SUMMARY

Let E be a non-archimedean normed space over a non-archimedean valued field F. We establish a formula for the distance $d(f,W)$ between a function $f \in \mathcal{C}(X;E)$, where X is a compact Hausdorff space, and a vector subspace $W \subset \mathcal{C}(X;E)$ which is a module over a subalgebra $A \subset \mathcal{C}(X;F)$. As a corollary we obtain several approximation results and a non-archimedean analogue of Bishop's generalization of the Stone-Weierstrass Theorem.

§1. PRELIMINARIES

Throughout this paper X stands for a *compact Hausdorff space*, and F stands for a *rank one valued field*, i.e. a field with a real-valued valuation, denoted by $t \to |t|$. The letters $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the fields of the real and the complex numbers. The symbol $C(X;F)$ denotes the algebra over F of all continuous F-valued functions on X. On $C(X;F)$ we shall consider the topology of *uniform convergence on* X, given by the sup-norm

$$f \to \|f\| = \sup\{|f(x)|;\ x \in X\}.$$

A subset $A \subset C(X;F)$ is said to be *separating over* X, or *to separate points*, if for any pair of points x and y in X, with $x \neq y$, there is a function $a \in A$ such that $a(x) \neq a(y)$. If the valued field F is non-archimedean, then $C(X;F)$ is separating over X if, and only if, the space X is 0-dimensional (see, for example, Théorème 1, §2, Chapitre II, Monna [10] or Theorem 2, section 4.9, Narici, Beckenstein and Bachman [13]).

We shall denote by $X|A$ the equivalence relation defined on X as follows: if $x,y \in X$, then $x \equiv y$ (modulo $X|A$) if, and only if, $a(x) = a(y)$, for all $a \in A$. Let Y be the quotient topological space of X modulo $X|A$ and let π be the quotient map of X onto Y; π is continuous and for each $x \in X$, $y = \pi(x)$ is the equivalence class of x modulo $X|A$. Hence, for each $a \in A$, there is a unique $b: Y \to F$ such that $a(x) = b(\pi(x))$, for all $x \in X$. We claim that $b \in C(Y;F)$. Indeed, for every open subset $G \subset F$, $a^{-1}(G)$ is open in X, and $a^{-1}(G) = \pi^{-1}(b^{-1}(G))$. By the definition of the quotient topology of Y, this means that $b^{-1}(G)$ is an open subset of Y. Let us define

$B \subset C(Y;F)$ by setting $B = \{b \in C(Y;F);\ a = b \circ \pi,\ a \in A\}$. It follows that B is a subalgebra (resp. a unitary subalgebra) of $C(Y;F)$, whenever A is a subalgebra (resp. a unitary subalgebra) of $C(X;F)$. Notice the important fact that B is separating over Y. This implies that Y is a compact Hausdorff space, which is 0-dimensional, whenever the field F is non-archimedean.

The following *separating version* of the Stone-Weierstrass theorem is well-known.

1.1. THEOREM. *Let* F *be any valued field except* $\mathbb{C}$. *Let* $A \subset C(X;F)$ *be a unitary subalgebra which is separating over* X . *Then* A *is uniformly dense in* $C(X;F)$.

For a proof, see Chernoff, Rasala and Waterhouse [3]. In fact they prove Theorem 1.1 in the more general case of arbitrary Krull valuations, i.e. not necessarily real-valued valuations. For a proof in the case of non-archimedean rank one valuations, see Theorem 2, section 4.10 of Narici, Beckenstein, and Bachman [13].

The first author to prove a Stone-Weierstrass Theorem for non archimedean valued fields was Dieudonné, who proved such a result in [4] for the field of p-adic numbers. Theorem 1.1 for the case of rank-one non-archimedean valuations is due to Kaplansky [7].

From Theorem 1.1 and the quotient construction described above, it is possible to derive a *general* version of the Stone- Weierstrass theorem, i.e. a description of the closure of a unitary subalgebra of $C(X;F)$.

1.2 THEOREM. *Let* F *be any valued field except* $\mathbb{C}$. *Let* $A \subset C(X;F)$

be a unitary subalgebra, and let $f \in C(X;F)$. *Then* f *belongs to the uniform closure of* A *in* $C(X;F)$ *if, and only if,* f *is constant on each equivalence class of* X *modulo* $X|A$.

PROOF. Necessity is clear. Let Y, π and B as before. Let now $f \in C(X;F)$ be constant on each equivalence class of X modulo $X|A$. There exists $g : Y \to F$ such that $f = g \circ \pi$. As in the proof that B is contained in $C(Y;F)$ it is easy to see that g belongs to $C(Y;F)$. By Theorem 1.1, B is dense in $C(Y;F)$. Therefore g belongs to the closure of B in $C(Y;F)$. Since the mapping $h \to h \circ \pi$ is an isometry of $C(Y;F)$ into $C(X;F)$, it follows that f belongs to the closure of A in $C(X;F)$.

The hypothesis that the algebra A be unitary can be very annoying, so let us remove it.

1.3. THEOREM. *Let* F *be any valued field except* $\mathbb{C}$. *Let* $A \subset C(X;F)$ *be a subalgebra, and let* $f \in C(X;F)$. *Then* f *belongs to the uniform closure of* A *in* $C(X;F)$ *if, and only if, the following conditions hold:*

(1) *given* $x,y \in X$ *with* $f(x) \neq f(y)$, *there exists* $g \in A$ *such that* $g(x) \neq g(y)$;

(2) *given* $x \in X$ *with* $f(x) \neq 0$, *there exists* $g \in A$ *such that* $g(x) \neq 0$.

PROOF: Necessity is clear. Let $f \in C(X;F)$ be a function satisfying conditions (1) and (2).

CASE I. There exists a point $x \in X$ such that $g(x) = 0$ for all $g \in A$. By condition (2), we have $f(x) = 0$ too. Let $B \subset C(X;F)$ be the subalgebra generated by A and the constants. The equivalence relations $X|A$ and $X|B$ are the same, and by condition (1), f is constant on each equivalence class of X modulo $X|A$. By Theorem 1.2, f belongs to the closure of B in $C(X;F)$. Let $\varepsilon > 0$ be given. There exists $g \in A$ and constant $\lambda \in F$ such that $|f(t) - g(t) - \lambda| < \varepsilon$, for all $t \in X$. Making $t = x$, we obtain $|\lambda| < \varepsilon$. If F is non-archimedean, this implies that for all $t \in X$, $|f(t) - g(t)| < \varepsilon$. If F is archimedean, then $|f(t) - g(t)| < 2\varepsilon$ for all $t \in X$. In any case, we see that f belongs to the closure of A .

CASE II. The algebra A has no common zeros. By Proposition 2, [3], A contains a function h vanishing nowhere on X. Now $1/h$ belongs to $C(X;F)$ and it is constant on each equivalence class modulo $X|B$. By Theorem 1.2, $1/h$ belongs to the closure of B in $C(X;F)$. On the other hand, $\bar{A}$ is a $\bar{B}$-module, so $1 = h(1/h) \in \bar{A}$. Therefore, $\bar{A}$ is a unitary subalgebra. Since A and $\bar{A}$ determine the same equivalence relations on X , by condition (1), f is constant on each equivalence class modulo $X|\bar{A}$. By Theorem 1.2, f belongs to $\bar{A}$.

§2. STONE-WEIERSTRASS THEOREM FOR MODULES.

Throughout this section E denotes a normed space over F, and we assume that $E \neq 0$. It follows that whenever E is non-archimedean, so is F. The space $C(X;E)$ of all continuous E-valued functions on X is endowed with the topology of *uniform convergence on* X, given by the sup-norm $f \to \|f\| = \sup \{\|f(x)\| \; ; \; x \in X\}$.

Let $A \subset C(X;F)$ be a subalgebra and let $W \subset C(X;E)$ be a vector subspace which is an A-module, i.e. $AW \subset W$. Our aim is to describe the closure of W in $C(X;E)$; or more generally, given a function $f \in C(X;E)$ to find the distance of f from W, i.e. to find

$$d(f;W) = \inf\{ \| f - g \| ; \quad g \in W\} .$$

To solve this problem, we need a "partition of unity" result. To this end, we shall adapt the proof of Rudin [15] , section 2.13, to the non-archimedean setting.

2.1 LEMMA: *Let* Y *be a* 0-*dimensional compact Hausdorff space, and let* $V_1, \ldots, V_n$ *be a finite open covering of* Y. *Let* F *be a non-archimedean valued field. There exist functions* $h_i \in C(Y;F)$, $i = 1,\ldots,n$, *such that*

(a) $h_i(y) = 0$ *for all* $y \notin V_i$, $i = 1,\ldots,n$;

(b) $\| h_i \| \leq 1$, $i = 1,\ldots,n$;

(c) $h_1 + \ldots + h_n = 1$ *on* Y.

PROOF: Each $y \in Y$ has a clopen (i.e., closed and open) neighborhood $W(y) \subset V_i$ for some i (depending on y). By compactness of Y, there are points $y_1,\ldots,y_m$ such that $Y = W_1 \cup \ldots \cup W_m$, where we have set $W_j = W(y_j)$ for each $j = 1,\ldots,m$. If $1 \leq i \leq n$, let H_i be the union of those W_j which lie in V_i . Let $f_i \in C(Y;F)$ be the characteristic function of H_i, $i = 1,\ldots,n$. Define

$$h_1 = f_1$$
$$h_2 = (1 - f_1)\, f_2$$
$$\cdots\cdots\cdots\cdots$$
$$h_n = (1 - f_1)(1 - f_2) \ldots (1 - f_{n-1})\, f_n$$

Then $H_i \subset V_i$ implies that $f_i(y) = 0$ for all $y \notin V_i$ and so $h_i(y) = 0$ for $y \notin V_i$ too, $i = 1,\ldots,n$. This proves (a). Clearly $\| h_i \| \leq 1$, $i = 1,\ldots,n$, since h_i takes only the values 0 and 1, which proves (b). On the other hand $Y = H_1 \cup \ldots \cup H_n$ and

$$h_1 + \ldots + h_n = 1 - (1 - f_1)(1 - f_2)\ldots(1 - f_n).$$

Hence, given $y \in Y$, at least one $f_i(y) = 1$ and therefore

$$h_1(y) + \ldots + h_n(y) = 1.$$

This proves (c).

2.2. THEOREM. *Let* E *be a non-archimedean normed space. Let* $A \subset C(X;F)$ *be a subalgebra and let* $W \subset C(X;E)$ *be a vector subspace which is an* A-*module. Let* $f \in C(X;E)$. *Then*

$$d(f;W) = \sup\{ d(f|S;\ W|S);\ \ S \in P_A \},$$

where P_A *denotes the set of all equivalence classes* $S \subset X$ *modulo* $X|A$.

Before proving Theorem 2.2, let us point out that it implies the following result.

2.3. THEOREM. *Let* E, A, W *and* f *be as in theorem 2.2. Then* f *belongs to the uniform closure of* W *in* $C(X;E)$ *if, and only if,* $f|S$ *is in the uniform closure of* $W|S$ *in* $C(S;E)$ *for each equivalence class* $S \subset X$ *modulo* $X|A$.

The above Theorem 2.3 contains the non-archimedean analogue of Nachbin's Stone-Weierstrass Theorem for modules (Nachbin [11], §19), and 2.2 is the "strong" Stone-Weierstrass Theorem for modules (terminology of Buck [2]).

PROOF OF THEOREM 2.2. Let us put $d = d(f;W)$ and

$$c = \sup\{ d(f|S;\ W|S)\ ;\ S \in P_A\}.$$

Clearly, $c \leq d$. To prove the reverse inequality, let $\varepsilon > 0$. Without loss of generality we may assume that A is unitary. Indeed, the subalgebra A' of $C(X;F)$ generated by A and the constants is unitary, and the equivalence relations $X|A$ and $X|A'$ are the same. Moreover, since W is a vector space, W is an A-module if, and only if, W is an A'-module.

Let Y be the quotient space of X modulo $X|A$, with quotient map π. For any $S \in P_A$, since $d(f|S;\ W|S) < c + \varepsilon$, there exists some function w_S in the A-module W such that $\|w_S(t) - f(t)\| < c + \varepsilon$ for all $t \in S$. Let $K_S = \{x \in X;\ \| w_S(x) - f(x) \| \geq c + \varepsilon\}$. Then K_S is compact and disjoint from S. Hence, for each $y \in Y$, $y \notin \pi(K_S)$, if $S = \pi^{-1}(y)$. This implies that

$$\bigcap \{\pi(K_S)\ ;\ S = \pi^{-1}(y),\ y \in Y\}$$

is empty. By the finite intersection property, there is a finite set $\{y_1,\ldots,y_n\} \subset Y$ such that $\pi(K_1) \cap \ldots \cap \pi(K_n) = \emptyset$, where $K_i = K_S$, for $S = \pi^{-1}(y_i)$, $i = 1,\ldots,n$. Let V_i be the open subset given by the complement of $\pi(K_i)$, $i = 1,\ldots,n$. Y is a 0-dimensional compact Hausdorff space. Hence, by Lemma 2.1, there exist functions $h_i \in C(Y;F)$, $i = 1,\ldots,n$, such that

(a) $h_i(y) = 0$ for all $y \notin V_i$, $i = 1,\ldots,n$;

(b) $\| h_i \| \leq 1$, $i = 1,\ldots,n$;

(c) $h_1 + \ldots + h_n = 1$.

Put $g_i = h_i \circ \pi$, so that we have $g_i \in C(X;F)$, $i = 1,\ldots,n$, and each g_i is constant on every equivalence class of X modulo $X|A$. By Theorem 1.2, g_i belongs to the closure of A in the space $C(X;F)$, for each $i = 1,2,\ldots,n$. Notice that $g_i(x) = 0$ for all $x \in K_i$, $i=1,\ldots,n$, since $h_i(y) = 0$ for all $y \in \pi(K_i)$, $i = 1,\ldots,n$. Moreover $\| g_i \| \leq 1$, $i = 1,\ldots,n$, and $g_1 + \ldots + g_n = 1$ on X. Let $g = \sum_{i=1}^{n} g_i w_i$ where $w_i = w_S$, with $S = \pi^{-1}(y_i)$, $i = 1,\ldots,n$. then $\| g(x) - f(x) \| < c+\varepsilon$, for all $x \in X$. Indeed, for any $x \in X$ we have

$$\| g(x) - f(x) \| = \| \sum_{i=1}^{n} g_i(x)(w_i(x) - f(x)) \|$$

$$\leq \max_{1\leq i\leq n} | g_i(x) | \cdot \| w_i(x) - f(x) \| .$$

Now, for each $1 \leq i \leq n$, either $x \in K_i$ and then $g_i(x) = 0$; or else $x \notin K_i$ and then

$$| g_i(x) | \cdot \| w_i(x) - f(x) \| \leq \|w_i(x) - f(x) \| < c + \varepsilon.$$

Let $M = \max \{ \| w_i \| ; i = 1,\ldots,n \}$ and choose $\delta > 0$ such that $\delta M < c + \varepsilon$. For each $i = 1,\ldots,n$, there is $a_i \in A$ such that $\|a_i - g_i\| < \delta$. Let us define $w = \sum_{i=1}^{n} a_i w_i$. Then $w \in W$ and for all $x \in X$,

$$\| w(x) - g(x) \| < c + \varepsilon.$$

Indeed, for any $x \in X$ we have

$$\| w(x) - g(x) \| = \| \sum_{i=1}^{n} (a_i(x) - g_i(x))w_i(x) \|$$

$$\leq \max_{1\leq i\leq n} | a_i(x) - g_i(x) | \cdot \| w_i(x) \|$$

$$\leq \delta M < c + \varepsilon .$$

Finally, notice that $\| w(x) - f(x) \| = \| w(x) - g(x) + g(x) - f(x) \| \leq$ $\leq \max (\| w(x) - g(x) \| , \| g(x) - f(x) \|) < c + \varepsilon$, for all $x \in X$. Hence $d < c + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $d \leq c$.

2.4. THEOREM. *Let* E , A *and* W *be as in Theorem 2.2. For each* $f \in C(X;E)$, *there exists an equivalence class* $S \subset X$ *modulo* $X|A$ *such that*

$$d(f;W) = d(f|S \, ; W|S).$$

PROOF. Let Y and π be as before. For each $g \in W$, the function

$$y \to \| f \, |\pi^{-1}(y) - g \, |\pi^{-1}(y) \|$$

is upper semicontinuous on Y, by Lemma 1, Machado and Prolla [9]. Hence

$$y \to \inf \{ \| f \, |\pi^{-1}(y) - g \, |\pi^{-1}(y) \| \, ; \, g \in W \}$$

is upper semicontinuous on Y too, and therefore attains its supremum on Y. By Theorem 2.2, this supremum is $d(f;W)$. Let then $y \in Y$ be the point where $d(f;W)$ is attained and let $S = \pi^{-1}(y)$. Then

$$d(f;W) = \inf \{ \| f|S - g|S \| , \; g \in W \} =$$

$$= d(f|S \, ; W|S),$$

as desired.

2.5 COROLLARY. *Let* E , A *and* W *be as in Theorem 2.2. Assume that* A *is separating over* X. *Then* W *is dense in* $C(X;E)$ *if and only if* $W(x) = \{g(x); \; g \in W \}$ *is dense in* E, *for each* $x \in X$. *More generally, for any* $f \in C(X;E)$, $f \in \overline{W}$ *if, and only if,* $f(x) \in \overline{W(x)}$ *in* E, *for each* $x \in X$.

Using the above corollary we can prove a result on ideals in function algebras. Let E be a non-archimedean normed non-associative algebra with unit over a (necessarily) non-archimedean field F; that is, E is a not necessarily associative linear algebra with unit e over F equipped with a non-archimedean norm satisfying

(1) $\|uv\| \leq \|u\| \cdot \|v\|$ and

(2) $\|e\| = 1$.

Condition (1) implies that multiplication is jointly continuous: If X is any compact Hausdorff space, C(X;E) with pointwise operations and sup norm becomes a non-archimedean normed algebra with unit too (over the same field F). Now the problem arises of characterizing the closed right (resp. left) ideals $I \subset C(X;E)$. Suppose that for every $x \in X$ a closed right (resp. left) ideal $I_x \subset E$ is given, and let us define

$$I = \{f \in C(X;E);\ f(x) \in I_x \text{ for all } x \text{ in } X\}.$$

Manifestly, I is a closed right (resp. left) ideal in C(X;E). We shall prove that *any* closed right (resp. left) ideal in C(X;E) has the above form. Namely we have the following.

2.6. THEOREM: *Let X be a 0-dimensional compact Hausdorff space. Let E be a non-archimedean normed algebra with unit e over a (necessarily non-archimedean) valued field F. Let $I \subset C(X;E)$ be a closed right (resp. left) ideal. For each $x \in X$, let I_x be the closure of I(x) in E. Then I_x is a closed right (resp. left) ideal* *in* E, *and*

$$I = \{f \in C(X;E);\ f(x) \in I_x \text{ for all } x \text{ in } X\}.$$

PROOF: For every x in X, I(x) is clearly a right (resp. left) ideal in E. Since the multiplication in E is jointly continuous, the closure I_x of I(x) is a right (resp. left) ideal in E. We claim that I is a C(X;F)-module. Indeed, let $f \in I$ and $g \in C(X;F)$ be given. Define $h \in C(X;E)$ to be $x \to g(x)e$, where e is the unit of E. If I is a right ideal, then for all $x \in X$,

$$g(x)f(x) = g(x)\,[f(x)e] = f(x)\,[g(x)e] = f(x)h(x).$$

Since $fh \in I$, gf belongs to I. (The case of a left ideal is treated similarly.) It remains to apply Corollary 2.5 to the separating algebra C(X;F) and the closed C(X;F)-module I.

2.7. COROLLARY: *Under the hypothesis of Theorem 2.6 assume that the algebra* E *is simple. Then any two-sided closed ideal consists of all functions vanishing on a closed subset of* X.

PROOF: We first recall that the unitary algebra E is said to be *simple* if it has no two-sided ideals other than 0 and E. Let $N \subset X$ be a closed subset of X. Clearly, the subset

$$Z(N) = \{f \in C(X;E);\ f(x) = 0 \text{ for all } x \text{ in } N\}$$

is a closed two-sided ideal of C(X;E).

Conversely, if I is a closed two-sided ideal in C(X;E), let us define $N = \{x \in X;\ f(x) = 0 \text{ for all } f \in I\}$. Clearly, N is closed in X and $I \subset Z(N)$. Conversely, let $f \in Z(N)$, and assume by contradiction that $f \notin I$. By Theorem 2.6, there is some $x \in X$ such that $f(x) \notin I_x$. Since I_x is a two-sided ideal, and E is simple, $I_x = \{0\}$. Therefore, $f(x) \neq 0$. Now $f \in Z(N)$, so $x \notin N$. However, $I_x = \{0\}$ implies $I(x) = 0$, and so $x \in N$. This contradiction shows that $f \in I$.

§3. SOME APPLICATIONS.

In this section E is a non-archimedean normed space over F, and we assume that $E \neq 0$. The vector subspace of $C(X;E)$ consisting of all finite sums of functions of the form $x \to f(x)v$, where $f \in C(X;F)$ and $v \in E$, will be denoted by $C(X;F) \otimes E$. Clearly, $C(X;F) \otimes E$ is a $C(X;F)$-module.

3.1. THEOREM. *Let X be a 0-dimensional compact Hausdorff space. Then $C(X;F) \otimes E$ is uniformly dense in $C(X;E)$.*

PROOF: Let $W = C(X;F) \otimes E$. Then W is a $C(X;F)$-module, and $C(X;F)$ is separating over X. For each $x \in X$, $W(x) = E$. By corollary 2.5 W is dense in $C(X;E)$.

If X and Y are two compact Hausdorff spaces, $C(X;F) \otimes C(Y;F)$ denotes the vector subspace of $C(X \times Y;F)$ consisting of all finite sums of functions of the form

$$(x,y) \to f(x)g(y)$$

where $f \in C(X;F)$ and $g \in C(Y;F)$. If both X and Y are 0-dimensional spaces, then $C(X;F) \otimes C(Y;F)$ is a separating unitary subalgebra of $C(X \times Y;F)$.

3.2. THEOREM. *Let X and Y be two 0-dimensional compact Hausdorff spaces. Then $(C(X;F) \otimes C(Y;F)) \otimes E$ is uniformly dense in $C(X \times Y;E)$.*

PROOF: Let $W = (C(X;F) \otimes C(Y;F)) \otimes E$. W is a $C(X;F) \otimes C(Y;F)$-module such that $W(x,y) = E$ for every pair $(x,y) \in X \times Y$. The result now follows from Corollary 2.5.

REMARK: When $E = F$, then the space $(C(X;F) \otimes C(Y;F)) \otimes E$ is just $C(X;F) \otimes C(Y;F)$ and one obtains Dieudonné's Theorem [4].

In [14] we studied polynomial algebras of functions with values in vector spaces over $\mathbb{R}$ or $\mathbb{C}$. To study the non-archimedean analogue let us adopt the following

3.3. DEFINITION. *A vector subspace* $W \subset C(X;E)$ *is called a polynomial algebra if* $A = \{u(f);\ u \in E',\ f \in W\}$ *is a subalgebra of* $C(X;F)$ *such that* $A \otimes E \subset W$.

Let us give an example of a polynomial algebra. Let

$$P_f(E;F) \subset C(E;F)$$

be the algebra over F generated by the topological dual E' of E. An element $p \in P_f(E;F)$ is called a *continuous polynomial of finite type from* E *into* F, and is of the form

$$(1) \qquad p = \sum_{|\kappa| \leq m} a_\kappa u^\kappa$$

where $\kappa = (\kappa_1,\dots,\kappa_n) \in \mathbb{N}^n$, $n \in \mathbb{N}^*$, $|\kappa| = \kappa_1 + \dots + \kappa_n$, $m \in \mathbb{N}$, $a_\kappa \in F$, $u = (u_1,\dots,u_n) \in (E')^n$, and we define

$$(2) \qquad u^\kappa(t) = (u_1(t))^{\kappa_1} \dots (u_n(t))^{\kappa_n}$$

for all $t \in E$. Let us now consider two non-archimedean normed spaces E_1 and E_2 over the same non-archimedean valued field F. We define $P_f(E_1,E_2)$ as the vector subspace of $C(E_1;E_2)$ generated by the functions of the form $t \in E_1 \to p(t)v$

where $p \in \mathcal{P}_f(E_1;F)$ and $v \in E_2$. Let now $A = \{u(p);\ u \in E_2';\ p \in \mathcal{P}_f(E_1;E_2)\}$. Clearly, $A \subset \mathcal{P}_f(E_1;F)$, and $A \otimes E_2 \subset \mathcal{P}_f(E_1;E_2)$. Suppose $(E_2)' \neq 0$. Then $A = \mathcal{P}_f(E_1;F)$ and $\mathcal{P}_f(E_1;E_2)$ is a polynomial algebra. Also, if $X \subset E_1$ is any compact subset, then $W = \mathcal{P}_f(E_1;E_2)|X$ is a polynomial algebra contained in $C(X;E_2)$. More generally, if $S \subset C(X;F)$ is any subset, let $A \subset C(X;F)$ be the subalgebra over F generated by S. If $E' \neq 0$, then $W = A \otimes E$ is a polynomial algebra. Indeed, in this case we have $A = \{u(f);\ u \in E',\ f \in W\}$. In particular, $C(X;E)$ is a polynomial algebra, when $E' \neq 0$ (e.g., when $E = F$).

When the field F is *spherically complete*, the Hahn-Banach Theorem is valid for any non-archimedean normed space E over F (see Ingleton [5]), and then E' is separating over E, and a fortiori, $E' \neq 0$.

Let us introduce the following notation. If $W \subset C(X;E)$ is an A-module, where $A \subset C(X;F)$, we denote by $L_A(W)$ the set of all $f \in C(X;E)$ such that the restriction $f|S$ is in the uniform closure of $W|S$ in $C(S;E)$, for each equivalence class $S \subset X$ modulo $X|A$. Thus, if $\overline{W}$ denotes the uniform closure of W in $C(X;E)$, the Theorem 2.3 may be stated as $f \in \overline{W} \Leftrightarrow f \in L_A(W)$.

3.4 THEOREM. *Let* E *be a non-archimedean normed space such that* E' *is separating over* E, *and let* $W \subset C(X;E)$ *be a polynomial algebra. Let* $A = \{u(g);\ u \in E',\ g \in W\}$. *Then, for every* $f \in C(X;E)$ *the following conditions are equivalent.*

(1) $f \in \overline{W}$;

(2) *given* $x, y \in X$ *and* $\varepsilon > 0$, *there is* $g \in W$ *such that* $\| f(x) - g(x) \| < \varepsilon$ *and* $\| f(y) - g(y) \| < \varepsilon$;

(3) (a) *given* $x, y \in X$, *with* $f(x) \neq f(y)$, *there is* $g \in W$ *such that* $g(x) \neq g(y)$; *and*

(b) *given* $x \in X$, *with* $f(x) \neq 0$, *there is* $g \in W$ *such that* $g(x) \neq 0$;

(4) $f \in L_A(A \otimes E)$.

PROOF: (1) => (2). Obvious.

(2) => (3). Let $x, y \in X$ with $f(x) \neq f(y)$. Define $\varepsilon = \| f(x) - f(y) \| > 0$. By (2) there is $g \in W$ such that

$$\| f(x) - g(x) \| < \varepsilon \quad \text{and} \quad \| f(y) - g(y) \| < \varepsilon.$$

If $g(x) = g(y)$, then $\varepsilon = \| f(x) - g(x) + g(y) - f(y) \| \leq$

$$\max (\| f(x) - g(x) \| , \| g(y) - f(y) \|) < \varepsilon,$$

a contradiction. This proves (a). The proof of (b) is similar.

(3) => (4). Let $S \subset X$ be an equivalence class modulo $X|A$, and let $x, y \in S$. If $f(x) \neq f(y)$, by (a) there is $g \in W$ such that $g(x) \neq g(y)$. Since E' is separating over E, there is $u \in E'$ such that $u(g(x)) \neq u(g(y))$. This is impossible , because $u(g) \in A$. Hence f is constant over S. Let $t \in E$ be its constant value. If $t = 0$, then $0 \in A \otimes E$ agrees with f over S. If $t \neq 0$, then, by (b) there is $g \in W$ such that $g(x) \neq 0$, where $x \in S$ is chosen arbitrarily. Let now $u \in E'$ be such that $u(g(x)) = 1$. Then the function $h = u(g) \otimes t$

belongs to $A \otimes E$ and agrees with f over S. Therefore

$$f \in L_A(A \otimes E).$$

(4) => (1). By Theorem 2.3 applied to the A-module $A \otimes E \subset C(X;E)$, f belongs to the uniform closure of $A \otimes E$ in $C(X;E)$. Since $A \otimes E \subset W$, the proof is complete.

3.5. COROLLARY: *Let X be a 0-dimensional compact Hausdorff space, and let E and W be as in Theorem 3.4. The following statements are equivalent.*

(1) *W is uniformly dense in $C(X;E)$;*

(2) *$W(x,y) = \{(g(x),g(y))\ ;\ g \in W\}$ is dense in $E \times E$, for every pair $x,y \in X$;*

(3) (a) *W is separating over X; and*

(b) *W is everywhere different from zero, i.e., given $x \in X$, there is $g \in W$ with $g(x) \neq 0$.*

(4) *Let $A = \{u(g);\ u \in E',\ g \in W\}$. Then A is separating over X and $W(x) = \{g(x);\ g \in W\} = E$ for every $x \in X$.*

PROOF: (1) => (2) => (3) are immediate from Theorem 3.4. (3) => (4) follows from the hypothesis that E' is separating over E and from $A \otimes E \subset W$.

Finally, (4) => (1) by Corollary 2.5 applied to the A-module $A \otimes E$, which is contained in W.

3.6. COROLLARY: (Weierstrass polynomial approximation) *Let E_1 and E_2 be two non-archimedean normed spaces over F such that E_i'*

is separating over E_i $(i = 1,2)$. *For every compact subset* $K \subset E_1$ *the set* $P_f(E_1;E_2)|K$ *is uniformly dense in* $C(K;E_2)$.

PROOF: Let $W = P_f(E_1;E_2)|K$. Since E_2' is separating over E_2, W is a polynomial algebra contained in $C(K;E_2)$. Now W contains the constants and it is separating over K, because E_1' is separating over E_1. It remains to apply the preceding Corollary.

As another application of the general results proved above, let us give a non-archimedean analogue of Blatter's Stone-Weierstrass Theorems for finite-dimensional non-associative real algebras (see Theorems 1.22 and 1.24 of [1]).

Let E be a finite-dimensional non-associative (i.e. not necessarily associative) linear algebra over a *complete* non-archimedean non-trivially valued field F. Since every field provided with a topology induced by a non-trivial valuation is *strictly minimal* (see Nachbin [12]), there is a unique Hausdorff topology on E that makes it a topological vector space over F, and moreover, under this topology, every linear transformation $T : E \to E$ is continuous. (See Nachbin [12], Theorems 7 and 9.) We shall always consider E endowed with its unique Hausdorff topology that makes it a topological vector space over F. This topology, called *admissible* in [12], can be defined as follows. If $\{e_1,\dots,e_n\}$ is a basis of E over F, then the non-archimedean sup-norm

$$\| v \| = \max \{|v_i| \ ; \ 1 \le i \le n \} ,$$

whenever $v = \sum_{i=1}^{n} v_i e_i$ is in E, defines the unique admissible topology of E.

If we define operations pointwise, $C(X;E)$ becomes a non-associative *algebra* over F too, as well a *bimodule* over E: if $v \in E$ and $f \in C(X;E)$ then the mappings $x \to vf(x)$ and $x \to f(x)v$ belong to $C(X;E)$. A vector subspace $W \subset C(X;E)$ is called a *submodule over* E if it is a bimodule over E, with the above operations. An algebra E is called a *zero-algebra* if $uv = 0$ for all $u,v \in E$. The algebra E is called *simple* if it is not a zero-algebra and has no subspaces invariant relative to the right and left multiplications, except 0 and E. Let $M(E)$ be the subalgebra of $L(E)$ generated by the set of all right and left multiplications. $M(E)$ is called the *multiplication algebra* of E. It follows that a non-zero-algebra is simple if, and only if, $M(E)$ is an irreducible algebra of linear transformations. The *centroid* of E is the set of all linear transformations $T \in L(E)$ which commute with all right and left multiplications. Clearly, all linear transformations of the form λI belong to the centroid of E, where $\lambda \in F$ and I is the identity map of E. We say that E is *central* if its centroid is just $\{\lambda I; \lambda \in F\}$.

3.7. THEOREM. *Let* F *be a complete and non-trivially valued non-archimedean field. Let* E *be a finite-dimensional central and simple non-associative algebra over* F. *Let* $W \subset C(X;E)$ *be an* F-*subalgebra which is a submodule over* E. *Then, for every* $f \in C(X;E)$, *conditions* (1) - (4) *of Theorem 3.4 are equivalent.*

PROOF: The proof consists in showing that, under the above hypothesis on E, any F-subalgebra $W \subset C(X;E)$ which is a submodule over E is a polynomial algebra.

By Theorem 4, Chapter X, Jacobson [6], we have $M(E) = L(E)$. Hence the submodule W is invariant under composition with any linear transformation $T \in L(E)$. Let $A = \{u(f);\ u \in E',\ f \in W\}$. By Lemma 1.1 of [13], extended to the case of F, A is a vector subspace of $C(X;F)$ and $A \otimes E \subset W$. It remains to prove that A is closed under multiplication. Since E is not a zero-algebra, choose a pair u_0, v_0 in E such that $u_0 v_0 \neq 0$. Let $u \in E'$ be such that $u(u_0 v_0) = 1$. Let $v(f)$ and $w(g)$ be in A. The mappings

$$x \to v(f(x))u_0 \qquad \text{and} \qquad x \to w(g(x))v_0$$

belong to W, since $A \otimes E \subset W$. By hypothesis, W is a subalgebra of $C(X;E)$. Therefore,

$$x \to [v(f(x))u_0] \cdot [w(g(x))v_0] = v(f(x))\ w(g(x))u_0 v_0$$

belongs to W. Call it h. Then $u(h) \in A$, and $u(h) = v(f)w(g)$, since $u(u_0 v_0) = 1$. Thus W is polynomial algebra.

§4. BISHOP'S THEOREM

Let F be a non-archimedean valued field, and let K be a finite extension of F. If F is *complete*, then the rank one valuation $t \to |t| \in \mathbb{R}_+$ of F can be extended from F to K in a unique way as a rank one valuation. If F is not complete, then its valua tion can be extended to a rank one valuation of K in finitely many non-equivalent ways.

4.1. DEFINITION. *Let* F *be a non-archimedean valued field; let* K *be a finite algebraic extension of* F, *endowed with a rank one valuation extending that of* F. *Let* $A \subset C(X;K)$ *be a subalgebra. A subset* $S \subset X$ *is called* A-*antisymmetric (with respect to* F) *if, for every* $a \in A$, $a|S$ *being* F-*valued implies that* $a|S$ *is constant.*

4.2. DEFINITION. *Let* $x,y \in X$. *We write* $x \equiv y$ *if there is an* A-*antisymmetric set* S *which contains both* x *and* y.

The equivalence classes modulo the equivalence relation $x \equiv y$ are called *maximal* A-*antisymmetric sets (with respect to* F).

The following result is the non-archimedean analogue of Machado's version of Bishop's Theorem [8]. In it, F is a non-archimedean valued field; K is a finite algebraic extension of F, and K is valued by one extension to K of the valuation of F; X is a compact Hausdorff space and E is a non-archimedean normed space over K.

4.3. THEOREM. *Let* $A \subset C(X;K)$ *be a subalgebra; let* $W \subset C(X;E)$ *be a vector subspace which is an* A-*module. For each* $f \in C(X;E)$, *there is a maximal* A-*antisymmetric set (with respect to* F) $S \subset X$ *such that*

$$d(f;W) = d(f|S \; ; \; W|S).$$

4.4. COROLLARY. *Let* A *and* W *be as in Theorem 4.3, and let* $f \in C(X;E)$. *Then* f *belongs to the closure of* W *in* $C(X;E)$ *if, and only if,* $f|S$ *belongs to the closure of* $W|S$ *in* $C(S;E)$, *for each maximal* A-*antisymmetric set* (*with respect to* F) $S \subset X$.

PROOF OF THEOREM 4.3.

Let $f \in C(X;E)$. Put $d = d(f;W)$. We can assume $d > 0$, the result being clear for $d = 0$, since $d(f|S\,;W|S) \leq d$ for any $S \subset X$. Let D be the set of all ordered pairs (P, S) such that

(i) P is a partition of X into non-empty pairwise disjoint and closed subsets of X ;

(ii) $S \in P$ and $d = d(f|S\,;W|S)$.

The pair $(\{X\}, X)$ belongs to D, so $D \neq \emptyset$. We partially order D by setting $(P, S) \leq (Q, T)$ if, and only if, the partition Q is finer than P, and $T \subset S$. The arguments in Machado's proof of Bishop's Theorem (see [8]) apply here, so that each chain in D has an upper bound. By Zorn's Lemma there is a maximal element $(Q, T) \in D$. We claim that T is A-antisymmetric (with respect to F). Indeed, let A_T be the set $\{a \in A;\ a|T \text{ is F-valued}\}$. By contradiction admit that $B = A_T|T$ contains non-constant functions. Since $B \subset C(T;F)$, and $W|T$ is a B-module, by Theorem 2.4 we may find an equivalence class $V \subset T$ (modulo $T|B$) such that

$$d(f|T\,;W|T) = d(f|V\,;W|V).$$

Since $d = d(f|T\,;W|T)$, and V is a proper subset of T, the partition

P of X consisting of the elements of Q distinct from T and by the equivalence classes of T modulo $T|B$ is strictly finer then Q, and therefore $(Q ; T) < (P , V)$, which contradicts the maximality of (Q , T). The *maximal* A-antisymmetric set S, which contains T, is then such that $d = d(f|S ; W|S)$.

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