$ and set $r(k) = (n-1)\, H_{i(k)}^n(k/(n-1))$. If $n-i = r(n-i)$ then:

(3.8) $\quad J_{n-i}^n(x) = J_{n-1}^n((H_{i(n-i)}^n(x) \vee x) \rightarrow (H_{i(n-i)}^n(x) \wedge x))$.

If $n-i < r(n-i)$, then:

(3.9) $\quad J_{n-i}^n(x) = J_{r(n-i)}^n(H_{i(n-i)}^n x)$.

The operators σ_i^n $(1 \leq i \leq n-1)$ and $F_{ij}^n((i,j) \in T_n)$ can now be defined as follows; for x,y in P_n:

(3.10) $\quad \sigma_i^n(x) = \bigvee_{j=1}^{\overset{i}{\cancel{n-1}}} J_{n-j}^n \qquad$ (cp (1.2))

(3.11) $\quad F_{ij}^n(x,y) = (x \rightarrow y) \wedge J_i^n(x) \wedge J_j^n(y)$

Note that in the algebras P_n we also have that:

(3.12) $\quad x \Rightarrow y = \sigma_1^n(x \rightarrow y) \vee y = J_{n-1}^n(x \rightarrow y) \vee y =$

$$= (\sim H_{n-1}^n(x \rightarrow y) \rightarrow y) \rightarrow y.$$

3.3. REMARKS: Explicit formulas defining the operators σ_i^n of the algebras P_n in terms of $\sim$ and $\rightarrow$, without introducing the operators J_i^n, where given in [13]. Note also that the fact that the operators $\vee, \wedge, \sigma_i^n$ and F_{ij}^n of the algebras P_n can be defineded in terms of the Lukasiewicz's negation $\sim$ and implication $\rightarrow$ is a consequence of a theorem of Mc Naughton [10]. However, Mc Naughton's result does not give explicit formulas for expressing $\vee, \wedge, \sigma_i^n$, F_{ij}^n in terms of $\sim$ and $\rightarrow$.

The above results show that in some sense, we can identify the matrices of the n-valued Lukasiewicz propositional calculi with the algebras P_n. More precisely, if we define the n-valued Lukasiewicz matrices M_n as algebras $\langle K_n, \rightarrow, \sim, 1 \rangle$ of type $\langle 2,1,0 \rangle$ where $x \rightarrow y = \min(1, 1-x+y)$, $\sim x = 1-x$ and $1 = (n-1)/(n-1)$, then we have that the algebras P_n and M_n are cryptoisomorphic in the sense of Birkhoff [2].

The following lemma is an application of our results on the operator $\rightarrow$ in the algebras P_n:

3.4. LEMMA: Let $A \in \mathcal{P}_n$. For each $k \in N$, define a unary operator $C_k^n : A \rightarrow A$ inductively as follows: $C_1^n(x) = \sigma_1 x$ and $C_{k+1}^n(x) = x \rightarrow C_k^n x$. Then we have that $C_k^n(x) = 1$ for each $k \geq n$.

PROOF: If follows from Corollary 2.10, that if the equation $C_k^n(x) = 1$ is satisfied in the algebra P_n, then it is satisfied in each $A \in \mathcal{P}_n$. Since $C_1^n(1) = \sigma_1^n(1) = 1$, it follows from (vii) of Proposition 3.2 that $C_k^n(1) = 1$ for each $k \in N$. Suppose now that $x = j/(n-1)$, with $0 \leq j \leq n-2$. By taking into account the definition of $\rightarrow$ in P_n, it is easy to prove, by induction on k, that $C_k^n(j/(n-1)) = \min(1, (k-1)((n-1-j)/n-1))$. If $k \geq n$, since $0 \leq j \leq n-2$, we have that $(k-1)((n-1-j)/n-1) \geq n-1-j \geq 1$. Consequently, the equations $C_k^n(x) = 1$, $k \geq n$ hold identically in P_n.

4. DEDUCTIVE SYSTEMS

Since n-valued Lukasiewicz algebras are Heyting algebras, it follows that for each $A \in \mathcal{P}_n$, $F \subseteq A$ is a filter if and only if the following conditions are satisfied: (1) $1 \in F$, (2) If x and $x \Rightarrow y$ belong to F, then $y \in F$. Consequently, $F \subseteq A$ is a Stone filter if and only if conditions (1), (2) and (3) if $x \in F$, then $\sigma_1^n x \in F$, are satisfied. We are going to see now that if $A \in P_n$, then Stone filters of A can be characterized by conditions of type (1) and (2) but with respect to the implication $\rightarrow$.

4.1. DEFINITION: Let $A \in P_n$. A deductive system of A is a subset $D \subseteq A$ that satisfies the following two conditions: (D1) $1 \in D$ and (D2). If x and $x \rightarrow y$ belong to D, then $y \in D$.

4.2. PROPOSITION: Let $A \in P_n$. The following are equivalent conditions for each $D \subseteq A$.

(i) D is a Stone filter of A.

(ii) D is a deductive system of A.

PROOF: (i) *implies* (ii). Let D be a Stone filter of A. Since $1 \in D$, we need to prove that (D2) is satisfied. Suppose x and $x \rightarrow y \in D$. Then $\sigma_1^n(x \rightarrow y) \in D$ and $x \Rightarrow y = \sigma_1^n(x \rightarrow y) \vee y \in D$. By the remarks at the beguining of this section, it follows that $y \in D$.

(ii) *implies* (i): Let D be a deductive system of A. Suppose that x and $x \Rightarrow y \in D$. Since $x \Rightarrow y \leq x \rightarrow y$, it follows that x and $x \rightarrow y$ belong to D and (D2) implies that condition (2) is satisfied. Since (1) is equivalent to (D1), we have that D is a filter of A. Lemma 3.4 implies that $C_n^n(x) = 1 \in D$, and several applications of (D2) yield that if $x \in D$, then $\sigma_1^n x \in D$.

As an application of the above result we have that:

4.3. PROPOSITION: Let D be a deductive system of $A \in P_n$. The following are equivalent conditions for each pair x,y of elements of A:

(1) $(x,y) \in \theta(D)$

(2) $x \to y \in D$ and $y \to x \in D$

(3) $x \Rightarrow y \in D$ and $y \Rightarrow x \in D$.

PROOF: (1) *implies* (2): Suppose $(x,y) \in \theta(D)$. Since $x \to y$ is a polynomial function of A, it follows that $(x \to y, x \to x) \in \theta(D)$ and $(y \to x, y \to y) \in \theta(D)$. Since $x \to x = y \to y = 1$, it follows that $x \to y \in D$ and $y \to x \in D$.

(2) *implies* (3): Suppose $x \to y \in D$ and $y \to x \in D$. Then $\sigma_1(x \to y) \in D$ and $\sigma_1(y \to x) \in D$. Consequently $x \Rightarrow y = \sigma_1(x \to y) \vee y \in D$ and $y \Rightarrow x = \sigma_1(y \to x) \vee x \in D$.

(3) *implies* (1): Suppose $x \Rightarrow y \in D$ and $y \Rightarrow x \in D$. Then $\sigma_1^n(x \Rightarrow y) \wedge \sigma_1^n(y \Rightarrow x) \in D$ and we have that:

$$x \wedge \sigma_1^n(x \Rightarrow y) \wedge \sigma_1^n(y \Rightarrow x) \leq y \wedge \sigma_1^n(x \Rightarrow y) \wedge \sigma_1^n(x \Rightarrow y) \leq$$

$$\leq x \wedge \sigma_1^n(x \Rightarrow y) \wedge \sigma_1^n(y \Rightarrow x).$$

Therefore, $(x,y) \in \theta(D)$.

5. LUKASIEWICZ n-VALUED PROPOSITIONAL CALCULI

Let F_n^* denote the algebra of (well formed) formulas constructed as usual from a denumerable set $\{p_n\}_{n \in N}$ of propositional variables by means of the binary connectives $\vee, \wedge, \Rightarrow, F_{ij}^n, (i,j) \in T_n$, and the unary connectives $\sim, \sigma_i^n$, $1 \leq i \leq n-1$, and let A_n denote the following set of axiom-schemes; where $\alpha \Leftrightarrow \beta$ is an abbreviation of $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$; and $J_i\alpha$ is an abbreviation of $\sigma_{n-i}^n\alpha \wedge \sim\sigma_{n-i-1}\alpha$, $1 \leq i \leq n-2$:

(A1) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$

(A2) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$

(A3) $\alpha \Rightarrow (\alpha \vee \beta)$

(A4) $\beta \Rightarrow (\alpha \vee \beta)$

(A5) $(\alpha \Rightarrow \gamma) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow ((\alpha \vee \beta) \Rightarrow \gamma))$

(A6) $(\alpha \wedge \beta) \Rightarrow \alpha$

(A7) $(\alpha \wedge \beta) \Rightarrow \beta$

(A8) $(\alpha \Rightarrow \beta) \Rightarrow ((\alpha \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow (\beta \wedge \gamma)))$

(A9) $\alpha \Leftrightarrow \sim (\sim \alpha)$

(A10) $\sigma_1^n(\alpha \Rightarrow \beta) \Leftrightarrow \sigma_1^n(\sim\beta \Rightarrow \sim\alpha)$

(A11) $\sigma_i^n(\alpha \vee \beta) \Leftrightarrow \sigma_i^n\alpha \vee \sigma_n^n\beta \qquad 1 \leq i \leq n-1$

(A12) $\sigma_i^n(\alpha \Rightarrow \beta) \Leftrightarrow \bigwedge_{j=1}^{n-1} (\sigma_j^n\alpha \Rightarrow \sigma_j^n\beta) \quad 1 \leq i \leq n-1$

(A13) $\sigma_i^n \sigma_j^n\alpha \Leftrightarrow \sigma_j^n \alpha \qquad 1 \leq i, j \leq n-1$

(A14) $\cancel{\sigma_1^n \alpha \vee \alpha \Leftrightarrow \alpha}$ $\sigma_1^n \alpha \Rightarrow \alpha$

(A15) $\sigma_i^n \alpha \Leftrightarrow \sim \sigma_{n-i}^n \sim\alpha$

(A16) $\sigma_1^n \alpha \vee \sim \sigma_1^n\alpha$

(A17) $\sim \sigma_k^n F_{ij}^n(\alpha,\beta) \quad 1 \leq k \leq i-j,\ (i,j) \in T_n$

(A18) $\sigma_k^n F_{ij}^n(\alpha,\beta) \Leftrightarrow J_i\alpha \wedge J_j\beta \quad i-j < k \leq n-1,\ (i,j) \in T_n$

Let C_n^* be the consequence operator defined on F_n^* by the set of axiom-schemes A_n and the rules of inference modus ponens: $\alpha, (\alpha \Rightarrow \beta)/\beta$ and the rule $(r_n) : \alpha/\sigma_1^n \alpha$. By the propositional calculus Luk_n^* $(n \in N, n \geq 2)$ we understand the pair $\langle F_n^*, C_n^* \rangle$.

5.1. LEMMA: The propositional calculus Luk_n^* belongs to the class

S of standard implicative calculi in the sense of Rasiowa ([11] Ch. VIII. §5).

PROOF: Since conditions S1) and S2) of the definition of the class S of the standard implicative calculi ([11], Ch. VIII, §5) are obviously satisfied, it follows from Theorem X. 1.1 and X. 1.2 of [11] that in order to complete the proof that $Luk_n^* \in S$ we need to show that, for each $X \subseteq F_n^*$ and α, β in F_n^* the following properties hold:

(i) If $\alpha \Leftrightarrow \beta \in C_n^*(X)$, then $\sim\alpha \Leftrightarrow \sim\beta \in C_n^*(X)$

(ii) If $\alpha \Leftrightarrow \beta \in C_n^*(X)$, then $\sigma_i^n\alpha \Leftrightarrow \sigma_i^n\beta$ for $i = 1, 2, \ldots, n-1$

(iii) If $\alpha \Leftrightarrow \alpha'$ and $\beta \Leftrightarrow \beta'$ belong to $C_n^*(X)$, then $F_{ij}^n(\alpha,\beta) \Leftrightarrow F_{ij}^n(\alpha',\beta') \in C_n^*(X)$ for each $(i,j) \in T_n$.

(i) follows from axioms A10) and A14) and (r_n). ~~is an immediate consequence of axiom A10~~). In order to prove (ii), note that if $\alpha \Rightarrow \beta \in C_n^*(X)$, then it follows from rule (r_n) and A12) that $\bigwedge_{i=1}^{n-1} (\sigma_i^n\alpha \Rightarrow \sigma_i^n\beta) \in C_n^*(X)$, and consequently, that $\sigma_i^n\alpha \Rightarrow \sigma_i^n\beta \in C_n^*(X)$ for $i = 1, 2, \ldots, n-1$. Observe that conversely, if $\sigma_i^n\alpha \Rightarrow \sigma_i^n\beta \in C_n^*(X)$ for $1 \leq i \leq n-1$, then $\sigma_1^n(\alpha \Rightarrow \beta) \in$ $\in C_n^*(X)$, and it follows from A14) and modus ponens that $\alpha \Rightarrow \beta \in C_n^*(X)$. Thus we have also proved:

(ii') If $\sigma_i^n\alpha \Rightarrow \sigma_i^n\beta \in C_n^*(X)$ for each $1 \leq i \leq n-1$, then $\alpha \Rightarrow \beta \in C_n^*(X)$.

Since $J_i^n\alpha \Leftrightarrow (\sigma_{n-i}^n\alpha \wedge \sim\sigma_{n-i-1}^n\alpha)$ it follows from (i), (ii) and (ii') that:

(ii) $\alpha \Leftrightarrow \beta \in C_n^*(X)$ if and only if $J_i(\alpha) \Leftrightarrow J_i(\beta) \in C_n^*(X)$ for $0 \leq i \leq n-1$.