SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR

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1. INTRODUCTION

Let M^n be an n-dimensional connected riemannian manifold isometrically immersed in an (n+p)-dimensional riemannian manifold $\tilde{M}^{n+p}(c)$, with constant sectional curvature c . Let B be the second fundamental form of the immersion, H the mean curvature vector and and S the square of the lenght of B .

In [11] Simons prove the following inequality in the case $\tilde{M}^{n+p}(c) = S^{n+p}$ and M^n minimal and compact

$$\int_{M^n} \{(2 - \frac{1}{p})S^2 - nS\}\, dv \geq 0 \tag{1.1}$$

where dv is the volume element of M^n. It follows that if M^n is not totally geodesic and $S \leq n|(2 - \frac{1}{p})$ then $S = n|(2 - \frac{1}{p})$. Using (1.1) Chern, Do Carmo and Kobayashi [4] determined all compact minimal submanifolds of S^{n+p} satisfying (*) $S = n|(2 - \frac{1}{p})$. The condition (*) was subsequentely generalized by Braidi and Hsiung [1].

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The purpose of the present paper is to determine all isometric immersions M^n of $\tilde{M}^{n+p}(c)$, $c=0$ or $c=-1$, with non-zero parallel mean curvature vector, such that S is constant (this condition is automatically satisfied if M^n is compact) and satisfies a condition analogous to (*).

In §2 we compute the Laplacian of the second fundamental form of a submanifold M^n of $\tilde{M}^{n+p}(c)$, with parallel mean curvature vector.

In §3 we give a integral inequality analogou to (1.1), while §4 and §5, contain the main theorems of this paper.

2. LAPLACIAN OF THE SECOND FUNDAMENTAL FORM

Let M^n be an n-dimensional riemannian manifold isometrically immersed in an (n+p)-dimensional riemannian manifold $\tilde{M}^{n+p}(c)$. We choose a local field of orthonormal frames $e_1,\dots,e_{n+p}$ in M^n such that, restricted to M^n, the vectors $e_1,\dots,e_n$ are tangent to M^n and, consequently, the remaining vectors $e_{n+1},\dots,e_{n+p}$ are normal to M^n. Unless otherwise stated, we shall make of the following convention on the ranges of indices:

$$1 \leq A,B,C,\dots, \leq n+p \quad , \quad 1 \leq i,j,k,\dots, \leq n \; ,$$

$$n+1 \leq \alpha,\beta,\gamma,\dots, \leq n+p$$

and we shall agree that repeated indices are summed over the respective ranges, let $\omega^1,\dots,\omega^{n+p}$ be the field of dual frames with respect to the frame field of $\tilde{M}^{n+p}$ chosen above. Then the structure equations of $\tilde{M}^{n+p}$ are given by

$$d\omega^A = -\sum \omega^A_B \wedge \omega^B , \quad \omega^A_B + \omega^B_A = 0 \tag{2.1}$$

$$d\omega^A_B = -\sum \omega^A_C \wedge \omega^C_B + \phi^A_B , \quad \phi^A_B = \frac{1}{2}\sum K^A_{BCD}\, \omega^C \wedge \omega^D$$
$$K^A_{BCD} + K^A_{BCD} = 0 . \tag{2.2}$$

If we restrict these forms to M^n, then

$$\omega^\alpha = 0 . \tag{2.3}$$

Since $0 = d\omega^\alpha = -\sum \omega^\alpha_i \wedge \omega^j$, by CARTAN'S lemma we may write

$$\omega^\alpha_i = \sum h_{ij}\omega^j , \quad h^\alpha_{ij} = h^\alpha_{ji} . \tag{2.4}$$

From these formulas, we obtain

$$d\omega^i = -\sum \omega^i_j \wedge \omega^j , \quad \omega^i_j + \omega^j_i = 0 \tag{2.5}$$

$$d\omega^i_j = -\sum \omega^i_k \wedge \omega^k_j + \Omega^i_j , \quad \Omega^i_j = \frac{1}{2}\sum R^i_{jk\ell}\, \omega^k \wedge \omega^\ell \tag{2.6}$$

$$R^i_{jk\ell} = K^i_{jk\ell} + \sum (h^\alpha_{ik}h^\alpha_{j\ell} - h^\alpha_{i\ell}h^\alpha_{jk}) \tag{2.7}$$

$$d\omega^\alpha_\beta = -\sum \omega^\alpha_\gamma \wedge \omega^\gamma_\beta + \Omega^\alpha_\beta , \quad \Omega^\alpha_\beta = \frac{1}{2}\sum R^\alpha_{\beta k\ell}\, \omega^k \wedge \omega^\ell \tag{2.8}$$

$$R^\alpha_{\beta k\ell} = K^\alpha_{\beta k\ell} + \sum (h^\alpha_{ik}h^\beta_{i\ell} - h^\alpha_{i\ell}h^\beta_{jk}) \tag{2.9}$$

The riemannian connection of M^n is defined by (ω^i_j). The form (ω^α_β)

defines a connection in the normal bundle of M^n. We call be $B = \Sigma h^\alpha_{ij}\omega^i\omega^j e_\alpha$ the *second fundamental form* of the immersed manifold M^n. We shall denote the second fundamental form by its components h^α_{ij}. We denote by $H = \Sigma_\alpha(\Sigma_i h^\alpha_{ii})e_\alpha$ the *mean curvature vector*.

If $\tilde{\nabla}$ is the riemannian connection of $\tilde{M}^{n+p}$, X an element of T_qM^n and V is a normal vector field along M^n we have

$$\tilde{\nabla}_X V = A_V(X) + \nabla^\perp_X V$$

where $A_V(X)$ and $\nabla^\perp_X V$ are respectively the tangent and normal components of $\tilde{\nabla}_X V$. We will say that H is parallel if $\nabla^\perp_X H = 0$ for all X of T_qM^n and all q of M^n.

We apply the exterior differential d on (2.4) and define h^α_{ijk} by

$$\Sigma h^\alpha_{ijk}\omega^k = dh^\alpha_{ij} - \Sigma h^\alpha_{i\ell}\omega^\ell_j - \Sigma h^\alpha_{\ell j}\omega^\ell_i + \Sigma h^\beta_{ij}\omega^\alpha_\beta \,. \tag{2.10}$$

Then

$$\Sigma(h^\alpha_{ijk} + \frac{1}{2} K^\alpha_{ijk})\omega^j \wedge \omega^k = 0 \tag{2.11}$$

$$h^\alpha_{ijk} - h^\alpha_{ikj} = K^\alpha_{ikj} = -K^\alpha_{ijk} \tag{2.12}$$

Similarly, we aplly d on (2.10) and define $h^\alpha_{ijk\ell}$ by

$$\Sigma h^\alpha_{ijk}\omega^\ell = dh^\alpha_{ijk} - \Sigma h^\alpha_{\ell jk}\,\omega^\ell_i - \Sigma h^\alpha_{i\ell k}\omega^\ell_j - \Sigma h^\alpha_{ij\ell}\omega^\ell_k + \Sigma h^\beta_{ijk}\,\omega^\alpha_\beta \tag{2.13}$$

Then

$$\Sigma(h^{\alpha}_{ijk\ell} - \frac{1}{2}\Sigma h^{\alpha}_{im}R^{m}_{jk\ell} - \frac{1}{2}\Sigma h^{\alpha}_{mj}R^{m}_{jk\ell} + \frac{1}{2}\Sigma h^{\beta}_{ij}R^{\alpha}_{\beta k\ell})\omega^{k}\wedge\omega^{\ell} = 0 \tag{2.14}$$

$$h^{\alpha}_{ijk\ell} - h^{\alpha}_{ij\ell k} = \Sigma h^{\alpha}_{im}R^{m}_{jk\ell} + \Sigma h^{\alpha}_{mj}R^{m}_{ik\ell} - \Sigma h^{\beta}_{ij}R^{\alpha}_{\beta k\ell} \ . \tag{2.15}$$

Since (ω^{i}_{j}) defines a connection in the tangent bundle $T=T(M^{m})$ [hence a connection in the cotangent bundle $T^{*} = T^{*}(M^{n})$ also] and $(\omega^{\alpha}_{\beta})$ defines a connection in the normal bundle $T^{\perp} = T^{\perp}(M^{n})$, we have covariant differentiation, which maps a section of $T^{\perp}\otimes T^{*}\otimes \ldots \otimes T^{*}$, ($T^{*}$: k times) to a section of $T^{\perp}\otimes T^{*}\otimes \ldots \otimes T^{*}\otimes T^{*}$, ($T^{*}$. k + 1 times). The second fundamental form h^{α}_{ij} is a section of the vector bundle $T^{\perp}\otimes T^{*}\otimes T^{*}$, and h^{α}_{ijk} is the covariant derivative of h^{α}_{ij}. Similarly $h^{\alpha}_{ijk\ell}$ is the covariant derivative of h^{α}_{ijk}.

Lemma 1. *Let* M^{n} *be a riemannian manifold isometrically immersed in* $\tilde{M}^{n+p}$ *with parallel mean curvature vector. Then*

$$\Sigma_{i}\, h^{\alpha}_{iij} = 0 \qquad \text{(for all } \alpha) \tag{2.16}$$

$$\Sigma_{i}\, h^{\alpha}_{iijk} = 0 \tag{2.17}$$

Proof. Let e_{α} be an orthonormal frame of the normal bundle such that $H = \|H\|\, e_{\alpha}$. Hence from the definition of H, we obtain

$$\Sigma_{i}\, h^{\beta}_{ii} = 0 \quad \text{if} \quad \beta \neq \alpha \tag{2.18}$$

$$\Sigma_{i}\, h^{\alpha}_{ii} = h \quad \text{if} \quad \beta = \alpha \tag{2.19}$$

where $h = \|H\|$.

From (2.10) and (2.19) we obtain

$$\Sigma_{i,k} h^{\alpha}_{iik}\omega^{k} = dh - 2\,\Sigma_{i,\ell} h^{\alpha}_{i\ell}\omega^{\ell}_{i} = dh \tag{2.20}$$

Since h is constant then by (2.20) we have

$$\Sigma_i h^{\alpha}_{iik} = 0 \quad \text{if} \quad \alpha = \beta \tag{2.21}$$

From (2.10) and (2.18) we obtain

$$\Sigma_{i,k} h^{\beta}_{iik}\omega^{k} = h\omega^{\beta}_{\alpha} \quad \text{if} \quad \beta \neq \alpha \tag{2.22}$$

The vector field e_α is parallel, i.e., $\omega^{\beta}_{\alpha} = 0$. Then by (2.22) we have

$$\Sigma_i h^{\beta}_{iik} = 0 \quad \text{if} \quad \beta \neq \alpha \tag{2.23}$$

From (2.21) and (2.23) we obtain (2.16). Now (2.17) follows from (2.13) and (2.16).

The Laplacian Δh^{α}_{ij} of the second fundamental form h^{α}_{ij} is defined by

$$\Delta h_{ij} = \Sigma_k h_{ijkk} . \tag{2.24}$$

For each α, let H_α denote the symmetric matrix (h^{α}_{ij}), and set

$$S_{\alpha\beta} = \Sigma_{i,j} h^{\alpha}_{ij} h^{\beta}_{ij} \tag{2.25}$$

Then the $(p \times p)$-matrix $(S_{\alpha\beta})$ is also symmetric and can be assumed to be diagonal for a suitable choise of $e_{n+1},\ldots,e_{n+p}$. We set

$$S_\alpha = S_{\alpha\alpha} \quad , \tag{2.26}$$

and denote the square of the length of the second fundamental form by S, i.e.,

$$S = \Sigma h^\alpha_{ij} h^\alpha_{ij} = \Sigma_\alpha S_\alpha \ . \tag{2.27}$$

In general, for a matrix $A(a_{ij})$ we denote by $N(A)$ the square of the norm of A, i.e.,

$$N(A) = \mathrm{trace}\ A\ {}^tA = \Sigma(a_{ij})^2 \ .$$

Clearly, $N(A) = N(T^{-1}AT)$ for any orthogonal matrix T.

Now if the ambient space $\tilde{M}^{n+p}(c)$ is of constant sectional curvature c, then

$$K^A_{BCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) \tag{2.28}$$

where δ_{AC} are the Kronecker delta

Proposition 1. *Let* M^n *be a riemannian manifold isometrically immersed in* $\tilde{M}^{n+p}(c)$ *with parallel mean curvature vector. Then*

$$\Sigma_{\alpha,i,j} h^\alpha_{ij}\Delta h^\alpha_{ij} = -\Sigma_{\alpha,\beta}\ N(H_\alpha H_\beta - H_\beta H_\alpha) - \Sigma_\alpha S^2 + ncS - c\,\|H\|^2 + \sigma \tag{2.29}$$

where $\sigma = \Sigma_{\alpha,i}\ \langle B[H^2_\alpha(e_i),\ e_i],\ H\rangle$.

Notice that σ *does not depend of the choice of the frame.*

Proof. The proposition follows from equation 3.8 of [1], using (2.17), the definition of H and the equality $\Sigma_{\alpha,i}\langle B[H^2(e_i),e_i],H\rangle = \Sigma_{\alpha,\beta}(\operatorname{tr} H_\beta)\operatorname{tr}(H_\alpha H_\beta H_\alpha)$.

§3. INTEGRAL FORMULAS

We need the following algebraic lemma

Lemma 2. (Chern, Do Carmo and Kobayashi [4]). *Let* A *and* B *be symmetric* $(n\times n)$*-matrices. Then*

$$N(AB - BA) \leq 2N(A)N(B),$$

and the equality holds for non-zero matrices A *and* B *if and only if* A *and* B *can be transformed simultaneously by an orthogonal matrix into scalar multiples of* $\tilde{A}$ *and* $\tilde{B}$ *respectively, where*

$$\tilde{A} = \left(\begin{array}{cc|c} 0 & 1 & \\ & & 0 \\ 1 & 0 & \\ \hline & 0 & 0 \end{array}\right), \quad \tilde{B} = \left(\begin{array}{cc|c} 1 & 0 & \\ & & 0 \\ 0 & -1 & \\ \hline & 0 & 0 \end{array}\right) \tag{3.1}$$

Moreover, if A_1, A_2 *and* A_3 *are* $(n\times n)$*-symmetric matrices and if*

$$N(A_\alpha A_\beta - A_\beta A_\alpha) = 2N(A_\alpha)N(A_\beta) \quad 1\leq\alpha, \beta\leq 3 \quad ,$$

then at least one of the matrices A_α *must be zero.*

Applying Lemma 2 to (2.29), we obtain

$$-\Sigma h^{\alpha}_{ij}\Delta h^{\alpha}_{ij} \leq 2\Sigma_{\alpha\neq\beta}\, N(H_\alpha)N(H_\beta) + \Sigma_\alpha S^2_\alpha - ncS + c\,\|H\|^2 - \sigma$$

$$= 2\Sigma_{\alpha\neq\beta} S_\alpha S_\beta + \Sigma_\alpha S^2_\alpha - ncS + c\,\|H\|^2 - \sigma$$

$$= (\Sigma_\alpha S_\alpha)^2 + 2\Sigma_{\alpha<\beta} S_\alpha S_\beta - ncS + \sigma \qquad (3.2)$$

$$= (p\sigma_1)^2 + p(p-1)\sigma_2 - ncS + c\,\|H\|^2 - \sigma$$

where

$$p\sigma_1 = \Sigma_\alpha S_\alpha = S\ ,\quad \frac{p(p-1)}{2}\,\sigma_2 = \Sigma_{\alpha<\beta}\, S_\alpha\, S_\beta \qquad (3.3)$$

It can be easily seen that

$$p^2(p-1)(\sigma_1^2 - \sigma_2) = \Sigma_{\alpha<\beta}(S_\alpha - S_\beta)^2 \geq 0 \qquad (3.4)$$

and therefore

$$-\Sigma h^{\alpha}_{ij}\Delta h^{\alpha}_{ij} \leq p^2\sigma_1^2 + p(p-1)\sigma_2 - ncS + c\|H\|^2 - \sigma$$

$$= p(2p-1)\sigma_1^2 - p(p-1)(\sigma_1^2 - \sigma_2) - ncS + c\|H\|^2 - \sigma$$

$$\leq p(2p-1)\sigma_1^2 - ncS + c\|H\|^2 - \sigma \qquad (3.5)$$

$$= (2 - \frac{1}{p})S^2 - ncS + c\|H\|^2 - \sigma$$

Theorem 1. *Let* M^n *be a compact oriented riemannian manifold isometrically immersed in* $\tilde{M}^{n+p}(c)$ *with parallel mean curvature* *vector. Then*

$$\int_{M^n} \{(2 - \frac{1}{p})S^2 - ncS + c\|H\|^2 - \sigma\}\, dv \geq 0 \tag{3.6}$$

where dv *denotes the volume element of* M^n.

Proof. This follows from (3.5) and the following lemma

Lemma 3. *If* M^n *is an oriented compact riemannian manifold in a riemannian manifold* $\tilde{M}^{n+p}$, *then*

$$\int_{M^n} (\Sigma h^{\alpha}_{ij} \Delta h^{\alpha}_{ij})\, dv = - \int_{M^n} \Sigma (h^{\alpha}_{ijk})^2\, dv \leq 0$$

Proof of Lemma 3. We have

$$\frac{1}{2}\Delta(\Sigma(h^{\alpha}_{ij})^2) = \Sigma(h^{\alpha}_{ijk})^2 + \Sigma h^{\alpha}_{ij}\Delta h^{\alpha}_{ij} \tag{3.7}$$

Integrating (3.7) over M^n and applying Green's theorem to the left hand side, we see that the integral of the left hand side and hence that of the right hand side also vanish.

Corollary 1. *Let* M^n *be a compact riemannian manifold isometrically immersed in* $\tilde{M}^{n+p}(c)$ *with parallel mean curvature vector. If*

$$(2 - \frac{1}{p})S^2 + c\|H\|^2 \leq ncS + \sigma$$

everywhere on M^n, *then* $(2 - \frac{1}{p})S^2 + c\|H\|^2 = ncS + \sigma$ *everywhere on* M^n.

Assume that $S = \Sigma(h^{\alpha}_{ij})^2$ is a constant, (3.6) implies

$$0 = \Sigma(h^{\alpha}_{ijk})^2 + h^{\alpha}_{ij}\Delta h^{\alpha}_{ij} ,$$

whether M^n is compact or not.

This combined with (3.4) yields

$$(2 - \frac{1}{p})S^2 - ncS + c\|H\|^2 - \sigma \geq \Sigma(h^{\alpha}_{ijk})^2$$

We may therefore conclude that if

$$(2 - \frac{1}{p})S^2 + c\|H\|^2 = ncS + \sigma$$

everywhere on M^n we have $h^{\alpha}_{ijk} = 0$. Then $\Delta h^{\alpha}_{ij} = 0$, and the terms at both ends of (3.5) vanish. It follows that all inequalities in (3.2), (3.4) and (3.5) are actually equalities. In deriving (3.2) from (2.29), we made use of the inequality $N(H_\alpha H_\beta - H_\beta H_\alpha) \leq 2N(H_\alpha)N(H_\beta)$. Hence,

$$N(H_\alpha H_\beta - H_\beta H_\alpha) = 2N(H_\alpha)\ N(H_\beta) \tag{3.8}$$

From (3.4) we obtain

$$p^2(p-1)(\sigma_1^2 - \sigma_2) = 0 . \tag{3.9}$$

From (3.8) and Lemma 2, we conclude that at most of the matrices H_α are non-zero, in which case they can be assumed to be scalar multiples of $\tilde{A}$ and $\tilde{B}$ in Lemma 2.

Let us fix some further notation for the case $p = 1$. Set

$h_{ij} = h_{ij}^{n+1}$. We choose our frame field in such a way that $h_{ij} = 0$ for $i \neq j$ and we set $h_i = h_{ii}$.

Lemma 4. *Let* M^n *be a hypersurface of the riemannian manifold* $\tilde{M}^{n+1}(c)$ *with parallel mean curvature vector* $H \neq 0$, *such that is* S *is constant an the condition*

$$S + c\|H\|^2 = ncS + \sigma$$

is verified. Then after a suitable renumbering of the base elements $e_1, \ldots, e_n$ *we have that either:*

(i) $h_1 = \ldots = h_n =$ *constant*

or

(ii) $h_1 = \ldots = h_k = \lambda =$ *constant*

$h_{k+1} = \ldots = h_n = \mu =$ *constant*, $(1 < k < n)$

$\lambda\mu + c = 0 \qquad \omega_j^i = 0$ for $1 \leq i \leq k$, $k+1 \leq j \leq n$.

Proof. Since $h_{ijk} = 0$, setting $i = j$ in (2.10) we obtain

$$0 = dh_i - 2\Sigma h_{i\ell}\omega_i^\ell = dh_i$$

which shows that h_i is a constant. Since $h_{ijk} = 0$ and $dh_{ij} = 0$, (2.10) implies

$$0 = \Sigma h_{i\ell}\omega_j^\ell + \Sigma h_{\ell j}\omega_i^\ell = (h_i - h_j)\omega_j^i$$

which shows that $\omega_j^i = 0$ whenever $h_i \neq h_j$. Thus, if $h_i \neq h_j$, then, by (2.2)

$$0 = d\omega_j^i = -\omega_k^i \wedge \omega_j^k - \omega_{n+1}^i \wedge \omega_j^{n+1} + c\,\omega^i \wedge \omega^j$$

the first sum of the equation above is zero, because $\omega_k^i = 0$ and $\omega_j^k \neq 0$ would imply $h_i = h_k = h_j$, contradicting the hypothesis . Hence,

$$0 = -\omega_{n+1}^i \wedge \omega_j^{n+1} + c\,\omega^i \wedge \omega^j$$

$$= \Sigma h_{ik} h_{j\ell} \omega^k \wedge \omega^\ell + c\,\omega^i \wedge \omega^j$$

$$= (h_i h_j + c)\omega^i \wedge \omega^j$$

which shows that if $h_i \neq h_j$, then $h_i h_j = -c$. Set $h_1 = \lambda$. Then we have either $h_1 = \ldots = h_n = \lambda$, which proves part (i) of Lemma 4, or, by renumbering the indices of $e_1, \ldots, e_n$ if necessary, $h_1 = \ldots = h_k = \lambda$ and $h_j \neq \lambda$ for $j > k$. In the latter case , $h_i h_j = -c$ for $j = k+1, \ldots, n$, and therefore $h_{k+1} = \ldots = h_n = \frac{c}{\lambda} = \mu$, proving part (ii)

Lemma 5. *Let* M^n *be a submanifold of the riemannian manifold* $\tilde{M}^{n+p}(c)$, $p \geq 2$, *with parallel mean curvature vector* $H \neq 0$ *such that* S *is constant and the condition*

$$(2 - \frac{1}{p})S^2 + c\|H\|^2 = ncS + \sigma \text{ .}$$

Then $p = 1$.

Proof. Since $p \geq 2$, Eq. (3.9) implies

$$\sigma_1^2 = \sigma_2 . \tag{3.10}$$

We know that at most two of H_α , $\alpha = n+1,\ldots,n+p$, are different from zero. Assume that only one of them, say H_α ,is different from zero. Then we have $\sigma_1 = \frac{1}{p} S_\alpha$ and $\sigma_2 = 0$, contradicting (3.10). Therefore we can assume that

$$H_{n+1} = \lambda\tilde{A} , \quad H_{n+2} = \mu\tilde{B} , \quad \lambda,\mu = 0$$

$$H_\alpha = 0 \quad \text{for} \quad \alpha \geq n+3 \tag{3.11}$$

where $\tilde{A}$ and $\tilde{B}$ are defined by (3.1). Moreover, we can verify that λ and μ are constants (see [4]). From (3.11). $\operatorname{tr} H_{n+1} = \operatorname{tr} H_{n+2} = 0$, then $H = 0$. This is a contradiction and therefore we obtain the lemma.

§4. SUBMANIFOLDS OF EUCLIDEAN SPACES WITH PARALLEL MEAN CURVATURE VECTOR.

The purpose of the present section is to determine all isometric immersions M^n of $\mathbb{R}^{n+p}$, with non-zero parallel mean curvature vector, and such that S is constant and satisfies the condition

$$(2 - \frac{1}{p}) S = \sigma \tag{4.1}$$

The first main results is the following

Theorem 2. *Let* M^n *be a complete riemannian manifold isometrically immersed in* $\mathbb{R}^{n+p}$ *with parallel mean curvature vector* $H \neq 0$, *such that* S *is constant and the condition* (4.1) *is satisfied. Then* $p = 1$ *and* M^n *is either a sphere of radius* r, $S^n(r)$, *or product* $S^k(r) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$. *Except for the case* $k=1$, *the immersion is an imbedding.*

Assuming $p = 1$ a theorem of the above type was proved by Nomizu and Smyth [10], under the condition of non negativity of the sectional curvature of M^n in place of our condition (4.1).

Lemma 6. *Let* M^n *be a hipersurface of* $\mathbb{R}^{n+1}$, *with parallel mean curvature vector* $H \neq 0$, *such that* S *is constant and the condition* (4.1) *is verified. Then either* M^n *is part of a sphere* $S^n(r)$ *or is locally a riemannian product* $M_1 \times M_2$ *of spaces* M_1 *and* M_2 *of constant sectional curvatures with* $\dim M_1 = k \geq 1$ *and* $\dim M_2 = n-k \geq 1$. *In the latter case, with respect an adapted frame field, the matrix of the connection forms* ω_B^A *of* $\mathbb{R}^{n+1}$ *restricted a* M^n, *is given by*

$$\left(\begin{array}{ccc|ccc|c}
\omega_1^1 & \cdots & \omega_k^1 & & & & -\lambda\omega^1 \\
 & \cdots & & & 0 & & \vdots \\
\omega_1^k & \cdots & \omega_k^k & & & & -\lambda\omega^k \\
\hline
 & & & \omega_{k+1}^{k+1} & \cdots & \omega_n^{k+1} & 0 \\
 & 0 & & & \cdots & & \vdots \\
 & & & \omega_{k+1}^n & \cdots & \omega_n^n & 0 \\
\hline
\lambda\omega^1 & \cdots & \lambda\omega^k & 0 & \cdots & 0 & 0
\end{array}\right) \tag{4.2}$$

where $\lambda = h/k$ and $h = \|H\|$

Proof of Lemma 6.

From (i) of lemma 4 we obtain that M^n is part of a sphere $S^n(r)$. Suppose now that condition (ii) of lemma 4 is valid and $A = H_{n+1}$, then we can define two distributions

$$T^1(q) = \{x \in T_q M^n \ ; \ Ax = \lambda x\}$$

and

$$T^2(q) = \{x \in T_q M^n \ ; \ Ax = 0\}$$

of dimensions k and $n-k$, respectively. Knowing that λ is a constant, it is easy to see that both distributions are differentiable, involutive and totally geodesic on M^n. Then every point of M^n has a neighborhood U which is a riemannian product $M_1 \times M_2$. From (2.7) we see that the curvatures of M_1 and M_2 are given by

$$R^i_{jm\ell} = \lambda^2(\delta_{im}\delta_{j\ell} - \delta_{i\ell}\delta_{jm}), \quad 1 \leq i,j,m,\ell \leq k$$

$$R^i_{jm\ell} = 0 \qquad\qquad , \ k+1 \leq i,j,m,\ell \leq n$$

If $k \geq 2$ (resp. $n-k \geq 2$), then M_1 (resp. M_2) is a space of constant curvature λ^2 (resp. 0). We know that H is parallel, then

$$h = \|H\| = \Sigma_i h_i$$

is constant. Since $h = k\lambda$ we obtain $\lambda = h/k$. Now the lemma follows from part (ii) of Lemma 4.

Proof of Theorem 2. We first assume that M^n is simply connected. Let $\phi : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion. If M^n is complete and $p=1$, by lemma 6 $\phi(M^n)$ is a sphere $S^n(r)$. Since M^n and $\phi(M^n)$ are simply connected, we conclude that ϕ is an imbedding (see [8] Theor. 4.6).

Now, we consider the hipersurface $S^k(r) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$, and prove that the matrix of the connection forms $\mathbb{R}^{n+1}$, restricted to $S^k(r) \times \mathbb{R}^{n-k}$ is given by (4.2). Now, we may use the theorem of local uniqueness of the form (ω^A) and (ω^A_B) and obtain that $M_1 \times M_2$ is an open set of the riemannian product $S^k(r) \times R^{n-k}$ (see [5] Theor. 1'). If $k \geq 2$, then $\phi(M^n)$ is simply connected and we can conclude that ϕ is an imbedding. If $k = 1$, then M^n may be $\mathbb{R} \times \mathbb{R}^{n-1}$, which is immersed onto $S^1(r) \times \mathbb{R}^{n-1}$ in $\mathbb{R}^{n+1}$.

In the general case, let $\tilde{M}^n$ be the universal covering manifold on M^n with projection $\pi : \tilde{M}^n \to M^n$. If M^n has the covering metric, then $\tilde{M}^n$ and $\tilde{\phi} = \phi \circ \pi$ satisfy the same assumptions as M^n and ϕ. Thus $\tilde{\phi}(\tilde{M}^n) = \phi(M^n)$ is of the form $S^k(r) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$. If $k \neq 1$, then $\tilde{\phi}$ is an imbedding and so is ϕ.

If $p \geq 2$ then by lemma 5 have that $p = 1$.

Corollary 2. *Let* M^n *be a compact hipersurface of* $\mathbb{R}^{n+1}$, *with parallel mean curvature vector* $H \neq 0$, *such that condition* (4.1) *is verified, then* M^n *is a sphere and the immersion is an imbedding.*

§5. SUBMANIFOLDS OF HYPERBOLIC SPACE WITH PARALLEL MEAN CURVATURE VECTOR.

Before determining all isometric immersions M^n of $\tilde{M}^{n+p}(c)$ c = -1, with non-zero parallel mean curvature vector and such tha S is constant and satisfies the condition

$$(2 - \frac{1}{p})S^2 + c\|H\|^2 = ncS + \sigma , \qquad (5.1$$

let us look at some examples.

Consider, in $\mathbb{R}^{n+1}$ the non-degenerate quadratic form

$$(x,y) = x_1y_1 + \ldots + x_ny_n - x_{n+1}y_{n+1}$$

and let

$$H^n(\tilde{c}) = \{(x_1,\ldots,x_{n+1}) \in R^{n+1}/\Sigma_i x_i^2 - x_{n+1}^2 = -\tilde{c}^2,\ x_{n+1} > 0\}$$

The above quadratic form induces a complete riemannian metri of constant sectional curvature $-\tilde{c}^2$, on $H^n(\tilde{c})$. Since $H^n(\tilde{c})$ is sim- ply connected, it can be identified with the hyperbolic space o curvature $-\tilde{c}^2$. The "flat" hypersurface.

$$F^n = \{(x_1,\ldots,x_{n+2}) \in H^{n+1}(1) / x_{n+2} = x_n + 1\} \text{ in } H^{n+1}(1)$$

is called a "*horosphere*". The second fundamental form of F^n ha n eigenvalues equal to 1 and so $\|H\| = S = n$, $S^2 + ncS = \|H\|^2 + \sigma = 2n$

For $0 \leq k \leq n$, $r > 0$, we define the hypersurface $M(r,k$ of $H^{n+1}(1)$ by

$$M(r,k) = S^k(r) \times H^{n-k}((1+r^2)^{1/2}) = \{(x_1,\dots,x_{n+2}) \in H^{n+1}(1)/\Sigma_i x_i^2 = r^2\} ,$$

If $k = n$ $M(r,k)$ is called a *"geodesic sphere"*

If $k = 0$ $M(r,k)$ is called an *"equidistant hipersurface"*

If $k \neq 0,n$ $M(r,k)$ is called a *"cilindric-type hypersurface"*

The second fundamental form of $M(r,k)$ has eigenvalues $(1+r^2)^{1/2}/r$ and $r/(1+r^2)^{1/2}$ with multiplicities k and $n-k$ respectively . Moreover

$$\|H\| = k(1+r^2)^{1/2}/r + (n-k)r/(1+r^2)^{1/2}$$

$$S = k(1+r^2)/r^2 + (n-k)r^2/(1+r^2)$$

$$S^2 + nS = \|H\|^2 + \sigma .$$

We now prove.

Theorem 3. *Let* M^n *be a complete riemannian manifold isometrically immersed in* $H^{n+p}(1)$ *with parallel mean curvature vector* $H \neq 0$, *such that* S *is constant and the condition* (5.1) *is satisfied. Then* $p = 1$ *and* M^n *is either a horsphere or a manifold of type* $M(r,k)$. *Except for the case* $k = 1$ *the immersion is an imbedding.*

We need, the following lemma.

Lemma 7. *Let* M^n *be a hipersurface of* $H^{n+1}(1)$, *with parallel mean curvature vector* $H \neq 0$, *such that* S *is constant and the condition* (5.1) *is verified. Then either* M^n *is part of a horsphere or part of a geodesic sphere or part of an equidistant hipersurface* *or it is locally a riemannian product* $M_1 \times M_2$ *of spaces* M_1 *and* M_2 *of constant sectional curvatures with* dim $M_1 = k \geq 1$ *and* dim $M_2 = n-k \geq 1$. *In the latter case with respect an adapted frame field, the matrix of the connection forms* ω^A_B *of* $H^{n+1}(1)$, *restricted to* M^n *is given by*

$$
\left(\begin{array}{ccc|ccc|c}
\omega^1_1 & \cdots & \omega^1_k & & & & -\lambda\omega^1 \\
 & \cdots & & & 0 & & \cdot \\
\omega^k_1 & \cdots & \omega^k_k & & & & -\lambda\omega^k \\
\hline
 & & & \omega^{k+1}_{k+1} & \cdots & \omega^{k+1}_n & -\mu\omega^{k+1} \\
 & 0 & & & \cdots & & \cdot \\
 & & & \omega^{k+1}_n & \cdots & \omega^n_n & -\mu\omega^n \\
\hline
\lambda\omega^1 & \cdots & \lambda\omega^k & \mu\omega^{k+1} & \cdots & \mu\omega^n & 0
\end{array}\right)
$$

where

$$\lambda = (h + (h^2 - 4k(n-k))^{1/2})/2k,$$

$$\mu = (h - (h^2 - 4k(n-k)^{1/2})/2(n-k) \quad \text{and} \quad h = \|H\|$$

Proof of Lemma 7.

From (i) of lemma 4 and by [13], Theor. 29, we obtain that either M^n is part of a horsphere ($h_1 = \ldots = h_n = 1$) or part of a geodesic sphere ($h_1 = \ldots = h_n = (1+r^2)^{1/2}/r$) or part of a equi-distant hypersurface ($h_1 = \ldots = h_n = r/(1+r^2)^{1/2}$). Let $A = H_{n+1}$, by (ii) of Lemma 4, we can define two distributions

$$T^1(q) = \{x \in T_qM^n \; ; \; Ax = \lambda x\}$$

and

$$T^2(q) = \{x \in T_qM^n \; ; \; Ax = \mu x\}$$

of dimensions k and $n-k$, respectively. Knowing that λ and μ are constants, it is easy to see that both distributions are differentiable, involutive and totally geodesic on M^n. Then every point of M^n has a neighborhood U which is a riemmanian product $M_1 \times M_2$. From (2.7) we see that the curvatures of M_1 and M_2 are given by

$$R^i_{jm\ell} = (\lambda^2 - 1)(\delta_{im}\delta_{j\ell} - \delta_{i\ell}\delta_{jm}), \quad 1 \leq i,j,m,\ell \leq k$$

$$R^i_{jm\ell} = (\mu^2 - 1)(\delta_{im}\delta_{j\ell} - \delta_{i\ell}\delta_{jm}), \; k+1 \leq i,j,m,\ell \leq n$$

If $k \geq 2$ (resp. $n-k \geq 2$), then M_1 (resp. M_2) is a sphere of constant curvature $\lambda^2 - 1$ (resp. $\mu^2 - 1$). We know that H is parallel, then

$$h = ||H|| = \Sigma_i h_i$$

is constant and

$$h = k\lambda + (n-k)\mu \quad .$$

This relation together with $\lambda\mu = 1$ imply

$$\lambda = h + (h^2 - 4k(n-k))^{1/2}/2k \; , \; \mu = h - (h^2 - 4k(n-k))^{1/2}/2(n-k) \quad (5.3)$$

or

$$\lambda = h - (h^2 - 4k(n-k))^{1/2}/2k \ , \quad \mu = h + (h^2 - 4k(n-k))^{1/2}/(2(n-k)$$

Replacing e_{n+1} by $-e_{n+1}$ if necessary, we may assume (5.3). Now the lemma follows from part (ii) of Lemma 4.

Proof of Theorem 3. We first assume that M^n is simply connected. Let $\phi: M^n \to H^{n+p}(1)$ be an isometric immersion. If M^n is complete and $p = 1$, by Lemma 7 $\phi(M^n)$ is either a horspehre or a geodesic sphere or an equidistant hypersurface. Since M^n and $\phi(M^n)$ are simply connected, we conclude that ϕ is an imbedding (see [8] Theor. 4.6)

We consider now the "cylindric-type hypersurface" $M(r,k) = S^k(r) \times H^{n-k}((1+r^2)^{1/2})$, $1 \leq k \leq n-1$, and show that the connection form of $H^{n+1}(1)$, restricted to $M(r,k)$ is given by (5.2).

We have that the second fundamental form of $M(r,k)$ has an eigenvalue λ of multiplicity k and an eigenvalue μ of multiplicity n-k, and

$$\lambda\mu = 1 \tag{5.4}$$

by the argument in the proof of Lemma 4. On the other hand, $S^k(r)$ (resp. $H^{n-k}(1+r^2)^{1/2}$) has constant sectional curvature $\lambda^2 - 1$ (resp. $\mu^2 - 1$) which is equal to $1/r^2$ (resp. $-1/(1+r^2)$) so that

$$\lambda^2 - 1 = 1/r^2 \ , \quad \mu^2 - 1 = -1/(1+r^2). \tag{5.5}$$

Thus by (5.4) and (5.5), without loss of generality, we have

$$\lambda = (1+r^2)^{1/2}/r \quad , \quad \mu = r/(1+r^2)^{1/2} \tag{5.6}$$

from which it follows immediately that

$$k(1+r^2)^{1/2}/r) + (n-k)(r/(1+r^2)^{1/2} = h \tag{5.7}$$

where $h = \|H\|$ and H is the mean curvature vector of $M(r,k)$. Solving (5.7) we therefore obtain, without loss of generality,

$$r^2 = (h^2 - 2nk - h(h^2 - 4n(n-k)))/2(n^2-h^2) \tag{5.8}$$

For simplicity, we shall denote by r_o the r given by (5.8). Hence $M(r_o,k)$ is a hypersurface in $H^{n+1}(1)$.

Now let $f_o, f_1, \ldots, f_k$ be an orthonormal frame field for $\mathbb{R}^{k+1}$ such that f_o is normal to $S^k(r_o)$, and $\phi^o, \phi^1, \ldots, \phi^k$ the dual frame field. For $H^{n-k}((1+r_o^2)^{1/2})$ in $\mathbb{R}^{n-k+1}$, we choose an orthonormal frame $f_{k+1}, \ldots, f_{n+1}$, (for instance see [5] P. 123) such that f_{n+1} is normal to $H^{n-k}((1+r_o^2)^{1/2})$, i.e.,

$$\langle f_i, f_j \rangle = \delta_{ij}, \ \langle f_{n+1}, f_i \rangle = 0; \ i,j=k+1,\ldots,n; \ \langle f_{n+1}, f_{n+1} \rangle = -1 \tag{5.9}$$

and $\phi^{k+1}, \ldots, \phi^{n+1}$ be the dual frame field. Let (ϕ^A_B) $A,B=0,1,\ldots,n+1$ be the connection form for $\mathbb{R}^{n+2}$ with respect to the dual frame field (ϕ^A) $A = 0,1,\ldots,n+1$. Then these forms ϕ^A_B, restricted to $M(r_o,k)$ satisfy

$$\phi^o = \phi^{n+1} = 0$$

$$\phi^o_i = -\phi^i_o = (1/r_o)\phi^i \ , \ i = 1,2,\ldots,k$$

$$\phi^j_{n+1} = \phi^{n+1}_j = (1/(1+r_o^2)^{1/2})\phi^j \ , \ j = k+1,\ldots,n \tag{5.10}$$

$$\phi^A_B = -\phi^B_A = 0 \quad \text{for} \quad A = 0,1,\ldots,k \text{ and } B = k+1,\ldots,n+1$$

The image of the imbedding $M(r_o,k) \to \mathbb{R}^{n+2}$ lies in the hyperbolic space $H^{n+1}(1)$. Now we take a new frame field $e_o, e_1, \ldots, e_{n+1}$ for $\mathbb{R}^{n+2}$, with $e_i = f_i$, $i = 1,\ldots,n$, e_o normal to $H^{n+1}(1)$, and e_{n+1} normal to $M(r_o,k)$, so that

$$e_o = r_o f_o + (1+r_o^2)^{1/2} f_{n+1}$$

$$e_i = f_i \quad , \quad i = 1,\ldots,n \tag{5.11}$$

$$e_{n+1} = (1+r_o^2)^{1/2} f_o + r_o f_{n+1}$$

Let $\omega^o, \omega^1, \ldots, \omega^{n+1}$ be the dual frame field. Then

$$\omega^o = -r_o\phi^o + (1+r^2)^{1/2}\phi^{n+1}$$

$$\omega^i = \phi^i \quad , \quad i = 1,\ldots,n \tag{5.12}$$

$$\omega^{n+1} = (1+r_o^2)^{1/2}\phi^o - r_o\phi^{n+1}$$

The connection form $(\omega^A_B) A,B = 0,1,\ldots,n+1$ for $\mathbb{R}^{n+2}$ with respect the dual frame field (ω^A) is then given by

$$\begin{aligned}
&\omega_j^o = \omega_o^j = -r_o\phi_o^j + (1+r_o^2)^{1/2}\phi_{n+1}^j,\ j = 1,\dots,n \\
&\omega_{n+1}^o = \omega_o^{n+1} = \phi_{n+1}^o \\
&\omega_j^i = \phi_j^i \qquad ,\ i,j = 1,\dots,n \\
&\omega_i^{n+1} = -\omega_{n+1}^i = (1+r_o^2)^{1/2}\phi_i^o + r_o\phi_i^{n+1}\ ,\ i = 1,\dots,n.
\end{aligned} \tag{5.13}$$

Substitution of (5.10) in (5.12), (5.13) shows immediately that the connection form $(\omega_B^A)A,B = 1,\dots,n+1$ of $H^{n+1}(1)$, restricted to $M(r_o,k)$, coincides with the form in (5.2). Now, we may use the theorem of local uniqueness of the forms (ω^A) and (ω^B) and obtain that $M_1 \times M_2$ is an open set of the riemannian product $M(r_o,k) = S^k(r_o) \times H^{n-k}((1+r_o^2)^{1/2})$ (see [5] P. 128). If $k \geq 2$, then $\phi(M^n)$ is simply connected and we can conclude that ϕ is an imbedding. If $k = 1$, then M^n may be $\mathbb{R} \times H^{n-1}((1+r_o^2)^{1/2})$ which is immersed onto $S^1(r_o) \times H^{n-1}((1+r_o^2)^{1/2})$ in $\mathbb{R}^{n+1}$.

In the general case, let $\tilde{M}^n$ be the universal covering manifold on M^n with projection $\phi:\tilde{M}^n \to M^n$. With respect to the covering metric $\tilde{M}^n$ and $\tilde{\phi} = \phi \circ \pi$ satisfy the same assumptions as M^n and ϕ. Thus $\tilde{\phi}(M^n) = \phi(M^n)$ is of the form $S^k(r_o)\ H^{n-k}(1+r_o^2)^{1/2})$, $1 \leq k \leq n-1$. If $k \neq 1$, then $\tilde{\phi}$ is an imbedding and so is ϕ.

If $p \geq 2$ then by Lemma 5, we have that $p = 1$.

Corollary 3. *Let* M^n *be a compact hypersurface of* $H^{n+1}(1)$, *with parallel mean curvature vector* $H \neq 0$, *such that the condition* (5.1) *is verified, then* M^n *is a geodesic sphere and the immersion is an imbedding.*

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