THE LATTICE STRUCTURE OF SOME LUKASIEWICZ ALGEBRAS

By

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INTRODUCTION: Many-valued logics were introduced by J. Lukasiewicz in 1920. With the aim of developing an algebraic theory of the n-valued Lukasiewicz propositional calculi, G. Moisil introduced in 1940 the n-valued Lukasiewicz algebras. These algebras are distributive lattices on which are defined a De Morgan negation and $n-1$ modal operators fulfilling some axioms that are given in §1.

The three - and four-valued Lukasiewicz algebras are the algebraic counterpart of the three and four-valued Lukasiewicz propositional calculi respectively, but this is not the case if $n \geq 5$. The adequate algebraic notion corresponding to the cases $n \geq 5$ was introduced by R. Grigolia [9] following some ideas of C. C. Chang. But these structures are not directly based on lattices, and so their comparison with algebraic structures corresponding to other logical calculi (for instance, classical, Post, intuitionistic calculi) is not easy.

Three-valued Lukasiewicz algebras were characterized in [2] as Kleene algebras such that the subalgebra of its complemented elements is relativelly complete and separating, and in [3] it was shown that if the complemented elements of a bounded distributive lattice satisfy the above conditions, then it admits a Kleene algebra structure. Thus three-valued Lukasiewicz algebras were characterized as bounded distributive lattices such that their complemented elements satisfy certain specific properties. These results motivated the characterization of three-valued Lukasiewicz algebras as a subvariety of the variety of double Stone algebras in [11].

On the other hand, it is well known that if $n \geq 4$, then the De Morgan negation cannot be defined in terms of the complemented elements and the lattice operations of an n-valued Lukasiewicz algebra, and that if $n \geq 6$, then there are Kleene algebras admitting two non-isomorphic structures of n-valued Lukasiewicz algebra.

The aim of this paper is to characterize five-valued Lukasiewicz algebras as De Morgan algebras such that their complemented elements satisfy certain conditions, and to obtain corresponding characterizations of four-valued and three-valued Lukasiewicz algebras as particular cases. We hope that this will be a step toward a possible algebraic characterization of n-valued Lukasiewicz logics in terms of lattices, which will permit to compare it with other propositional calculis and to evaluate possible applications to the theory of circuits and programming.

In §1, after recalling the definitions and some properties of De Morgan, Kleene and Lukasiewicz algebras, we show that a bounded distributive lattice admits at most one structure of n-valued Lukasiewicz algebra for $n = 3$, 4 and 5 (and that it is not the case if $n \geq 6$). Our proof depends strongly on properties of the prime filters of the

Lukasiewicz algebras that were established in [4], and on the fact that the modal operators that appear in the definition of n-valued Lukasiewicz algebras for $n = 3$, 4 and 5 have an intrinsic definition in terms of the subjacent De Morgan algebra structure. The main properties of these modal operators are studied in §2, where we consider them as defined on a De Morgan algebra. We think that the results of this section are of an independent interest. Finally, in §3 we show that if such modal operators can be defined on a De Morgan algebra and if they satisfy a relation known as the Moisil determination principle, then they authomatically fulfill all the other requirements of the definition of a five-valued Lukasiewicz algebra. We also give the extra conditions necessary to characterize four-valued and three-valued Lukasiewicz algebras.

1. DE MORGAN, KLEENE AND LUKASIEWICZ ALGEBRAS:

We shall consider bounded distributive lattices as algebras $\langle L, \vee, \wedge, 0, 1\rangle$ of type (2,2,0,0), i.e., the zero 0 and unit 1 are considered as nullary operations defined on L, so by a sublattice we shall always understand a sublattice containing 0 and 1, and by a homomorphism a lattice homomorphism preserving 0 and 1. B(L) will denote the boolean sublattice of all complemented elements of L, and if $x \in B(L)$ the complement of x will be denoted by $-x$. If S is a sublattice of L and x, y are elements of L, we are going to denote (following G. Epstein and A. Horn [7] [8]) the largest (smallest) element z of S such that $x \wedge z \leq y$ $(x \vee z \geq y)$, in case it exists, by $x \xrightarrow{S} y$ ($x \xleftarrow{S} y$). We are going to write $x \to y$ ($x \leftarrow y$) instead of $x \xrightarrow{L} y$ ($x \xleftarrow{L} y$) and $x \Rightarrow y$ ($x \Leftarrow y$) instead of $x \xrightarrow{B(L)} y$ ($x \xleftarrow{B(L)} y$). Note that the pseudocomplement x^* of an element $x \in L$, if it exists, coincides with $x \to 0$ and analogously, the dual pseudocomplement x^+ is equal to $x \leftarrow 1$.

A De Morgan algebra $\langle A, \vee, \wedge, \sim, 0, 1 \rangle$, or simply A, is an algebra such that $\langle A, \vee, \wedge, 0, 1 \rangle$ is a distributive lattice and $\sim$ is a unary operation defined on A fulfilling the following conditions:

(M1) $\sim\sim x = x$ and (M2) $\sim(x \vee y) = \sim x \wedge \sim y$

A De Morgan algebra A such that the operation $\sim$ fulfills the condition:

(K) $x \wedge \sim x \leq y \vee \sim y$ for any x, y in A

is called a Kleene algebra.

A systematic treatment of De Morgan and Kleene algebras can be found in [1, Chapter XI] (see also [5]).

If S is a De Morgan subalgebra of A, it is plain that $x \xrightarrow{S} y$ exists if and only if $\sim x \xleftarrow{S} \sim y$ exists, and in this case $\sim(x \xrightarrow{S} y) = \sim x \xleftarrow{S} \sim y$. Analogously we have that $\sim(x \xleftarrow{S} y) = \sim x \xrightarrow{S} \sim y$. In particular we have that $x \to y = \sim(\sim x \leftarrow \sim y)$, $x^* = \sim(\sim x)^+$, $x^+ = \sim(\sim x)^*$ and since B(A) is always a De Morgan subalgebra of A [5, 2.1] we also have that $x \Rightarrow y = \sim(\sim x \Leftarrow \sim y)$, $x \Leftarrow y = \sim(\sim x \Rightarrow \sim y)$.

Let $K(A) = \{z \in B(A) \mid \sim z = -z\}$; $K(A)$ is both a De Morgan subalgebra of A and a Boolean subalgebra of $B(A)$ [5, Lemma 2.1]. If $z \in B(A)$ and $-z$ is comparable with $\sim z$, then $z \in K(A)$. For, suppose, for instance, that $-z \leq \sim z$. Then $1 = -z \vee z \leq \sim z \vee z$, and this also implies that $0 = \sim(\sim z \vee z) = \sim z \wedge z$. Therefore $\sim z = -z$. This result, together with the next lemma, provide an inmediate proof of the following result of A. Monteiro (see [2, 2.2]): In a Kleene algebra A, $K(A) = B(A)$.

1.1. LEMMA: In a Kleene algebra A, if $x \wedge y = 0$, then $y \leq \sim x$.

PROOF: If $x \wedge y = 0$, we have that,

$$y = y \wedge (\sim x \vee \sim y) = (y \wedge \sim x) \vee (y \wedge \sim y) \leq (y \wedge \sim x) \vee (x \vee \sim x) = x \vee \sim x$$

Therefore: $y = y \wedge (x \vee \sim x) = (y \wedge x) \vee (y \wedge \sim x) = y \wedge \sim x$.

This lemma also shows that if in a Kleene algebra A x^* (or x^+) exists, then we have that $x^* \leq \sim x \leq x^+$.

The notion of an n-valued Lukasiewicz algebra was introduced by G. Moisil [10] in the following way: An n-valued Lukasiewicz algebra (n an integer ≥ 2) is an algebra $(L, \vee, \wedge, \sim, \sigma_1^n, \ldots, \sigma_{n-1}^n, 0, 1)$ such that $\langle L, \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra and σ_i^n $(1 \leq i \leq n-1)$ are unary operations defined on L fulfilling the following conditions:

L1) $\sigma_i^n(x \vee y) = \sigma_i^n x \vee \sigma_i^n y$

L2) $\sigma_i^n x \vee \sim\sigma_i^n x = 1$

L3) $\sigma_i^n \sigma_j^n x = \sigma_j^n x$

L4) $\sigma_i^n(\sim x) = \sim\sigma_{n-i}^n x$

L5) $\sigma_1^n x \leq \sigma_2^n x \leq \ldots \leq \sigma_{n-1}^n x$

L6) If $\sigma_i^n x = \sigma_i^n y$ for $i = 1,2,\ldots,n-1$, then $x = y$.

Condition L6 is known as "the Moisil determination principle" and it was shown in [4] (see also [1, Chapter XI]) that it can be replaced by the conditions:

L7) $x \leq \sigma_{n-1}^n x$, and

L8) $x \wedge \sim\sigma_i^n x \wedge \sigma_{i+1}^n y \leq y$ for $i = 1,2,\ldots,n-2$.

Note that the axioms L1) - L5), L7) and L8) give an equational characterization of n-valued Lukasiewicz algebras.

It is easy to prove that in an n-valued Lukasiewicz algebra the following properties hold true (see [4] or [1, Chapter XI]):

L9) $\sigma_i^n(x \wedge y) = \sigma_i^n x \wedge \sigma_i^n y$.

L10) $x \leq y$ if and only if $\sigma_i^n x \leq \sigma_i^n y$ for $i = 1,2,\ldots,n-1$.

L11) $\sigma_1^n x \leq x$

L12) $\sigma_i^n 1 = 1$, $\sigma_i^n 0 = 0$.

It is also well known that Lukasiewicz algebras are Kleene algebras, and that $z \in B(L)$ if and only if there is an $i \in \{1,2,\ldots,n-1\}$ such that $\sigma_i^n z = z$, and in this case $\sigma_j^n z = z$ for $j = 1,2,\ldots,n-1$. It follows, in particular, that $\sigma_i^n x \in B(L)$ for each $x \in L$ and every $i \in \{1,\dots,n-1\}$ and hence that $\sim\sigma_i^n x = -\sigma_i^n x$.

As an easy application of these facts we can prove that the structure of n-valued Lukasiewicz algebra determines the De Morgan negation $\sim$. More precisely, we have:

1.2. LEMMA: Let $\langle L,\vee,\wedge,0,1\rangle$ be a distributive lattice, $\sim$ and $\neg$ unary operations defined on L such that $\langle L,\vee,\wedge,\sim,0,1\rangle$ and $\langle L,\vee,\wedge,\neg,0,1\rangle$ are De Morgan algebras. If we can define n-1 unary operations $\sigma_1^n,\dots,\sigma_{n-1}^n$ such that $\langle L,\vee,\wedge,\sim,\sigma_1^n,\dots,\sigma_{n-1}^n,0,1\rangle$ and $\langle L,\vee,\wedge,\neg,\sigma_1^n,\dots,\sigma_{n-1}^n,0,1\rangle$ are both n-valued Lukasiewicz algebras, then $\neg x = \sim x$ for each x in L.

PROOF: For each $i \in \{1,2,\dots,n-1\}$, we have $\sigma_i^n \sim x = -\sigma_{n-i}^n x = \sigma_i^n \neg x$, and the Moisil determination principle implies that $\sim x = \neg x$.

It was proved in [4] that the operations σ_1^n and σ_{n-1}^n are intrinsically defined by the lattice structure of an n-valued Lukasiewicz algebra. Indeed, we have that:

L13) $\sigma_1^n x = 1 \Rightarrow x$, and

L14) $\sigma_{n-1}^n x = 0 \Leftarrow x$

1.3. LEMMA: Let L be an n-valued Lukasiewicz algebra. If n is even, we have that $\sigma_{n/2}^n x = \sim x \Rightarrow x = \sim x \Leftarrow x$. If n is odd, we have that $\sigma_{(n-1)/2}^n x = \sim x \Leftarrow x$ and $\sigma_{(n+1)/2}^n x = \sim x \Rightarrow x$.

PROOF: Suppose that n is even. If $i \leq n/2$, by using L3), L9), L5) and L4) we get that:

$$\sigma_i^n(\sim x \wedge \sigma_{n/2}^n x) = \sigma_i^n(\sim x) \wedge \sigma_{n/2}^n x \leq \sigma_{n/2}^n(\sim x) \wedge \sigma_{n/2}^n x = 0 \leq \sigma_i^n x.$$

If $i > n/2$, then $\sigma_i^n(\sim x \wedge \sigma_{n/2}^n x) \leq \sigma_{n/2}^n x \leq \sigma_i^n x$, therefore: $\sigma_i^n(\sim x \wedge \sigma_{n/2}^n x) \leq \sigma_i^n x$ for $i = 1,2,\ldots,n-1$, and L10) implies that $\sim x \wedge \sigma_{n/2}^n x \leq x$.

On the other hand, if $z \in B(L)$ is such that $\sim x \wedge z \leq x$, by applying L10) we get that $\sigma_{n/2}^n(\sim x \wedge z) \leq \sigma_{n/2}^n x$; by L9) and the properties of the Boolean complement in Lukasiewicz algebras we have that $-\sigma_{n/2}^n x \wedge z \leq \sigma_{n/2}^n x$, which implies: $z \leq \sigma_{n/2}^n x$. Therefore we have proved that $\sigma_{n/2}^n x = \sim x \Rightarrow x$.

To complete the proof of the case n even, observe that from L4) we get that:

$$\sim x \Rightarrow x = \sigma_{n/2}^n x = \sim\sigma_{n/2}^n \sim x = \sim(x \Rightarrow \sim x) = \sim x \Leftarrow x.$$

The case n odd can be proved by similar arguments. When $n = 3$, 4 or 5, the above lemmas, together with properties L13) and L14), give a complete characterization of all the operations σ_i^n, $i = 1,2,\ldots,n-1$. That is:

For $n = 3$, $\sigma_1^3 x = 1 \Rightarrow x = \sim x \Rightarrow x$; $\sigma_2^3 x = 0 \Leftarrow x = \sim x \Leftarrow x$

For $n = 4$, $\sigma_1^4 x = 1 \Rightarrow x$; $\sigma_2^4 x = \sim x \Rightarrow x = \sim x \Leftarrow x$; $\sigma_3^4 x = 0 \Leftarrow x$

For $n = 5$, $\sigma_1^5 x = 1 \Rightarrow x$; $\sigma_2^5 x = \sim x \Leftarrow x$; $\sigma_3^5 x = \sim x \Rightarrow x$; $\sigma_4^5 x = 0 \Leftarrow x$.

This characterization implies, in particular, that the structure of n-valued Lukasiewicz algebra ($n=3, 4$ or 5) is determined by the structure of De Morgan algebra: We cannot define two different structures of n-valued Lukasiewicz algebra ($n=3$, 4 or 5) on a De Morgan algebra.

It is also remarkable that the cases $n=3$ and $n=4$ are both particular cases of the case $n=5$.

We have proved that in an n-valued Lukasiewicz algebra the lattice operations together with the operations σ_i^n determine the De Morgan negation (Lemma 1.2) and that for $n=3$, 4 or 5 the lattice operations and the De Morgan negation determine the operations σ_i^n. The following proposition shows that for $n=3$, 4 or 5 the De Morgan negation and the operations σ_i^n are determined by the lattice operations.

1.4. PROPOSITION: There is at most one structure of 5-valued Lukasiewicz algebra that can be defined on a distributive lattice $L = \langle L, \vee, \wedge, 0, 1 \rangle$.

PROOF: Suppose L admits two structures of 5-valued Lukasiewicz algebras: $\langle L, \vee, \wedge, \sim, \sigma_1^5, \ldots, \sigma_4^5, 0, 1 \rangle$ and $\langle L, \vee, \wedge, \neg, \alpha_1^5, \ldots, \alpha_4^5, 0, 1 \rangle$. By L13) and L14) we have that $\alpha_1^5 x = 1 \Rightarrow x = \sigma_1^5 x$ and $\alpha_4^5 x = 0 \Leftarrow x = \sigma_4^5 x$.

Suppose there is $x \in L$ such that $\sigma_2^5 x \not\leq \alpha_2^5 x$. By the well known Theorem of Stone there is an ultrafilter U of B(L) such that:

(1) $\sigma_2^5 x \in U$ and (2) $\alpha_2^5 x \notin U$.

If we define $U_i = \{x \in L : \sigma_i^5 x \in U\}$ and $U_i^1 = \{x \in L : \alpha_i^5 x \in L\}$, then (1) and (2) imply respectively:

(3) $x \in U_2$ and (4) $x \notin U_2^1$.

It is plaim that $U_1 = U_1^1$, $U_4 = U_4^1$ and that $U_1 \subseteq U_2 \subseteq U_3 \subseteq U_4$, $U_1^1 \subseteq U_2^1 \subseteq U_3^1 \subseteq U_4^1$. We are going to prove now that $U_1 = U_2 (U_1^1 = U_2^1)$ is equivalent to $U_3 = U_4 (U_3^1 = U_4^1)$. Indeed, suppose $U_1 = U_2$ and $U_3 \neq U_4$. Then there is $y \in U_4$ such that $y \notin U_3$, that is, $\sigma_4^5 y \in U$ and $\sigma_3^5 y \notin U$. But $\sigma_4^5 y \in U$ implies (U ultrafilter) that $\sigma_1^5 \sim y = -\sigma_4^5 y \notin U$, i.e., $\sim y \notin U_1 = U_2$. Hence $-\sigma_3^5 y = \sigma_2 \sim y \notin U$ and, since U is an ultrafilter of $B(L)$, $\sigma_3^5 y \in U$, i.e. $y \in U_3$, a contradiction. Therefore $U_3 = U_4$ (because $U_3 \subseteq U_4$). By a symmetrical reasonning we can prove that $U_3 = U_4$ implies $U_1 = U_2$.

It is easy to check that U_i^1 are prime filters of L and that $U_i^1 \cap B(L) = U$ for $i = 1,2,3,4$. Then, according to the results of [4] there is an $i \in \{1,2,3,4\}$ such that $U_2^1 = U_i$.

Since (3) and (4) imply that $U_2 \not\subseteq U_2^1$, we cannot have that $U_2^1 = U_i$ for $i = 2,3$ or 4. Suppose $U_2^1 = U_1 = U_1^1$. Then we would also have $U_3^1 = U_4^1$. By applying the results of [4] to the prime filter U_2, we get that there is $j \in \{1,2,3,4\}$ such that $U_2 = U_j^1$, and since $U_2 \not\subseteq U_2^1$, j should be 3 or 4, i.e. $U_2 = U_3^1 = U_4^1 = U_4$ and we would have $U_1 = U_2 = U_3 = U_4 = U_1^1 = U_2^1 = U_3^1 = U_4^1$, a contradiction. Therefore we must conclude that $\sigma_2^5 x \leq \alpha_2^5 x$. Interchanging the roles of the σ_i^5 and α_i^5, we finally prove that $\sigma_2^5 x = \alpha_2^5 x$, for each x in L, and equations L4) imply that $\sigma_3^5 x = \alpha_3^5 x$ for each x. Therefore we have proved that $\sigma_i^5 x = \alpha_i^5 x$ for each x in L and $i=1,2,3,4$. The proof is completed by applying Lemma 1.2.

1.5. COROLLARY: There is at most one structure of 3(4)-valued Lukasiewicz algebra that can be defined on a distributive lattice $L = \langle L, \vee, \wedge, 0, 1\rangle$.

PROOF: Suppose $n=3$ and that L admits two structures of 3-valued Lukasiewicz algebra:

$$\langle L,\vee,\wedge,\sim,\sigma_1^3,\sigma_2^3,0,1\rangle \quad \text{and} \quad \langle L,\vee,\wedge,\neg,\alpha_1^3,\alpha_2^3,0,1\rangle$$

we can associate two 5-valued structures by defining $\sigma_1^5=\sigma_2^5=\sigma_1^3$, $\sigma_3^5=\sigma_4^5=\sigma_2^3$ and $\alpha_1^5=\alpha_2^5=\alpha_1^3$, $\alpha_3^5=\alpha_4^5=\sigma_2^3$. By the Proposition we have $\sigma_1^3=\alpha_1^3$ and $\sigma_2^3=\alpha_2^3$.

The same proof applies to the case $n=4$, if we consider $\sigma_1^5=\sigma_1^4$, $\sigma_2^5=\sigma_3^5=\sigma_2^4$, $\sigma_4^5=\sigma_3^4$ and $\alpha_1^5=\alpha_1^4$, $\alpha_2^5=\alpha_3^5=\alpha_2^4$; $\alpha_4^5=\alpha_3^4$.

REMARK: If $n\geq 6$, we may have two non-isomorphic n-valued Lukasiewicz algebras defined on the same Kleene algebra. For instance, let L_4 be the four-element chain with the natural structure of a 4-valued Lukasiewicz algebra (see [4] or [1]). We can introduce two non-isomorphic structures of 6-valued Lukasiewicz algebras on L_4 by defining $\sigma_1^6=\sigma_1^4$, $\sigma_2^6=\sigma_3^6=\sigma_4^6=\sigma_2^4$, $\sigma_5^6=\sigma_3^4$ and $\alpha_1^6=\alpha_2^6=\sigma_1^4$, $\alpha_3^6=\sigma_2^4$, $\alpha_4^6=\alpha_5^6=\sigma_3^4$.

We are going to close this paragraph with an exemple that shows that σ_3^5 (or σ_2^5) cannot be defined in terms of the De Morgan algebra operations, σ_1^5 and σ_4^5.

EXEMPLE: Let $L = L_4 \times L_4$, S the subset of L formed by the elements (0,0), (1/3, 1/3), (1/3, 2/3), (2/3, 1/3), (2/3, 2/3) and (1.1). S is a De Morgan subalgebra of L closed under σ_1^5 and σ_4^5, but it is not closed under σ_2^5 (in this case, $\sigma_2^5 = \sigma_3^5$). Indeed $\sigma_2^5(1/3, 2/3) = (0,1) \notin S$.

2. SOME OPERATORS ON DE MORGAN ALGEBRAS

Lemma 1.3 suggest the study of the following operators on a De Morgan algebra A:

$$\alpha(x) = \sim x \Leftarrow x \qquad \text{and} \qquad \beta(x) = \sim x \Rightarrow x.$$

Since we have the relations

(1) $\alpha(x) = \sim\beta(\sim x)$ and (2) $\beta(x) = \sim\alpha(\sim x)$

it is plain that $\alpha(x)$ exists for each x in A if and only if $\beta(x)$ exists for each x in A. In this notation Lemma 1.3 can be reformulated by saying that if n is even, $\sigma_{n/2}^n x = \alpha(x) = \beta(x)$ and if n is odd, $\sigma_{n-1/2}^n x = \alpha(x)$ and $\sigma_{n+1/2}^n x = \beta(x)$. If A is a Kleene algebra, $B(A) = K(A)$ and therefore in this case we have that

(3) $\alpha(x) = \sim x \xleftarrow{K(A)} x$ and (4) $\beta(x) = \sim x \xrightarrow{K(A)} x$

It is interesting to note that relations (3) and (4) are true in any De Morgan algebra.

More precisely, we have that:

2.1. PROPOSITION: Let A be a De Morgan algebra such that $\alpha(x)$ and $\beta(x)$ exist for each x in A. Then the following properties hold true:

(i) If $x \leq y$, then $\alpha(x) \leq \alpha(y)$ and $\beta(x) \leq \beta(y)$.

(ii) $x \leq \sim x$ if and only if $\alpha(x) = 0$ and $\sim x \leq x$ if and only if $\beta(x) = 1$.

(iii) If $z \in B(A)$, then $\alpha(z) = \sim - z \wedge z$ and $\beta(z) = \sim - z \vee z$.

(iv) $z \in K(A)$ if and only if for every x in A: $\alpha(x \wedge z) = \alpha(x) \wedge z$ (respectively $\beta(x \vee z) = \beta(x) \vee z$) if and only if $\alpha(z) = z$ (respectively $\beta(z) = z$).

(v) $\alpha(x) = \sim x \xleftarrow{K(A)} x$ and $\beta(x) = \sim x \xrightarrow{K(A)} x$, for each x in A.

(vi) $\alpha(\alpha(x)) = \alpha(x)$; $\alpha(\beta(x)) = \beta(x)$; $\beta(\alpha(x)) = \alpha(x)$; $\beta(\beta(x)) = \beta(x)$.

(vii) If A is a Kleene algebra, then $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$ and $\beta(x \wedge y) = \beta(x) \wedge \beta(y)$ for each pair x, y of elements of A.

PROOF: We are going to prove just the results corresponding to β, because those corresponding to α follow from formulae (1) and (2) and the fact that $z \in K(A)$ if and only if $\sim z \in K(A)$.

Parts (i) and (ii) are inmediate consequences of the definition of β.

(iii) It is easy to see that: $\sim z \wedge (\sim -z \vee z) \leq z$. Also, is trivial that $\sim z \wedge b \leq z$ implies $b \leq -\sim z \vee z$, that is $b \leq \sim -z \vee z$ (see [5], Lemma 2.1). Then $\beta(z) = \sim -z \vee z$.

(iv) Let be $z \in K(A)$, that is: $\sim z = -z$. For every x in A from the definition of $\beta(x)$ it follows that:

$$\sim(x \vee z) \wedge (\beta(x) \vee z) = \sim x \wedge -z \wedge \beta(x) \leq x \wedge -z \leq x \vee z.$$

If $z' \in B(A)$ is such that $\sim(x \vee z) \wedge z' \leq x \vee z$, then $\sim x \wedge -z \wedge z' \leq$ $\leq x \vee z$, i.e. $\sim x \wedge -z \wedge z' \leq x$, and since $-z \wedge z' \in B(A)$ it follows that $-z \wedge z' \leq \beta(x)$, i.e. $z' \leq \beta(z) \vee z$.

Suppose now that for each $x \in A$ is $\beta(x \vee z) = \beta(x) \vee z$. In particular for $x = 0$ this implies that: $\beta(x) = z$.

Finally, we are going to prove that for every x in A, $\beta(x)$ is in $K(A)$. Indeed, since; $\sim x \leq \sim x \vee -\beta(x) = (\sim x \wedge \beta(x)) \vee -\beta(x) \leq x \vee -\beta(x)$, we have that $\sim x \wedge \sim -\beta(x) \leq x$. Then, from the definition of $\beta(x)$ it follows that $\sim -\beta(x) \leq \beta(x)$, i.e. $\sim\beta(x) \leq -\beta(x)$, and this implies that $\beta(x) \in K(A)$ (see §1). This fact proves that if $\beta(z) = z$, then $z \in K(A)$.

(v) and (vi) are obvious consequences of the fact that $\alpha(x)$ and $\beta(x)$ belong to $K(A)$.

(vii) If follows at once from (i) that $\beta(x \wedge y) \leq \beta(x) \wedge \beta(y)$. In order to prove the other inequality, we have to show that:

(1) $\sim(x \wedge y) \wedge \beta(x) \wedge \beta(y) \leq x \wedge y$. From the definition of $\beta(x)$, we obtain that (2) $\sim x \wedge \beta(x) \wedge \beta(y) \leq x \wedge \beta(y) \leq x$ and by condition (K) (see § 1):

$$(3) \quad \sim x \wedge \beta(x) \wedge \beta(y) \leq \sim x \wedge x \wedge \beta(y) \leq (\sim y \vee y) \wedge \beta(y)$$

$$\leq (\sim y \wedge \beta(y)) \vee (y \wedge \beta(y)) \leq y.$$

From (2) and (3) it follows that

$$\sim x \wedge \beta(x) \wedge \beta(y) \leq x \wedge y$$

and by interchanging x and y, we also get that

$$\sim y \wedge \beta(x) \wedge \beta(y) \leq x \wedge y$$

and the last two inequalities are equivalent to (1).

REMARKS: In the statement of (vii) of the above proposition we need to require that A be a Kleene algebra, because it is not hard to find a finite De Morgan algebra A, with $K(A) = B(A) = \{0, 1\}$ and such that $\beta(x \wedge y) \neq \beta(x) \wedge \beta(y)$ for a pair of elements x, y of A. It is also easy to find examples of finite Kleene algebras in which $\beta(x \vee y) \neq \beta(x) \vee \beta(y)$.

In Lukasiewicz algebras are important the operators $\sigma_1^n x = 1 \Rightarrow x$ and $\sigma_{n-1}^n x = 0 \Leftarrow x$ (see formulae L13) and L14)). Since Lukasiewicz algebras are Kleene algebras, we have also the equalities

$$\sigma_1^n x = 1 \xrightarrow{K(A)} x \quad \text{and} \quad \sigma_{n-1}^n x = 0 \xleftarrow{K(A)} x.$$

But it is easy to find examples of (finite) De Morgan algebras in which $1 \Rightarrow x \neq 1 \xrightarrow{K(A)} x$ and $0 \Leftarrow x \neq 0 \xleftarrow{K(A)} x$. This remarks motivates the introduction of the following operators in a De Morgan algebra A:

$$\Delta x = 1 \xrightarrow{K(A)} x \quad \text{and} \quad \nabla x = 0 \xleftarrow{K(A)} x .$$

We have the relations

(5) $\Delta x = \sim\nabla\sim x$ and (6) $\nabla x = \sim\Delta\sim x$

and from them it follows that Δx exists for each x in A if and only if ∇x exists for each x. It is well known (and easy to check) that ∇ is a closure operator and Δ and interior operator defined on A. Explicity, we have:

2.2. PROPOSITION: Let A be a De Morgan algebra such that Δx and ∇x are defined for each x in A. Then the following properties hold:

(i) $\Delta x \leq x$ and $x \leq \nabla x$

(ii) $\Delta\Delta x = \Delta x$ and $\nabla\nabla x = \nabla x$

(iii) $\Delta(x \wedge y) = \Delta x \wedge \Delta y$ and $\nabla(x \vee y) = \nabla x \vee \nabla y$

(iv) $z \in K(A)$ if and only if for every x in A: $\Delta(x \vee z) = \Delta x \vee z$ (respectively $\nabla(x \wedge z) = \nabla x \wedge z$) if and only if $\Delta(z) = z$ (respectively $\nabla z = z$).

2.3. DEFINITION: We are going to say that a De Morgan algebra A satisfies property (Bl) if Δx and $\alpha(x)$ exist for each x in A.

Note that this is equivalent to require the existence of any one of the pairs Δx and βx; ∇x and αx or ∇x and βx for each x in A.

2.4. LEMMA: If A satisties property (B1), then the following inequalities hold:

(i) $\Delta x \leq \alpha(x) \leq \nabla x$

(ii) $\Delta x \leq \beta(x) \leq \nabla x$

PROOF: (i) Since $\Delta x \leq x$, we have that $\alpha\Delta x \leq \alpha x$ and since $\Delta x \in K(A)$, $\alpha\Delta x = \Delta x$. On the other hand, since $\sim x \vee \nabla x \geq \nabla x \geq x$, and $\nabla x \in K(A)$, it follows that $\alpha(x) \leq \nabla x$. The proof of (ii) now follows from relations (1), (2) (begining of §2) (5) and (6).

REMARK: It is not hard to find a finite Kleene algebra A in which $\alpha(x) \not\leq \beta(x)$ for some x in A.

3. THE MOISIL DETERMINATION PRINCIPLE

3.1. DEFINITION: We say that a De Morgan algebra A satisfies property (B2) if it satisfies (B1) and the following determination principle: *If* x, y *are in* A *and* $\Delta x \leq \Delta y$, $\alpha x \leq \alpha y$, $\beta x \leq \beta y$ *and* $\nabla x \leq \nabla y$, *then* $x \leq y$.

3.2. LEMMA: If a De Morgan algebra A fulfills condition (B2), then:

(i) $\alpha(x) \leq \beta(x)$ for each x in A, and

(ii) A is a Kleene algebra.

PROOF: (i) Accordingly to the definition of α, the inequality $\alpha(x) \leq \beta(x)$ is equivalent to

(1) $x \leq \sim x \vee \beta(x)$.

Since $\Delta x \leq \beta x \leq \sim x \vee \beta x$, it is plain that

(2) $\Delta x \leq \Delta(\sim x \vee \beta x)$.

From properties (iv) of Proposition 2.1 it follows at once that:

(3) $\beta(x) \leq \beta(\sim x) \vee \beta(x) = \beta(\sim x \vee \beta x)$

and from propertis (vi) and (i) we have that:

$$1 = -\beta x \vee \beta x = \alpha(\sim x) \vee \alpha(\beta x) \leq \alpha(\sim x \vee \beta x).$$

Thus, using Lemma 2.4, we get that:

(4) $\alpha x \leq \alpha(\sim x \vee \beta x) = 1$

and

(5) $\nabla x \leq \nabla(\sim x \vee \beta x) = 1$.

Now (i) follows from (2), (3), (4), (5) and (B2).

(ii) Since $\sim(x \vee \sim x) = \sim x \wedge x \leq \sim x \vee x$, it follows from (ii) of Proposition 2.1 that $\beta(x \vee \sim x) = 1$. Analogously $\alpha(x \wedge \sim x) = 0$. By Lemma 2.4 we also have $\Delta(x \wedge \sim x) = 0$ and $\nabla(\sim x \vee x) = 1$. Therefore a simple application of (B2) shows that $x \wedge \sim x \leq y \vee \sim y$ for each pair x, y of elements of A.

We recall now some definitions and results from [6], that we are going to use.

3.3. DEFINITION: Let S be a sublattice of the bounded distributive lattice L. A filter F of L is called an S-filter in case that for any x in F there is an element $s \in S$ such that $S \in F$ and $s \leq x$. In other words, F is an S-filter if and only if it is the filter of L generated by a filter F_1, of S, and in this case $F_1 = F \cap S$. A filter F of L is said to be S-prime if it is the filter generated by a prime filter of S.

3.4. DEFINITION: A bounded distributive lattice L is called S-completely normal if S is a sublattice of L and if given x, y in L, there are elements s, t in S such that $x \wedge s \leq y$, $y \wedge t \leq x$ and $s \vee t = 1$.

3.5. THEOREM: [6, Th 1.3] L is an S-completely normal lattice if and only if S is a sublattice of L and each proper filter of L that contains an S-prime filter is prime in L.

We are not going to use directly the above theorem, but the following:

3.6. COROLLARY: Let L be an S-completely normal lattice and $f : L \to S$ such that:

(i) $f(x \wedge y) = f(x) \wedge f(y)$ for any x, y in L, and

(ii) $s \leq f(s)$ for each $s \in S$.

Then $f(x \vee y) = f(x) \vee f(y)$ for each x, y in L.

PROOF: By hypothesis (i), f is monotonic, so we need to prove that $f(x \vee y) \leq f(x) \vee f(y)$. Suppose that there are elements u, v in L such that $f(u \vee v) \not\leq f(u) \vee f(v)$. By the well known Birkhoff - Stone theorem, there is a prime filter P of S such that $f(u \vee v) \in P$ and $f(u) \vee f(v) \notin P$. Let $P' = f^{-1}(P)$. Hypothesis (i) implies that P' is a filter of L and hypothesis (ii) that P'', the filter of L generated by P, is contained in P'. Since P'' is an S-prime filter, it follows from the theorem quoted above that P' is a prime filter of L, and since $u \vee v \in P'$, we have that $u \in P'$ or $v \in P'$, in contradiction with $f(u) \vee f(v) \notin P$.

Our interest in S-completely normal lattices is due to the following:

3.7. PROPOSITION: Let A be a De Morgan algebra satisfying (B2). Then A is a $B(A)$-completely normal lattice.

PROOF: Let $x, y \in L$ and set:

$$z = -\nabla x \vee (\nabla x \wedge -\beta x \wedge \nabla y) \vee (\beta x \wedge -\alpha x \wedge \beta y) \vee (\alpha x \wedge -\Delta x \wedge \alpha y) \vee (\Delta x \wedge \Delta y).$$

In order to prove the proposition, it will be enough to prove that $x \wedge z \leq y$ and $y \wedge -z \leq x$. (See [6, Th. 2.1])

It is easy to check that $x \wedge z = u_1 \vee u_2 \vee u_3 \vee u_4$, where

$$u_1 = x \wedge -\beta x \wedge \nabla y$$

$$u_2 = x \wedge \beta x \wedge -\alpha x \wedge \beta y$$

$$u_3 = x \wedge \alpha x \wedge -\Delta x \wedge \alpha y \quad \text{and}$$

$$u_4 = \Delta x \wedge \Delta y.$$

Since $\Delta u_1 \leq \alpha u_1 \leq \beta u_1 = 0$ and $\nabla u_1 \leq \nabla y$, (B2) implies that $u_1 \leq y$. Analogously, from $\Delta u_2 \leq \alpha u_2 = 0$ and $\beta(u_2) \leq \nabla u_2 \leq \beta y \leq \nabla y$ we get that $u_2 \leq y$ and from $\Delta u_3 = 0$, $\alpha u_3 \leq \alpha y$, $\beta u_3 \leq \nabla u_3 \leq \alpha y \leq \beta y \leq \nabla y$, that $u_3 \leq y$. Since it is obvious that $u_4 \leq y$, we have proved that $x \wedge z \leq y$.

After some computations we can see that: $y \wedge -z = v_1 \vee v_2 \vee v_3$, where:

$$v_1 = \Delta x \wedge -\Delta y \wedge y$$

$$v_2 = \alpha x \wedge -\alpha y \wedge y \quad \text{and}$$

$$v_3 = \beta x \wedge -\beta y \wedge y.$$

It is obvious that $v_1 \leq x$, and (B2) an the relations $\Delta v_2 \leq \alpha v_2 = 0$, $\beta v_2 \leq \nabla v_2 \leq \alpha x \leq \beta x \leq \nabla x$; $\Delta v_3 \leq \alpha v_3 \leq \beta v_3 = 0$, $\nabla v_3 \leq \beta x \leq \nabla x$, imply that $v_2 \leq x$ and $v_3 \leq x$. Thus we also have that $y \wedge -z \leq x$.

We are now in conditions of establishing our main result:

3.8. THEOREM: If $A = \langle A, \vee, \wedge, \sim, 0, 1\rangle$ is a Morgan algebra satisfying property (B2), then $\langle A, \vee, \wedge, \sim, \Delta, \alpha, \beta, \nabla, 0, 1\rangle$ is a 5-valued Lukasiewicz algebra.

PROOF: The equations $\Delta(x \vee y) = \Delta x \vee \Delta y$ and $\beta(x \vee y) = \beta x \vee \beta y$ follow from Corollary 3.6 and Proposition 3.7, taking into account properties (iv) and (vii) of Proposition 2.1, (iii) and (iv) of Proposition 2.2, and Lemma 3.2. That $\alpha(x \vee y) = \alpha x \vee \alpha y$ follows from (vii) of Proposition 2.1 and $\nabla(x \vee y) = \nabla x \vee \nabla y$ was established in (iii) of Proposition 2.2. The inequalities $\Delta x \leq \alpha x \leq \beta x \leq \nabla x$ follow from Lemmas 2.4 and 3.2. Properties L2) L3) and L4) in the definition of Lukasiewicz algebras follow at once from Propositions 2.1 and 2.2 and property L6) coincides with (B2).

3.9. COROLLARY: In the conditions of 3.8, if $\alpha = \beta$, then $\langle A, \vee, \wedge, \sim, \Delta, \alpha, \nabla, 0, 1\rangle$ is a 4-valued Lukasiewicz algebra. The structure is unique.

REMARK: Let be (B2)' the condition defined by:

(B2)' is condition (B1) together with the following determination principle: if x, y are in A and $\Delta x \leq \Delta y$, $\alpha x \leq \alpha y$, $\nabla x \leq \nabla y$, then $x \leq y$.

Then, it is interesting to observe that the hypothesis

(B2)' is not sufficient to define the structure, that is, not necessarily $\alpha = \beta$.

The example is a 3-valued Lukasiewicz algebra. Indeed, we are proved (see Lemma 1.3) that $\sigma_1^3 = \Delta = \alpha$ and $\sigma_2^3 = \nabla = \beta$.

It is also false that (B1) and the condition $\alpha = \beta$ imply (B2)'. It suffices to consider the chain of six elements.

3.10. COROLLARY: In the conditions of 3.8, if $\Delta = \alpha$, $\beta = \nabla$, then $\langle A, \vee, \wedge, \sim, \Delta, \nabla, 0, 1 \rangle$ is a 3-valued Lukasiewicz algebra. The structure is unique.

REFERENCE

1. R. BALBES and P. DWINGER: "Distributive Lattices", University of Missouri Press, Columbia, Missouri 65201, 1974.

2. R. CIGNOLI: "Boolean Elements in Lukasiewicz Algebras I", Proc. Japan Acad., 41(1965), 670 - 675.

3. R. CIGNOLI and A. MONTEIRO: "Boolean Elements in Lukasiewicz Algebras II", Proc. Japan Acad., 41(1965), 676 - 680.

4. R. CIGNOLI: "Moisil Algebras", Notas de Lógica Matemática, Nº 27, Instituto de Matemática, Universidad Nac. Del Sur, Bahia Blanca, Argentina, 1970.

5. R. CIGNOLI: "Injective De Morgan and Kleene Algebras", Proc. Amer. Math. Soc., Volume 47, Number 2, 1975.

6. R. CIGNOLI: "The Lattice of Global Sections of Sheaves of Chains over Boolean Spaces", Algebra Universalis, 8(1978) 357 - 373.

7. G. EPSTEIN and A. HORN: "P-Algebras, an Abstraction from Post Algebras", Algebra Universalis, 4(1974), 195 - 206.

8. G. EPSTEIN and A. HORN: "Logics which are Characterized by Subresiduated Lattices", Z. Math. Logik Grundlagen Math., 22(1976), 199 - 210.

9. R. GRIGOLIA: "Algebraic Analysis of Lukasiewicz - Tarski's n-Valued Logical Systems". In: "Selected Papers on Lukasiewicz Sentential Calculi" (R. Wójcicki and G. Malinowski, Editors) Ossolineum, Wroclaw, Warszawa, 1977 81 - 92.

10. G. MOISIL: " Notes sur les Logiques Non-Chryssippiennes" Ann. Sci. Univ. Jassy, 27(1941), 86 - 98.

11. J. VARLET: "Algèbres de Lukasiewicz Trivalentes", Bull. Soc. Roy. Sci. Liège, 37(1968), 399 - 408.

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