ABSTRACT

A concept of integration appropriate to the conceptual framework of Non-deterministic analysis is introduced and studied. A number of desirable theorems are demonstrated. Also Non-deterministic Analysis is discussed in relation to the theory of Carathéodory Measure. Finally, some physical applications of Non-deterministic Analysis are considered.

"Les géomètres, qui ne sont que géomètres ont donc l'esprit droit, mais pourvu qu'on leur exprime bien toutes choses par définitions et principes; autrement ils sont faux et insupportables, car ils ne sont droits que sur les principes bien éclaircis".

B. Pascal

§I

Introduction to Non-deterministic Analysis

1. By Non-deterministic Analysis (NDA) we mean a mathematical system or structure in which the usual concepts we associate with analysis, such as continuity, differentiability, etc., are expressed in a "non-deterministic" way. By the word "non-deterministic" we mean we take as basic elements open sets rather than points. Functions then operate on open sets rather than points. (We call this kind of function an n-function and (2,Def.1) gives its precise definition. We use the notational convention of prefixing the name of the usual concept with a "n" to express the same concept in NDA). The use of the word non-deterministic is justified if we think of how we express mathematically the path of a particle. We usually give a specific position coordinate for each time coordinate. In NDA we as

sociate with each time interval some space interval (i.e. an open set). So the path of the particle is only approximately determined relative to the usual way we express a path. Further discussions and intuitive background to the main ideas of NDA can be seen at [6].

2. Definition I: Let X and Y be two topological spaces and $\mathcal{V}$ and $\mathcal{V}'$ be two families of open collections of sets X and Y respectively.

Suppose for each $\mu \in \mathcal{V}$ we can associate some $\mu' \in \mathcal{V}'$ such that $A \in \mu$ is associated with some $A' \in \mu'$. We call this association a n-function and denote it by

$$f: (X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}')$$

and

$$f_{\mathcal{V}}: \mu \in \mathcal{V} \longrightarrow \mu' \in \mathcal{V}' \quad \text{or} \quad f_{\mathcal{V}}(\mu) = \mu'$$

and

$$f_{\mu}: A \in \mu \longrightarrow A' \in \mu' \quad \text{or} \quad f_{\mu}(A) = A' \ , \ \mu' = f_{\mathcal{V}}(\mu).$$

So a n-function consists of a function, $f_{\mathcal{V}}: \mathcal{V} \longrightarrow \mathcal{V}'$, and a family of functions, $f_{\mu}: \mu \longrightarrow \mu'$, $\mu \in \mathcal{V}$.

Notation I: Let A be a subset of the real line, R. Define

$\$ A = (\inf A, \sup A)$, an open interval.

If A is a singleton set then $\$A = A$. The dollar sign is almost from the superposition of the first letters of Inf and Sup.

Notation II: $\forall A \epsilon \mathcal{V}$ is equivalent to $\forall A \epsilon \sigma,\ \forall \sigma \in \mathcal{V}$.

Notation III: Given two collections σ and τ of subsets of X, we say that τ refines σ, $\tau \geq \sigma$, if any set $A \in \tau$ is contained in some set $B \in \sigma$.

In the special case where Y is the real line we allow $\mathcal{V}$ to be an arbitrary collection of covers with sets that are either open or single points. For convenience we call this a real valued n-function and denote it by

$$f: (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R].$$

If we want to make it clear that $\mathcal{V}_R$ only contains open sets we will use ( ) instead of [ ] as for the general case. Throughout the rest of this paper, we use $\mathcal{V}_R$ to be the collection of all connected covers of R, in the sense that any cover $\sigma_R \in \mathcal{V}_R$ is made up of open intervals or isolated points. In previous work, f was called a special g-function.

<u>Definition II</u>: We say that the n-function f,

f: $(X,\mathcal{V}) \longrightarrow (Y,\mathcal{V}')$, is continuous if for all μ, $\lambda \in \mathcal{V}$, $\lambda \geq \mu$ and for $A \in \mu$, $B \in \lambda$ with $B \subset A$ we have

a) $f_{\mathcal{V}}(\lambda) \geq f_{\mathcal{V}}(\mu)$

b) $f_{\lambda}(B) \subset f_{\mu}(A)$.

For a real n-function, the definition of continuity is slightly different:

<u>Definition II'</u>: A real n-function

f: $(X,\mathcal{V}) \longrightarrow [R,\mathcal{V}_R]$

is continuous if

a) $\tau \geq \sigma \Rightarrow f_{\mathcal{V}}(\tau) \geq f_{\mathcal{V}}(\sigma)$

b) If $B \subset A$, $B \in \tau$, $A \in \sigma$, $\tau \geq \sigma$ then

i) $f_\tau(B) \subset f_\sigma(A)$ if $f_\tau(B)$ and $f_\sigma(A)$ are both open or points.

ii) $f_\tau(B) \in \overline{f_\sigma(A)}$, if $f_\tau(B)$ is a point and $f_\sigma(A)$ is open.

3. We know something about the relations between n-functions and usual functions. We have specific ways of "associating" n-func

tions with usual functions and vice-versa which we give now very briefly. More details can be seen in [4].

<u>Definition III</u>: Let Q be a usual function from X to Y and $f: (X,\mathcal{V}) \longrightarrow (Y,\mathcal{V}')$ an n-function. We say that f <u>generates</u> Q if for all $x \in X$ and any neighborhood W in Y with $Q(x) \in W$, $\exists\ \mu \in \mathcal{V}$ and $A \in \mu$ such that $x \in A$, $Q(x) \in \overline{f_\mu(A)}$ and $f_\mu(A) \subset W$.

<u>Definition IV</u>: Let Q be a map from X to Y such that $Q(U)^o$, the interior of $Q(U)$, is not empty for all open non-empty sets U of X. Let $\mathcal{V}$ be a family of open coverings of X and define $\mathcal{V}'$ as

$$\mathcal{V}' = \{\{Q(A)^o : A \in \mu\} ; \mu \in V\}$$

Then the n-function $f: (X,\mathcal{V}) \longrightarrow (Y,\mathcal{V}')$ defined by taking $f_{\mathcal{V}}(\mu) = \{Q(A)^o : A \in \mu\}$ and $f_\mu(A) = Q(A)^o$ for all $A \in \mu$, is called the n-function associated with Q by the <u>image method</u>.

In the special case where $Y = R$ and Q is an arbitrary function we let

$$f_\mu(A) \begin{cases} = (\inf Q(A), \sup Q(A)) & \text{if } Q(A) \text{ is not a singleton} \\ = Q(A) & \text{if } Q(A) \text{ is a singleton} \end{cases}$$

Also $\mathcal{V}' = \{\{f_\mu(A) ; a \in \mu\} : \mu \in \mathcal{V}\}$.

The real valued n-function

$$f: (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}']$$

so defined is called the real n-function associated with Q by the image method.

It is clear from the definition that any n-function constructed from a map Q by the image method will be continuous. Details about these facts can be seen in [3].

We remark that it is essential to have the particular definition II' for continuity of real n-functions to guarantee its continuity when generated by the image method just described. Indeed, consider the usual real function

$$Q(x) = \begin{cases} x \ , & 0 \le x \le 1 \\ 1 \ , & x > 1 \end{cases}$$

Considering an open interval I containing x = 1 we see that by the image method this would produce an open interval I' on the y-axis having 1 as least upper bound and so any other open interval $J \subset I$ to the right of x = 1 would produce a point, y = 1, by the image method and this point does not belong to I', but does belong to $\bar{I}'$.

4. As a preliminary to the definition of n-derivative of an n-function let us introduce the concept of Gauss space.

Definition V: A standard family of coverings, $\mathfrak{F}$ in a to pological space X, is a family of collections α, of subsets of X such that

a) Any set A of $\alpha \in \mathcal{F}$ is the closure of an open set of X.

b) Given $\alpha \in \mathcal{F}$ and two distinct sets $A_1, A_2 \in \alpha$, then $\mathring{A}_1 \cap \mathring{A}_2 = \emptyset$ (^{0} denotes the interior).

c) Any $\alpha \in \mathcal{F}$ is a covering of X.

d) Given any point $x \in X$ there is a neighborhood N of x such that any α has only a finite number of sets intersecting N.

e) Given any open set 0 of X, there is a covering $\alpha \in \mathcal{F}$ such that α has a set $A \subset 0$.

f) Ordered by refinement, $\mathcal{F}$ is a directed set.

<u>Definition VI</u>: <u>A Gauss space</u> is a topological space X with a standard family of coverings $\mathcal{F}$. We note this by $(X, \mathcal{F})$.

The reason for the above nomenclature is due to the fact that a standard family of coverings is a generalization of a system of Gauss coordinates on a surface S.

A detailed study of Gauss spaces has been done by O.T. Alas and part of her results are given in Reference [1].

<u>Definition VII</u>: <u>A Gauss transformation</u> from the Gauss

space $(X, \mathcal{F})$ into the Gauss space $(Y, \mathcal{F}')$ is a function $G: \mathcal{F} \longrightarrow \mathcal{F}'$ compatible with the order of refinement of $\mathcal{F}$ and $\mathcal{F}'$ i.e. $\alpha, \beta \in \mathcal{F}$, $\alpha \leq \beta$ then $G(\alpha) \leq G(\beta)$.

<u>Definition VIII:</u> A continuous n-function $f: (X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}')$ is called n-differentiable relative to the Gauss transformation $G: \mathcal{F} \longrightarrow \mathcal{F}'$ and the standard families of coverings $\mathcal{F}$ and $\mathcal{F}'$ of X and Y respectively, if for any $\mu \in \mathcal{V}$, $\alpha \in \mathcal{F}$, $A \in \mu$ the number of sets of α which intersect A is finite and the same for $\alpha' = G(\alpha)$, $\mu' = f_{\mathcal{V}}(\mu)$, $A' = f_{\mu}(A)$. We denote these numbers by $n(A, \alpha)$ and $n(A', \alpha')$ respectively.

<u>Definition IX:</u> Let the n-function $f: (X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}')$ be n-differentiable relative to $\mathcal{F}$, $\mathcal{F}'$ and G. The <u>n-derivative</u> of f is a real n-function $Df: (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$ defined as follows:

a) Let $n(A, \alpha)$ as before, denote the number of sets of α that intersect the set A, which is an element of $\mu \in \mathcal{V}$.

b) Let $Df_{\mu}(A, \alpha) = \dfrac{n(A', \alpha')}{n(A, \alpha)}$ where $f_{\mathcal{V}}(\mu) = \mu'$, $f_{\mu}(A) = A'$ and $G(\alpha) = \alpha'$.

c) Let $\overline{Df}_\mu(A) = \overline{\lim_{\alpha \in \mathcal{F}}} \; Df_\mu(A,\alpha)$

$\underline{Df}_\mu(A) = \underline{\lim_{\alpha \in \mathcal{F}}} \; Df_\mu(A,\alpha)$

d) So for each $B \in \lambda$ and each $\lambda \in \mathcal{V}$ we have two real numbers (possibly infinite) i.e. $\overline{Df}_\lambda(B)$ and $\underline{Df}_\lambda(B)$. Let us call $\overline{\underline{Df}}_\mu(A)$ the set of all such numbers for $B \subset A$, with $B \in \lambda$ and $\lambda \geq \mu$. i.e. we define

$\overline{\underline{Df}}_\mu(A) = \{\underline{Df}_\lambda(B), \overline{Df}_\lambda(B) : B \subseteq A, B \in \lambda, \lambda \geq \mu, \lambda \in \mathcal{V}\}$

e) Finally we define $Df_\mu(A)$ to be the open interval $(\inf \overline{\underline{Df}}_\mu(A), \sup \overline{\underline{Df}}_\mu(A))$ or the point in R $\inf \overline{\underline{Df}}_\mu(A) = \sup \overline{\underline{Df}}_\mu(A)$. with our former notation I we can write $Df_\mu(A) = \$\, \overline{\underline{Df}}_\mu(A)$.

f) So for each $A \in \mu$ we get a point or interval in R. We denote this collection $Df_\mu(A)$ by μ_R. And as μ runs through $\mathcal{V}$ we get a family $\mathcal{V}_R'$ of such collections.

We then have defined a real n-function

$$Df: \quad (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$$

which we call the n-derivative, Df, of f. If $Df_\sigma(A)$ is a finite interval $\forall A \in \sigma \in \mathcal{V}$, we say f is finite differentiable.

We note that by the definition of n-derivative Df is always a continuous real n-function. So providing Df is n-differentiable and we give a standard family of coverings, $\mathcal{F}_R$ on R with a Gauss transformation $G_R: \mathcal{F} \longrightarrow \mathcal{F}_R$ we can define higher derivatives inductively.

$$D^n f = D(D^{n-1} f) \qquad n = 2,3,...$$

5. We now want to make several remarks concerning the definition.

a) The definition of n-derivative depends on the Gauss transformation G and this is natural, because intuitively speaking G connects the "measure of motion" in $(X, \mathcal{F})$ with that in $(Y, \mathcal{F}')$ or in other words is analogous to the relation between scales in two lines.

b) It can be observed that as far as the definition of n-functions, continuity and differentiability go, the is no reason to restrict the concept of n-function to families of collections of open sets. These concepts would still be mathematically mean ingful for arbitrary families of collections of sets. But on intuitive grounds we prefer to restrict ourselves to open sets.

c) In the definition of n-differentiability we require both $n(A,\alpha)$ and $n(A',\alpha')$ to be finite for all $A \in \mu \in \mathcal{V}$ and $A' = f_{\mu}(A)$. But we need only require $n(A',\beta')$ to be finite for all $A \in \mu \in \mathcal{V}$ if we take $\frac{n(A',\alpha')}{n(A,\alpha)} = 0$ when $n(A,\alpha)$ is infinite.

d) Again looking at the definition of n-derivative we required f to be continuous on intuitive grounds since mathematically the definition would still be meaningful for a non-continuous f.

e) We notice that in the definition of n-derivative we constructed the interval $Df_{\mu}(A)$ by counting sets, so we always get $Df_{\mu}(A) \in R+$. If we have the additional structure of a linear ordering on X we can introduce a concept of a signed set on X and consequently we can then speak of a n-function as increasing or decreasing. Then in an analogous way to the classical case we can introduce a sign in the construction of our n-derivative, so $Df_{\mu}(A)$ can then contain negative points of R.

This is essential to do if we want to get a correspondence between higher n-derivative and usual derivatives in R. This is due to the fact that without a sign our interval $Df_{\mu}(\Lambda)$, would be smaller than it should be, exactly analogous to what the situation would be with usual derivatives if we only allowed positive values for it.

Several results showing the correspondence between n-derivatives and usual derivatives for Euclidean spaces can be seen in [2] or [3].

6. Notation IV: The standard family of coverings we will be using on R will be denoted by $\mathcal{F}_R$ and is called the canonical standard family of coverings on R. It is defined by

$$\mathcal{F}_R = \{\alpha_i\}_{i=1}^{\infty}, \quad \alpha_i = \{I_{ij}\}_{j=-\infty}^{\infty}$$

$$I_{ij} = \{x \in R: \frac{j}{2^{i-1}} \leq x \leq \frac{j+1}{2^{i-1}}, \quad j = 0, \pm 1, \pm 2 \ldots\}$$

This is simply the family of closed intervals of length $\frac{1}{2^{i-1}}$ starting from the origin, which have only their end points in common.

Notation V: The standard family of coverings that we will use on R^n will be denoted by $\mathcal{F}^n_R$ and is called the canonical standard family of coverings on R^n.

$$\mathcal{F}^n_R = \{\{\prod_{i=1}^{n} A_i : A_i \in \alpha\} : \alpha \in \mathcal{F}_R\}$$

Notation VI: We will use G^n_R to denote what we call the canonical Gauss transformation which is the identity Gauss transformation.

i.e. $G^n_R : \mathcal{F}^n_R \longrightarrow \mathcal{F}^n_R$ s.t. $G(\alpha_i) = \alpha_i$

We note that on any Euclidean space any implicity reference to a standard family of coverings is to the canonical families.

We now introduce the concept of two n-functions being almost equal.

Definition X: Let X be any space and suppose Γ is an open covering of X. We say that two subsets A,B of X are Γ-equal, $A \underset{\Gamma}{\cong} B$ if

a) $A \cap B = \emptyset$ implies that there is a set $H \in \Gamma$ such that $A \cup B \subset H$;

b) $A \cap B \neq \emptyset$ implies that there are two sets $H, G \epsilon \Gamma$ such that

$$A - B \subseteq H$$

$$B - A \subset G$$

Definition XI: Two n-functions

$$f: \ (X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}')$$

$$g: \ (X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}')$$

are said to be Γ-equal, $f \underset{\Gamma}{\simeq} g$, if there exists a $\mu \in \mathcal{V}$ such that for all $\lambda \geq \mu$ and for all $A \in \lambda$ we have $f_\lambda(A) \underset{\Gamma}{\simeq} g_\lambda(A)$.

Definition XII: Two n-functions f and g as in Definition XI are said to be almost equal, $f \simeq g$, if for any Γ, $f \underset{\Gamma}{\simeq} g$.

Of course, f equals g, $f = g$, means that for any $\mu \in \mathcal{V}$ and any $A \in \mu$, $f_\mu(A) = g_\mu(A)$.

For convenience we specialize the form of the previous three definitions for the case where $X = R$, it being an important special case.

Definition XIII: Let $\varepsilon > 0$. We say two subsets A,B of R are ε-equal, $A \underset{\varepsilon}{\simeq} B$, if

a) $A \cap B = \emptyset$ implies diameter $(A \cup B) < \varepsilon$

b) $A \cap B \neq \emptyset$ implies

$$\text{diameter } (A-B) < \varepsilon$$

$$\text{diameter } (B-A) < \varepsilon$$

Definition XIV: Two special n-functions.

$$f: \quad (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}']$$

$$g: \quad (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}'']$$

are said to be ε-equal, $f \underset{\varepsilon}{\simeq} g$, if there exists a $\mu \in \mathcal{V}$ such that for all $\lambda \geq \mu$, $\lambda \in \mathcal{V}$ and for all $A \in \lambda$ we have

$$f_\lambda(A) \underset{\varepsilon}{\simeq} g_\lambda(A)$$

Definition XV: Two special n-functions f and g as in Definition XIV are said to be almost equal, $f \simeq g$, if $f \underset{\varepsilon}{\simeq} g$ for all numbers $\varepsilon > 0$.

We now give the definitions of product and relative Gauss spaces.

Definition XVI: Let $(X, \mathcal{F})$ be a Gauss space and U an open subset of X. Then we have a result showing $(U, \mathcal{F}_U)$ is a Gauss space with the relative topology on U and

$$\mathcal{F}_U = \{\{U \cap A: \quad A \in \alpha\}: \ \alpha \in \mathcal{F}\}$$

We call $(U, \mathcal{F}_U)$ the relative Gauss space. See [1].

Definition XVII: Let $(X_i, \mathcal{F}_i)_{i\in I}$ be a non-empty family of Gauss spaces. Let $X = \prod_{i\in I} X_i$ have the product topology. We then have a result showing $(X, \mathcal{F})$ is a Gauss space where $\mathcal{F}$ is given by:

$$\mathcal{F} = \{\{\prod_{i\in I} A_i : A_i \in \alpha_i, \forall i \in J \text{ and } A_i = X_i \ \forall i \in I - J\}:$$

$$J \text{ is a finite non-empty subset of } I, (\alpha_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i\}$$

We call $(X, \mathcal{F})$ the product Gauss space of the $(X_i, \mathcal{F}_i)_{i\in I}$ and use the notation $\mathcal{F} = \prod_{i\in I} \mathcal{F}_i$. See [1].

Algebraic Operations: Any algebraic operation is always defined pointwise on the image space. Letting f and g be two n-functions

$$f: (X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}')$$

$$g: (X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}'')$$

with Y an additive group, then f + g means the n-function $h: (X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}''')$ as follows:

$$\forall A \in \sigma \in \mathcal{V}, \ h_\sigma(A) = f_\sigma(A) + g_\sigma(A),$$

where the addition is taken to be pointwise and $\mathcal{V}'''$ is defined in the obvious way.

If k is real number and f: $(X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$

we can define h= kf to be the real valued n-function

$$h_\sigma(A) = kf_\sigma(A) \ , \quad A \in \sigma \in \mathcal{V}$$

<u>Definition XVIII</u>: Let $\{A_n\}$ be a sequence of sets of real numbers. Define the <u>set limit</u>, A, of this sequence as follows:

$$A = \lim_{n\to\infty} A_n$$

where $x \in A$ iff there exists a sequence of points $\{x_n\}$ s.t. $x_n \in A_n$ for all n and $\lim_{n\to\infty} x_n = x$. We say the limit exists if A is not empty. Kuratowski [5] (Page 241) gives a number of properties of this type of limit, which he calls lower limit. Also note that the above definition is equivalent to saying $X \in A$ iff every nbhd of x intersects every A_n from a sufficiently great index onward.

§ II

Gauss Structure and Measure Theory

1. Our aim in this section is to show how we can build a measure from a Gauss space if suitable hypothesis are considered. We start by introducing the category of Gauss spaces. The objects of this category are Gauss spaces $(X, \mathfrak{F})$, $(Y, \mathfrak{F}')$ etc. and the morphisms are given by

Definition I: A morphism of the Gauss space $(X, \mathfrak{F})$ into the Gauss space $(Y, \mathfrak{F}')$ is

a) a Gauss transformation

$$G: \mathfrak{F} \longrightarrow \mathfrak{F}'$$

b) for every $\alpha \in \mathfrak{F}$ a surjective function

$$G_\alpha: \alpha \longrightarrow \alpha' = G(\alpha)$$

such that if $\alpha < \beta$ and $F \in \alpha$, $H \in \beta$ with $H \subset F$, then

$$G_\beta(H) \subset G_\alpha(F)$$

We notice the strong analogy between the concept of morphism just defined and that of continuous n-functions. The main difference lies on the fact that for n-functions we deal with open sets, which is not the case of definition I above.

Clearly, the composition of two morphisms of Gauss space is again a morphism of the same kind and the identity morphism can be defined in a obvious way. Therefore we have a Category of Gauss spaces.

If ϕ: $X \longrightarrow Y$ is a homeomorphism onto, it induces an isomorphism of Gauss space in a natural way. Indeed, if $(X, \mathfrak{F})$ is a Gauss space, the image by ϕ of all sets in $\alpha \in \mathfrak{F}$ for all α defines in Y a structure of Gauss space, as well as an isomorphism of Gauss space.

If Φ is a morphism of Gauss spaces we use the notation

$$\Phi: (X, \mathfrak{F}) \longrightarrow (Y, \mathfrak{F}')$$

We also indicate the funtion G in (a) by Φ.

2. The first question we shall consider is the following: given an isomorphism of Gauss space

$$\Phi: (X, \mathfrak{F}) \longrightarrow (Y, \mathfrak{F}')$$

and a family $\mathcal{V}$ of collections of open sets in X we shall build in a particular way, a continuous n-function.

$$f: (X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}')$$

where $\mathcal{V}'$ is determined by $\mathcal{V}$ and Φ, which we shall call n-function generated by Φ. To do that, take any $A \in \sigma \in \mathcal{V}$ and any $\alpha \in \mathfrak{F}$. Call

$$A_\alpha = \{F \in \alpha: F \cap A \neq \emptyset\}$$

and

$$A'_\alpha = \bigcup_{F \in A_\alpha} \Phi_\alpha(F)$$

Let us show that

$$\mathrm{Int}\ (\bigcap_{\beta<\alpha} A'_\beta) \neq \emptyset \qquad (1)$$

Indeed, let α be such that there is an $F \in \alpha$ and $F \subset A$, which is possible by definition of a Gauss structure. Let $B \in \mathcal{F}$ with $\beta > \alpha$, then, due to the definition of Gauss structure, there is a collection of sets $F^\beta \subset \beta$ whose union is F what implies that

$$A'_\beta \supset \bigcup_{H \in F^\beta} \Phi_\beta(H) \supseteq \Phi_\alpha(F)$$

due to the definition of Φ. In other words

$$\Phi_\alpha(F) \subset \bigcap_{\beta<\alpha} A'_\beta$$

and as

$$\mathrm{Int}\ \Phi_\alpha(F) \neq \emptyset$$

relation (1) is established.

Next we shall prove that for any α, $\beta \in \mathcal{F}$, we have

$$\bigcap_{\gamma>\alpha} A'_\gamma = \bigcap_{\delta>\beta} A'_\delta \qquad (2)$$

Indeed, let $\varepsilon > \alpha, \beta$, which gives

$$\bigcap_{\gamma>\alpha} A'_\gamma \subset \bigcap_{\eta>\varepsilon} A'_\eta$$

and

$$\bigcap_{\delta>\beta} A'_\delta \subset \bigcap_{\eta>\varepsilon} A'_\eta$$

But from the definition of A'_α, A'_β, etc. if λ, $\theta \in \mathcal{F}$ are such that $\lambda>\theta$ then

$$A'_\lambda \subset A'_\theta$$

so for every A'_γ, $\gamma>\alpha$, there is an A'_η, $\eta>\varepsilon$, with

$$A'_\eta \subset A'_\gamma$$

This implies that also

$$\bigcap_{\gamma>\alpha} A'_\gamma \supset \bigcap_{\eta>\varepsilon} A'_\eta$$

and

$$\bigcap_{\delta>\beta} A'_\delta \supset \bigcap_{\eta>\varepsilon} A'_\eta$$

what gives (2).

Therefore we can associate to A the open set

$$A' = \mathrm{Int}\ \left(\bigcap_{\beta>\alpha} A'_\beta\right)$$

with $\alpha \in \mathcal{F}$ arbitrary. Call $\sigma' = \{A'\}$ and

$$\mathcal{V}' = \{\sigma' : \sigma \in \mathcal{V}\}$$

Define

$$f_{\mathcal{V}}: \mathcal{V} \longrightarrow \mathcal{V}'$$

by

$$\forall\ \sigma \in \mathcal{V},\ f_{\mathcal{V}}(\sigma) = \sigma' \text{ obtained as above and}$$

define

$$f_\sigma: \ \sigma \longrightarrow \sigma'$$

by

$$\forall\ A \in \sigma,\ f_\sigma(A) = A' \in \sigma' \text{ defined as above.}$$

We have in this way defined an n-function.

$$f: \ (X, \mathcal{V}) \longrightarrow (Y, \mathcal{V}')$$

Let us show that f is continuous. Take $\sigma < \tau$ and let $B' \in f_{\mathcal{V}}(\tau)$, which means that

$$B' = \operatorname{Int} \ \left(\bigcap_{\beta > \alpha} B'_\beta \right)$$

for some $B \in \tau$. As $\sigma < \tau$ there is $A \in \sigma$ with $B \subset A$, which for every $\alpha \in \mathcal{F}$ implies that $B_\alpha \subset A_\alpha$ and so by the definition of Φ, $B'_\alpha \subset A'_\alpha$, which gives $B' \subset A'$ and so

$$f_{\mathcal{V}}(\sigma) < f_{\mathcal{V}}(\tau)$$

The proof that $B \subset A$ implies $f_\tau(B) \subset f_\sigma(A)$ for $B \in \tau$, $A \in \sigma$, $\sigma < \tau$ is the same as above and therefore f is continuous.

3. Now we investigate the behavior of derivatives of n-functions under morphisms of Gauss spaces. The main result is contained in the theorem which follows.

Theorem 1: Let $(X_1, \mathcal{F}_1)$, $(X_2, \mathcal{F}_2)$, $(Y_1, \mathcal{F}_1')$, $(Y_2, \mathcal{F}_2')$ be Gauss spaces and

$$\Phi_1: (X_1, \mathcal{F}_1) \longrightarrow (X_2, \mathcal{F}_2)$$
$$\Phi_2: (Y_1, \mathcal{F}_1') \longrightarrow (Y_2, \mathcal{F}_2')$$

isomorphisms of Gauss spaces. Let

$$f: (X_1, \mathcal{V}_1) \longrightarrow (Y_1, \mathcal{V}_1')$$

be a continuous n-function and let

$$f^1: (X_1, \mathcal{V}_1) \longrightarrow (X_2, \mathcal{V}_2)$$
$$f^2: (Y_1, \mathcal{V}_1') \longrightarrow (Y_2, \mathcal{V}_2')$$

be the continuous n-function generated by Φ_1 and Φ_2 respectively.

Let

$$g: (X_2, \mathcal{V}_2) \longrightarrow (Y_2, \mathcal{V}_2')$$

be an n-function such that

$$\begin{array}{ccc} (X_1, \mathcal{V}_1) & \xrightarrow{f} & (Y_1, \mathcal{V}_1') \\ {\scriptstyle f^1}\downarrow & & \downarrow{\scriptstyle f^2} \\ (X_2, \mathcal{V}_2) & \xrightarrow{g} & (Y_2, \mathcal{V}_2') \end{array}$$

Commutes.

Let $G_1: (X_1, \mathfrak{F}_1) \longrightarrow (Y_1, \mathfrak{F}_1')$ and

$G_2: (X_2, \mathfrak{F}_2) \longrightarrow (Y_2, \mathfrak{F}_2')$

be Gauss transformations such that

$$\begin{array}{ccc} \mathfrak{F}_1 & \xrightarrow{G_1} & \mathfrak{F}_1' \\ {\scriptstyle \Phi_1}\downarrow & & \downarrow{\scriptstyle \Phi_2} \\ \mathfrak{F}_2 & \xrightarrow{G_2} & \mathfrak{F}_2' \end{array}$$

commutes. Then

$$Df = D(gof^1)$$

<u>Proof</u>: We have to show that for any $\sigma_1 \in \mathcal{V}_1$ and every $A_1 \in \sigma_1$ we have

$$Df_{\sigma_1}(A_1) = Dg_{\sigma_2}(A_2) \qquad (1)$$

where

$$\sigma_2 = f^1_{\mathcal{V}_1}(\sigma_1)$$

and

$$A_2 = f^1_{\sigma_1}(A_1) .$$

We first show that $\alpha_1 \in \mathcal{F}_1$ implies

$$n(A_1,\alpha_1) = n(A_2,\alpha_2) , \tag{2}$$

where $\alpha_2 = \phi_1(\alpha_1)$.

Take $F_1 \in \alpha_1$ with $F_1 \cap A_1 \neq \emptyset$ and let $\beta_1 \in \mathcal{F}_1$, with $\beta_1 > \alpha_1$ having an element $\Omega_1 \subset F_1 \cap A_1$. This implies that

$$\phi_1\, \beta_1\, (\Omega_1) \subset \phi_1\, \alpha_1\, (F_1)$$

and by the definition of f^1

$$\phi_1\, \beta_1\, (\Omega_1) \subset A_2$$

which implies

$$\phi_1\, \alpha_1\, (F_1) \cap A_2 \neq \phi$$

This shows that

$$n(A_1,\alpha_1) \leq n(A_2,\alpha_2)$$

because Φ_1 is an isomorphism. So using Φ_1^{-1} we also get

$$n(A_2,\alpha_2) = n(A_1,\alpha_1)$$

which gives (2). Analogously we also have

$$n(A_2',\alpha_2') = n(A_1',\alpha_1') .$$

Now looking to the comutative diagram above we have

$$(3) \qquad \frac{n(A_2',\alpha_2')}{n(A_1,\alpha_1)} = \frac{n(A_1',\alpha_1')}{n(A_1,\alpha_1)} \cdot \frac{n(A_2',\alpha_2')}{n(A_1',\alpha_1')} = \frac{n(A_1',\alpha_1')}{n(A_1,\alpha_1)}$$

with

$$A_2' = g_{\sigma_2}(A_2) \qquad\qquad A_1' = f_{\sigma_1}(A_1)$$

$$\alpha_1' = G_1(\alpha_1) \qquad\qquad \alpha_2' = G_2(\alpha_2)$$

and also

$$(4) \qquad \frac{n(A_2',\alpha_2')}{n(A_1,\alpha_1)} = \frac{n(A_2',\alpha_2')}{n(A_2,\alpha_2)} \cdot \frac{n(A_2,\alpha_2)}{n(A_1,\alpha_1)} = \frac{n(A_2',\alpha_2')}{n(A_2,\alpha_2)}$$

So (3) and (4) give

$$\frac{n(A_1',\alpha_1')}{n(A_1,\alpha_1)} = \frac{n(A_2',\alpha_2')}{n(A_2,\alpha_2)}$$

and by definition of derivative this gives (1).

<u>Definition II:</u> The n-function g considered in theorem 1 above is called n-function image of f by the pair (Φ_1, Φ_2) and

will be denoted by $\tilde{f}$.

Corollary 1: If $\tilde{f}$ is the image by f of a pair (Φ_1, Φ_2) then if Df is pointwise cofinal(.) the same is true for $D\tilde{f}$.

Remarks:

a) If Φ: X $\rightarrow$ Y is a surjective homeomorphism, it defines in a natural way an isomorphism of Gauss structures. More precisely if X has a structure of Gauss space given by $\mathfrak{F}$, then the image by Φ of all, $F \in \alpha$ for all $\alpha \in \mathfrak{F}$ will give a Gauss structure $\mathfrak{F}'$ in Y and Φ: $(X, \mathfrak{F}) \rightarrow (Y, \mathfrak{F}')$ is then defined by the image by Φ of all $F \in \alpha$ for all $\alpha \in \mathfrak{F}$.

b) In general an isomorphism of Gauss structure does not come from a homeomorphism as described in (a). An easy example is given by the real line X with the canonical Gauss structure and the same X, considering now only the rational points, denoted by Y. The isomorphism Φ is given by associating to each $F \in \alpha \in \mathfrak{F}$, the same F without irrational points.

Corollary 2: If Φ_1 and Φ_2 are induced by a homeomorphism ϕ_1 and ϕ_2 then if f generates a map ϕ: X $\rightarrow$ Y, $\tilde{f}$ also generates a map $\phi = \phi_2 \circ \phi \circ \phi_1^{-1}$.

4. We intend to investigate now the relations between Gauss structures and measure. Let us start with some general con

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(.) On n-function f: $(X, \mathcal{V}) \rightarrow (Y, \mathcal{V}')$ is pointwise cofinal if given any cover Γ of Y by open sets and any $x \in X$ there is $\sigma \in \mathcal{V}$ and $A \in \sigma$, with $x \in A$ and $f_\sigma(A) \subset V$ for some $V \in \Gamma$. For the analogous definition for a real n-function (see [4], p. 2).

siderations.

If we have a figure in the plane and want to measure its area it is a tradition in mathematics, as conceived in the Western Culture, to approach the given figure with figures whose area is known and then to define the area as a limit, whenever it makes sense. This process was regarded by the Greeks as something not precise, and was good only for practical purposes due to their geometrical approach to mathematical problems and aversion to limit processes in general. However, in both mathematics - the Greek and the Western - a basic assumption underlying the process of measuring areas is the homogeneity of space, in consequence of which a rectangle, by rigid motion has its area unchanged and hence we can consider in particular the square of side one as unit of area, because it does not matter where we place it in the plane, its area is always the same. But suppose we imagine a world where concept of space is not homogeneous, i.e., the concept of area might change drastically when we move from one point to another. In this case, the best we can do is to subdivide such space in pieces with each of them having a known area and if one of the pieces is subdivided, the area changes accordingly. That is the "leit-motiv" for theory of measure in Gauss spaces which we intend to develop here.

We start by introducing several concepts.

<u>Definition III</u>: A base in $(X, \mathfrak{F})$ is a set of coverings in $\mathfrak{F}$ cofinal in $\mathfrak{F}$. A base of $\mathfrak{F}$ is denoted by $\mathfrak{B}(\mathfrak{F})$.

<u>Definition IV</u>: A Gauss space $(X, \mathfrak{F})$ is measurable relative to a base $\mathfrak{B}(\mathfrak{F})$ if the following conditions are satisfied:

(I) Every set $F \in \alpha \in \mathfrak{B}(\mathfrak{F})$ for any α is the union of at most countable sets of $\beta \in \mathfrak{B}(\mathfrak{F})$ if $\beta > \alpha$.

(II) To every set $F \in \alpha \in \mathfrak{B}(\mathfrak{F})$ of every $\alpha \in \mathfrak{B}(\mathfrak{F})$

it is associated a non-negative real number $m(F)$, called measure of F, such that, if F is a union of at most countably many sets F_i belonging to coverings $\alpha \in \mathfrak{B}(\mathfrak{F})$, no two of them with overlapping interiors, then

$$m(F) = \sum_i m(F_i)$$

Remark: We emphasize that in condition (II) above, the sets F_i do not belong necessarily to the same covering $\alpha \in \mathfrak{B}(\mathfrak{F})$ and the statement of that condition should also be understood in the sense that every time F is written as a union of at most countably many sets F_i, etc.

Definition V: A figure in X is a pair (H,Δ) where Δ is a collection, at most countable, of elements of coverings of $\mathfrak{B}(\mathfrak{F})$ such that no two of them have overlapping interiors and H is the union of all elements in Δ. We call Δ a decomposition of H.

Remarks:

a) We emphasize that in Δ we have in general sets of different coverings of $\mathfrak{B}(\mathfrak{F})$.

b) Sometimes we just say that "H is a figure", meaning that it can have a decomposition Δ, without making mention of this decomposition.

5. Now we want to show how a measurable Gauss space $(X,\mathfrak{F})$ generates a measure in X. We start with the particular case of a figure (H,Δ) contained in some measurable Gauss space $(X,\mathfrak{F})$.

Assume

$$\Delta = \{F_1, F_2 \ldots F_i \ldots\}, \quad F_i \in \mathcal{B}(\mathcal{F})$$

and define

$$m(H,\Delta) = \sum_{i=1}^{\infty} m(F_i)$$

as the measure of the figure (H,Δ) which may be finite or infinite.

Lemma A - If an element of $F \in \alpha \in \mathcal{B}(\mathcal{F})$ is contained in the union of at most countable many sets $K_i \in \beta_i \in \mathcal{B}(\mathcal{F})$, two by two with disjoint interiors, then

$$m(F) \leq \sum_{i=1}^{\infty} m(K_i) \qquad (1)$$

Proof - Indeed, for each K_i we can find $\gamma_i \in \mathcal{B}(\mathcal{F})$ such that $\gamma_i > \beta_i, \alpha$. Consider the set, at most countable by (I), Def. IV,

$$E_i = \{K \in \gamma_i, K \subset K_i\}$$

It can be decomposed in two sets

$$E_i' = \{K \in E_i, K \subset K_i \cap F\}$$

$$E_i'' = E_i - E_i'$$

which is a consequence of the concept of refinement in Gauss spaces. Then we have

$$m(K_i) \geq \sum_{k \in E_i'} m(K)$$

and so

$$\sum_{i=1}^{\infty} m(K_i) \geq \sum_{i=1}^{\infty} \sum_{k\in E_i'} m(K) = m(F)$$

Lemma B - If Δ_1 and Δ_2 are two decomposition of the same figure then

$$m(H,\Delta_1) = m(H,\Delta_2)$$

Proof - First we show that if a figure (H,Δ) is contained in another figure (H', Δ'), i.e., $H \subset H'$, then

$$m(H,\Delta) \leq m(H',\Delta') \qquad (2)$$

Indeed, for each $F_j \in \Delta$ let us associate the set

$$E_{ij}' = \{K\in\gamma_{ij},\ K \subset K_i \cap F_j\}$$

where $K_i \in \Delta'$ and $K_i \in \beta_i \in \mathcal{B}(\mathfrak{F})$, $F_j \in \alpha_j \in \mathcal{B}(\mathfrak{F})$ and $\gamma_{ij} > \alpha_j, \beta_i$.

We have (.)

$$i\sum_{K\in E_{ij}'} m(K) = m(F_j)$$

(.) the notation $i\Sigma$ means that the index i is variable and all the others involved remain constant.

$$\sum_j \sum_{K \in E'_{ij}} m(K) \leq m(K_i)\ .$$

Then

$$m(H,\Delta) = \sum_j \sum_i \sum_{K \in E'_{ij}} m(K) = \sum_i \sum_j \sum_{K \in E'_{ij}} m(K) \leq \sum_{K_i \in \Delta'} m(K_i)$$

$$= m(H',\Delta').$$

Finally if Δ, Δ' are two decompositions of H, we have

$$m(H,\Delta) \leq m(H,\Delta')$$
$$m(H,\Delta') \leq m(H,\Delta)$$

and therefore

$$m(H,\Delta) = m(H,\Delta')$$

and the lemma is proved.

As a consequence of this lemma, we can define the measure of any figure H of $(X,\mathfrak{F})$ which is the underlying space of a figure by

$$m(H) = m(H,\Delta)$$

with Δ an arbitrary decomposition of H, as a figure.

Definition VI: If E is an arbitrary subset of $(X,\mathfrak{F})$ the exterior measure of E is

$$m_e(E) = \inf_{H \supset E} m(H)$$

with H a figure in $(X, \mathfrak{F})$, if there is at least one figure $H \supset E$.

Otherwise

$$m_e(E) = \infty .$$

6. Our final goal is to show that m_e defined above is indeed an exterior measure in the sense of Carathéodory ([15], Chapter II) and so it will generate a measure in X.

<u>Lemma C</u> - Let (H_i, Δ_i) be figures, $i = 1,2,\ldots,$ in $(X, \mathfrak{F})$. Then it is possible to find a decomposition Δ for

$$H = \bigcup_{i=1}^{\infty} H_i$$

in such a way that we have a figure (H, Δ).

<u>Proof</u>: Let us order all sets of all Δ_i in a single row $F_1, F_2, \ldots, F_n, \ldots$. Take F_1 and F_2 and as a consequence of the concept of refinement in Gauss spaces we can find a decomposition of $F_1 \cup F_2$ as a figure. Now looking to $F_1 \cup F_2 \cup F_3$ we provide in the same way, a decomposition for it as a figure. By proceeding in this way, we finally get H as the union of at most countably many sets of coverings of $\mathcal{B}(\mathfrak{F})$ two by two without common interior points. Therefore if Δ is the collection of all these sets (H, Δ) is a figure.

<u>Lemma D</u> - Let $(H_i\ , \Delta_i)$ be a sequence of figures in

(X, F) and let (H,Δ) be defined as in lemma B. Then

$$m(H) \leq \sum_{i=1}^{\infty} m(H_i)$$

Proof: By the way (H,Δ) is defined, any set $F \in \Delta$ is contained in some H_i and so by lemma B and relation (2) we have

$$m(F) \leq m(H_i).$$

Now writing all sets of Δ in a row $F_1, F_2 \ldots F_n \ldots$ we have that $m(F_1) \leq m(H_{i_1})$. Looking to F_2 we have that $m(F_2) \leq m(H_{i_2})$ and if H_{i_1} happens to be equal to H_{i_2} we keep H_{i_1} and so we have

$$m(F_1) + m(F_2) = m(F_1 \cup F_2) \leq m(H_{i_1})$$

Proceeding in this way for any integer n we get

$$m(F_1) + m(F_2) + \ldots + m(F_n) \leq \sum_{j=1}^{p} m(H_{i_j}), \ p \leq n,$$

where the sets H_{i_j}, $1 \ldots p$, are all distincts. From this we conclude that

$$m(H) = \sum_{i}^{\infty} m(F_n) \leq \sum_{j=1}^{p} m(H_{i_j}) \leq \sum_{i=1}^{\infty} m(H_i), \ p \leq \infty.$$

Theorem 2: The exterior measure m_e in $(X, \mathfrak{F})$ introduced in definition VII is an exterior measure in the sense of Carathe̅ódory in X.

Proof: We have to show that m_e satisfies the conditions:

a) for any two subsets $A, B \subset X$, with $A \subset B$ we have

$$m_e(A) \leq m_e(B),$$

b) for any sequence $\{E_i\}$ of subsets of X, if

$$E = \bigcup_{i=1}^{\infty} E_i$$

then $$m_e(E) \leq \sum_{i=1}^{\infty} m_e(E_i).$$

The proof of (a) is an immediate consequence of the definition of exterior measure m_e.

To prove (b), take an arbitrary $\varepsilon > 0$ and associate to each E_i a figure (H_i, Δ_i) with $E_i \subset H_i$ s.t. the below equation is satisfied. If this is not possible for some E_i, then $m_e(E_i) = \infty$ and (b) is proved. Otherwise, consider

$$m_e(E_i) > m(H_i) - \delta_i$$

for $\delta_i > 0$ with

$$\delta = \sum_{i=1}^{\infty} \delta_i < \varepsilon .$$

Now E is contained in $H = \bigcup H_i$ which by lemma B can be considered as a figure (H,Δ). By lemma C we have

$$m_e(E) \leq m(H) \leq \sum_{i=1}^{\infty} m(H_i) < \sum_{i=1}^{\infty} m_e(E_i) + \varepsilon ,$$

As ε is arbitrary we conclude that

$$m_e(E) \leq \sum_{i=1}^{\infty} m_e(E_i).$$

Definition VII: The measure $m_e(E)$ induces, in the same sense of Carathéodory, a measure $m(E)$ in X, called, measure in $(X,\mathfrak{F})$ induced by the Gauss structure $\mathfrak{F}$.

7. Before we discuss some examples to illustrate our theory, we prove some general results, interesting in themselves, as well as, helpful in building examples of measurable spaces.

Definition VIII: A Gauss space $(X,\mathfrak{F})$ is a measurable union, of a family of measurable Gauss spaces $(X_a,\mathfrak{F}_a)$, $a \in A$, with A an arbitrary set of indices and such that for $a \neq b$, $X_a \cap X_b$ has empty interior in both X_a and X_b, if

(I) for every $\alpha \in \mathfrak{F}$, the collection

$$\alpha \cap X_a = \{F \cap X_a, \ F \in \alpha\}$$

belongs to $\mathfrak{F}_a$, for every $a \in A$.

(II) for every $F \in \alpha \in \mathfrak{F}$, arbitrary α, $F \cap X_a \neq \phi$ for at most countably many $a \in A$ and if m_a is the measure in $(X_a,\mathfrak{F}_a)$ we have

$$\sum_{a\in A} m_a(H_a) < \infty,$$

where $$H_a = F \cap X_a .$$

Theorem 3: If $(X, \mathcal{F})$ is measurable n-union of the measurable Gauss spaces $(X_a, \mathcal{F}_a)$, $a \in A$, then $(X, \mathcal{F})$ is also measurable.

Proof: We must show that both conditions of Def. IV are satisfied.

Let us consider condition (I). Take any $F \in \alpha \in \mathcal{F}$ and let $\beta > \alpha$, $\beta \in \mathcal{F}$. By Def. VIII we have that $\beta \cap X_a$ and $\alpha \cap X_a$ both belong to $\mathcal{F}_a$ for each $a \in A$ and also

$$B \cap X_a > \alpha \cap X_a.$$

Call

$$\beta_F = \{K \in \beta\colon K \subset F\}$$

and let $F = \bigcup H_a$ a countable union, with $H_a = F \cap X_a$.

Each set $F_a \in H_a$ intersects at most countably many sets $K \in \beta_F$. Indeed, $F_a \in \gamma_a \in \mathcal{F}_a$ and so there is $\mu_a \in \mathcal{F}_a$ with $\mu_a > \gamma_a$, $\mu_a > \beta \cap X_a$ and, therefore, as $(X_a, \mathcal{F}_a)$ is measurable, there is at most countably many sets of μ_a whose union is F_a and the same applies for every set $K \cap F_a$ where $K \in \beta_F$. Therefore F_a intersects at most countably many sets K. Now as each set H_a is union of at most countably many sets and as F is the union of at most countably many H_a we conclude that β_F is at most countable, what proves (I).

To prove (II) let us define for any $F \in \alpha \in \mathcal{F}$,

$$m(F) = \sum_{a \in A} m_a(H_a)$$

for H_a defined as above. This definition depends only on F because if $F \cap X_a$ can be expressed by another figure K_a instead of H_a we know that $m_a(K_a) = m_a(H_a)$.

Finally, let F be expressed as the countable union of sets F_i belonging to coverings of $\mathcal{F}$, no two of them with overlapping interiors. Then each F_i can be written as

$$F_i = \bigcup_{a \in A} H_a^i$$

with H_a^i a figure in $(X_a, \mathcal{F}_a)$ with

$$m(F_i) = \sum_{a \in A} m_a(H_a^i)$$

Then

$$F = \bigcup_i \bigcup_{a \in A} H_a^i = \bigcup_{a \in A} \bigcup_i H_a^i = \bigcup_{a \in A} K_a$$

where K_a is a figure in $(X_a, \mathcal{F}_a)$ by 6, lemma C and moreover as all H_a^i have two by two non-overlapping interior, from the definition of measure of a figure it fallows that

$$m_a(K_a) = \sum_i m_a(H_a^i) .$$

Now

$$m(F) = \sum_{a \in A} m_a(K_a) = \sum_{a \in A} \sum_i m_a(H_a^i) = \sum_i \sum_{a \in A} m_a(H_a^i) = \sum_i m(F_i)$$

which proves the theorem.

The measure m in $(X, \mathfrak{F})$ induced by $\mathfrak{F}$ is called <u>sum of the measures</u> $\underline{m_a}$.

We can also study the connections between measurable spaces and their cartesian multiplication, their inverse limits, etc., but we do not consider these questions now. We content ourselves with a few examples.

<u>Example 1</u>: Let $X = R^n$ and F the canonical Gauss structure as defined in §I, 6, Not. V. Define for any $F \in \alpha \in \mathfrak{F}$, m(F) as its usual n-dimensional volume. Then the measure generated by $\mathfrak{F}$ is nothing else but the Lebesgue measure in R^n. It is enough to observe that an open set in R^n is a figure in the sense considered before. A generalization of this example should be the case of an n-dimensional manifold M^n with a Gauss structure $\mathfrak{F}$ having a base $\mathfrak{B}(\mathfrak{F})$ such that any $F \in \alpha \in \mathfrak{B}(\mathfrak{F})$ implies that F has an n-dimensional area. That such Gauss structures do exist when M^n is differentiable can be seen at (16), §II, 2.

<u>Example 2</u>: Let M^n be a topological manifold not necessarily differentiable. In [13], Chapter 5, 5.2, it is introduced the concept of canonical Gauss structure $\mathfrak{F}$ in M^n having the property that if $F \in \alpha$ for any α in $\mathfrak{F}$, then $\overset{o}{F}$=interior of F, is homeomorphic to an open set A in R^n and then we define m(F)

as the Lebesgue measure of A in R^n. This will define $(M^m, \mathcal{F})$ as a measurable Gauss space. We remark that by the definition of canonical Gauss structure in M^n the set A is uniquely defined by F.

Example 3: Let λ be an arbitrary ordinal number and consider the set of all ordinals from 0 up to λ. Attach a segment of the real line in between each two of these ordinals starting with 0 going up to λ. The result is a connected space X, irreducible between 0 and λ. Consider each segment referred to above as a Gauss space $(X_a, \mathcal{F}_a)$ with the canonical Gauss structure which induces a measure in X_a, so that really $(X_a, \mathcal{F}_a)$ is a measurable space. Now if $\mathcal{F}$ is the Gauss structure in X given by all $\mathcal{F}_a$, then $(X, \mathcal{F})$ is the measurable union of all $(X_a, \mathcal{F}_a)$ and hence by theorem 3 above it is also a measurable space.

Example 4: In general if we have a measurable Gauss space $(X, \mathcal{F})$ and consider the measure m generated by its Gauss structure we might have that the measure of a single point in X is not necessarily zero. Indeed, let X be the real line and consider in X the canonical Gauss structure $\mathcal{F}$.

Let $f\colon R \to R$ be defined by

$$f(x) = \begin{cases} x, & x \leq 0 \\ x + 1, & x > 0 \end{cases}$$

and define for every $\alpha \in \mathfrak{J}$ and every $F \in \alpha$,

$$m(F) = f(b) - f(a)$$

where $a < b$ are the extremits of the interval F.
Now for every F, not containing 0, $m(F)$ coincides with its usual lenght. However the point 0 itself has measure 1.

Example 5: For later use we need to consider measures in a finite product of Gauss spaces. Let $(X_i, \mathfrak{J}_i)$, $i = 1..n$ be measurable Gauss spaces with measures m_i and consider

$$X = \prod_{i=1}^{n} X_i$$

with the corresponding Gauss structure $\mathfrak{J}$ given by the cartesian multiplication of the $\mathfrak{J}_i$, $i = 1,..n$. If $F \in \alpha \in \mathfrak{J}$ is given by

$$F = F_1 \times F_2 \times \ldots \times F_n$$

with $F_i \in \alpha_i \in \mathfrak{J}_i$ we define

$$m(F) = m_1(F_1) \cdot m_2(F_2) \cdot \ldots \cdot m_n(F_n)$$

This makes $(X, \mathfrak{J})$ a measurable Gauss space and m is called the product of the measures m_i, $i=1,\ldots n$.

Example 6: Let $(X, \mathfrak{J})$ be a Gauss space and let m be the measure induced by $\mathfrak{J}$. Then any figure (H, Δ) in $(X, \mathfrak{J})$

is measurable and its measure now coincides with the one previously defined. Indeed, it is enough to show that any $F \in \alpha \in \mathfrak{F}$ for any α is measurable. First if F is not contained in any figure of $B(\mathfrak{F})$ them $m(F) = \infty$. Otherwise if F is contained in a figure (H,Δ) of $B(\mathfrak{F})$ we write all sets of Δ in a row.

$$F_1, F_2, \ldots F_n, \ldots$$

Consider F_1 and look to $\beta_1 \in \mathcal{B}(\mathfrak{F})$ refinining both α and α_1, where α_1 is such that $F_1 \in \alpha_1$. Then $F_1 \cap F$ is the union of at most countable many sets of α_1 and therefore it is measurable. Do the same for $F_2, F_3 \ldots F_n$ - and get F as the union of at most countable many measurable sets with non overlopping interiors and so F is measurable.

We finish here by calling attention of the behaviour of measure in the category of Gauss spaces. More precisely we have: if

$$\Phi: \ (X, \mathfrak{F}) \rightarrow (Y, \mathfrak{F}')$$

is an isomorphism of Gauss spaces and if $(X, \mathfrak{F})$ is a measurable with measure m them $(Y, \mathfrak{F}')$ can be made a measurable space with measure defined by m. Indeed, for any $F \in \alpha \in \mathcal{B}(\mathfrak{F})$ define

$$m' \ (\Phi_\alpha(F)) = m(F)$$

In this case we say that the notion of measurable space is invariant by isomorphisms of Gauss spaces.

Many other interesting questions can be investigated by approaching the concept of derivative of n-functions with that of the concept of measure. For instance, we might investigate conditions under which the derivative of an n-function can be expressed by a quotient of the measure of two sets, namely, if

$$f: (X, \mathcal{V}) \to (Y, \mathcal{V}')$$

and if $(X, \mathfrak{F})$ and $(Y, \mathfrak{F}')$ are measurable spaces with measures m amd m', then when is it true that, for $A \in \sigma \in \mathcal{V}$

$$Df_\sigma(A) = \frac{m'\left[f_\sigma(A)\right]}{m(A)}$$

In future publications we shall study these questions.

§ III

The Integral of an n-Function

In this section we define the concept of integration for an n-function, prove a number of the usually desired properties of the integral, and finally give several results showing the relationship between the integral of an n-function and the integral of a usual function when the n-function is "generated" by that usual function.

1. Some Assumptions and Notation.

The concepts of NDA are very general, applying to at least regular topological spaces. In order to avoid too many technical details, we have made no effort to always try to frame each result in the most general context that seemed possible. It is convenient to here establish the below blanket assumptions and notation.

a. When we are dealing with any Euclidean space we always assume the Canonical Gauss Structure defined in Section I.6, which we use for both n-differentiation and n-integration. As stated in Section II.6, Example 1, this gives us the Lebesque measure.

b. $(X,\mathfrak{F})$ is both a Gauss space and a measure space with measure m. The measure m is not necessarily constructed from $\mathfrak{F}$ by the methods of Section II. Also $\mathfrak{F}$ is countable and of finite type. That is, $n(F,\beta) < \infty$ for all F in α in $\mathfrak{F}$ and for all α,β in $\mathfrak{F}$. Further $m(F)$ is finite for all F in α in $\mathfrak{F}$.

c. $\mathcal{V}$ is a directed countable family of locally finite coverings such that each σ in $\mathcal{V}$ is countable, $m(A)$ and $n(A,\alpha)$ is finite for all A in σ in $\mathcal{V}$ and all α in $\mathcal{F}$.

d. By definition the <u>canonical family of open coverings on the real line</u>, R, is defined by

$$\mathcal{V} = \{\sigma_i\}_{i=1}^{\infty} \quad , \quad \sigma_i = \{I_{i_j}\}_{j=-\infty}^{\infty}$$

$$I_{i_j} = \{x \in R: \frac{j-1}{j^{i-1}} < x < \frac{j+1}{j^{i-1}}, \; j= 0, \pm 1, \pm 2 \ldots\}$$

The <u>canonical family of open converings on R^n</u> is defined by

$$\mathcal{V}^n = \{\{\prod_{k=1}^{n} A_k : A_k \in \sigma_i\} : \sigma_i \in \mathcal{V}\}$$

e. We use the product measure $m' = m \times \lambda$ on the product space $X \times R$ where λ is the Lebesque measure on R and m is the given measure on X. We use the symbol m for both the measure on X and $X \times R$ whenever it is clear.

f. The notation

$$\overline{f_\sigma(A)} = (0, \sup f_\sigma(A)]$$

$$\underline{f_\sigma(A)} = (0, \inf f_\sigma(A)]$$

is convenient to use, where $(0, \sup f_\sigma(A)]$ is the half-open interval with extremities 0 and $\sup\{f_\sigma(A)\}$. Here f is assumed to be a real n-function.

2. <u>The n-Integral of a Positive Real n-Function.</u>

<u>Definition I:</u> Let

$$f: (X, \mathcal{V}) \longrightarrow [R^{+}, \mathcal{V}_{R^{+}}]$$

be a non-negative n-function. Define

$$Sf_{\sigma}(A) = A \times (0, \sup f_{\sigma}(A)]$$

$$If_{\sigma}(A) = A \times (0, \inf f_{\sigma}(A)]$$

with $A \in \sigma \in \mathcal{V}$ and $(0, \sup f_{\sigma}(A)]$ the half-open interval with extremities 0 and $\sup\{f_{\sigma}(A)\}$. Let

$$\begin{aligned} Sf_{\sigma} &= \{\bigcup_{A} Sf_{\sigma}(A) : A \in \sigma\} \\ If_{\sigma} &= \{\bigcup_{A} If_{\sigma}(A) : A \in \sigma\} \end{aligned} \qquad (1)$$

If Sf and If are m-measurable and $E \subset X$ is m-measurable, then define

$$\begin{aligned} S(f_{\sigma}, E) &= m[Sf_{\sigma} \cap (E \times R)] \\ I(f_{\sigma}, E) &= m[If_{\sigma} \cap (E \times R)] \end{aligned} \qquad (2)$$

and

$$S(f,E) = \underline{\lim}_{\sigma \in \mathcal{V}} S(f_\sigma,E)$$
$$I(f,E) = \overline{\lim}_{\sigma \in \mathcal{V}} I(f_\sigma,E) \qquad (3)$$

where the previous limits are the limit inferior and the limit superior respectively. Then the integral of f on E is defined as the open interval

$$\int_E f dm = \between\{I(f,E),\ S(f,E)\} \qquad (4)$$

We say f is integrable on E if the sets in Equation (1) are measurable and the limits in Equation (3) are finite.

The geometric significance of this definition is easily seen by making a diagram for f an n-function from R to R.

For convenience in the remainder of this section, we assume, that when the n-integral exists, that the limit inferior and the limit superior in Equation (3a) and (3b) respectively can be replaced by a simple limit.

We remark that the symbols defined in Equation (2) of Definition (I) can also be written as

$$S(f_\sigma,E) = m\{\bigcup_A (A \cap E) \times (0,\ \sup f_\sigma(A)] : A \in \sigma\}$$
$$I(f_\sigma,E) = m\{\bigcup_A (A \cap E) \times (0,\ \inf f_\sigma(A)] : A \in \sigma\} \qquad (2')$$

Definition II: Let

$$f\colon (X,\mathcal{V}) \longrightarrow (Y,\mathcal{V}')$$

be an n-function. Consider a continuous non-negative real-valued n-function

$$g: \quad (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$$

Then define

$$\int_E f dg \quad = \quad \int_E g \circ f dm$$

which is a type of Lebesque-Stieltjes integral.

3. <u>Some Properties of the n-Integral</u>

<u>Definition III</u>: Let $E \subset X$ be m-measurable and finite. Def ine the characteristic n-function,

$$f: \quad (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$$

on E by

$$f_\sigma^E(A) = \begin{cases} 1 & A \cap E \neq \emptyset \\ 0 & A \cap E = \emptyset \end{cases}$$

<u>Theorem 1</u>:

$$\int_E f^E dm = m(E).$$

<u>Proof</u>: By definition

$$Sf_{\sigma}^{E}(A) := \begin{cases} A \times (0, \sup f_{\sigma}^{E}(A)] & A \cap E \neq \emptyset \\ \emptyset & A \cap E = \emptyset \end{cases}$$

$$= \begin{cases} A \times (0,1] & A \cap E \neq \emptyset \\ \emptyset & A \cap E = \emptyset \end{cases}$$

So

$$S(f_{\sigma}^{E},E) = m\{\bigcup_{A}(A \cap E) \times [0,1]: A \in \sigma\}$$

$$= m\{ E \times [0,1]\} =$$

$$= m(E).$$

Likewise

$$I(f_{\sigma}^{E},E) = m(E),$$

and therefore

$$\int_E f^E dm = \$\{S(f_{\sigma}^{E},E), I(f_{\sigma}^{E},E)\} = m(E).$$

<u>Definition IV:</u> Let $K \in R$ and f,g real-valued n-functions

$$f: \quad (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$$

$$g: \quad (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$$

Define Kf as an n-function

$$Kf: \quad (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$$

with

$$(Kf)_\sigma(A) = Kf_\sigma(A) = \{Kx: x \in f_\sigma(A)\}$$

Define h = f + g as an n-function

$$h: \quad (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$$

with

$$h_\sigma(A) = f_\sigma(A) + g_\sigma(A) = \{x+y: x \in f_\sigma(A),\ y \in g_\sigma(A)\}$$

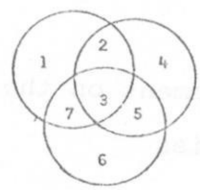
The following definition and two lemmas are helpful in the proof of several theorems.

Definition V: Let $\sigma \in \mathcal{V}$ and define the disjoint cover σ' of σ as follows:

$$\sigma' = \{\{ \bigcap_{i=1}^{n} A_i - \bigcup_{j=1}^{\infty} B_j : \forall\, B_j \in \sigma \text{ s.t. } B_j \neq A_i,\ i=1,n\} : A_i \in \sigma,\ n \leq N\}$$

where N is the order of the coverings σ, which by assumption is locally finite.

This complicated expression for σ' is simply the new cover constructed (no longer open) from σ in such a way as to obtain a disjoint cover. For example, the cover consisting of the three sets



gives the cover consisting of the seven numbered sets.

Lemma A: If each $A \in \sigma$ is measurable then each $A' \in \sigma'$ is also measurable.

Proof: Since measure is preserved under countable intersections, unions and complementation, then each element of A' is measurable if all $A \in \sigma$ are measurable.

Notation I:

$$\overline{f}_{\sigma'}(B) = \{\bigcup \overline{f_\sigma(A)} : \forall \ A \in \sigma,\ A \supset B\}$$

$$\underline{f}_{\sigma'}(B) = \{\bigcup \underline{f_\sigma(A)} : \forall \ A \in \sigma,\ A \supset B\}$$

with $B \in \sigma'$, σ' the disjoint cover constructed from σ.

Lemma B:

(i) $$m\{\bigcup_B (B \times \overline{f}_{\sigma'}(B)) \cap (E \times R) : \forall \ B \in \sigma'\} = m\{\bigcup_A (A \times \overline{f_\sigma(A)}) \cap (E \times R) : \forall \ A \in \sigma\}$$

$$= S(f_\sigma, E)$$

(ii) $$m\{\bigcup_B (B \times \underline{f}_{\sigma'}(B)) \cap (E \times R) : \forall \ B \in \sigma'\} =$$

$$m\{\bigcup_A (A \times \underline{f_\sigma}(A)) \cap (E \times R) : \forall \ A \in \sigma \} =$$

$= I(f_\sigma, E)$.

Proof: Let x be an element on the set on the left-hand-side of Equation (i). This implies

$$x \in B \times \overline{f}_{\sigma'}(B), \quad \text{some } B \in \sigma'.$$

So

$$x \in A \times \overline{f_\sigma(A)} \quad \text{some } A \in \sigma .$$

and therefore x is an element of the set on the right-hand-side of Equation (i) since each B is a subset of some A.

Let $x = (x_1, x_2)$ be an element of the set on the right-hand-side of Equation (i). So

$$x \in A \times \overline{f_\sigma(A)}, \quad \text{some } A \in \sigma$$

Let $B \in \sigma'$ be such that $x_1 \in B \subset A$. Let A_1 be any other element of σ containing B. $x_1 \in B$ and

$$\overline{f}_{\sigma'}(B) = \{\bigcup_A \overline{f_\sigma(A)}: A \in \sigma, A \supset B\}$$

imply $x_2 \in \overline{f}_{\sigma'}(B)$.

Since the sets on the two sides of Equation (i) are equal, the equation is true.

Expression (ii) follows in a similiar manner.

Theorem 2: Let f and g be non-negative real-valued n-functions

$$f,g: \quad (X,\mathcal{V}) \longrightarrow [R^{+},\mathcal{V}_{R^{+}}]$$

and k a positive constant. Then

$$\text{(i)} \qquad k \int_E fdm \;=\; \int_E kfdm$$

$$\text{(ii)} \quad \int_E fdm \;+\; \int_E gdm \;\geq\; \int_E (f+g)dm$$

where we assume the integrals on the left-hand sides of the previous two expressions exist. The symbol "$\geq$" is given in Definition VI.

Proof: Demonstration of (ii). First we show that for any $\sigma \in \mathcal{V}$

$$S(f_{\sigma},E) \;+\; S(g_{\sigma},E) \;\geq\; S((f+g)_{\sigma},E)$$

Indeed

$$S(f_{\sigma},E) \;=\; m\{\bigcup_A (Ax\overline{f_{\sigma}(A)}) \cap (ExR):\ A \in \sigma\}$$

$$= \; m\{\bigcup_B (Bx\overline{f}_{\sigma'}(B)) \cap (ExR):\ B \in \sigma'\}$$

by Lemma B. Since σ' is disjoint

$$S(f_{\sigma},E) = \sum_B n\ \{(Bx\overline{f}_{\sigma'}(B)) \cap (ExR)\}\ ,\quad B \in \sigma'$$

Similarly

$$S(g_\sigma,E) = \sum_B m\{(Bx\overline{g}_{\sigma'}(B)) \cap (ExR)\},\ B \in \sigma'$$

and

$$S(h_\sigma,E) = \sum_B n\{(Bx\overline{h}_{\sigma'}(B)) \cap (ExR)\},\ B \in \sigma'$$

where

$$h = f + g$$

Then we have

$$h_{\sigma'}(B) = \{\bigcup_A \overline{h_\sigma(A)} : A \in \sigma,\ A \supset B\}$$

$$= \{\bigcup_A (0, \sup h_\sigma(A)] : A \in \sigma,\ A \supset B\}$$

$$= \{\bigcup_A (0, \sup f_\sigma(A) + \sup g_\sigma(A)] : A \in \sigma,\ A \supset B\}$$

$$= \{\bigcup_A (0, \sup f_\sigma(A)] + (0, \sup g_\sigma(A)] : A \in \sigma,\ A \supset B\}$$

$$\{\bigcup_A (0, \sup f_\sigma(A)] : A \in \sigma,\ A \supset B\} + \{\bigcup_A (0, \sup g_\sigma(A)] : A \in \sigma,\ A \supset B\}$$

Making the observation that the right-hand side of the previous inequality is of the form

$$(0,a] + (0,b],$$

we then obtain

$$\lambda(\overline{h}_{\sigma'}(B)) \leq \lambda(\overline{f}_{\sigma'}(B)) + \lambda(\overline{g}_{\sigma'}(B))$$

with λ the Lebesque measure on R. This implies

$$m(B)\ \lambda\ (\overline{h}_{\sigma'}(B) \leq m(B)\ \lambda\ (\overline{f}_{\sigma'}(B)) + m(B)\ \lambda\ (\overline{g}_{\sigma'}(B))$$

and

$$m(B \times \overline{h}_{\sigma'}(B)) \leq m(B \times \overline{f}_{\sigma'}(B)) + m(B \times \overline{g}_{\sigma'}(B)).$$

So we have

$$\sum_B m(B \times \overline{h}_{\sigma'}(B)) \leq \sum_B \left[m(B \times \overline{f}_{\sigma'}(B)) + m(B \times \overline{g}_{\sigma'}(B))\right]$$

$$\leq \sum_B m(B \times \overline{f}_{\sigma'}(B)) + \sum_B m(B \times \overline{g}_{\sigma'}(B))$$

which is valid since the integrals of f and g exist. Therefore we have the stated result

$$S(h_\sigma, E) \leq S(f_\sigma, E) + S(g_\sigma, E)$$

we can show

$$I(f_\sigma, E) + I(g_\sigma, E) \geq I(h_\sigma, E)$$

by the previous procedure and due to the fact that

$$\{\bigcup_A (0, \inf f_\sigma(A)] : A \in \sigma, A \supset B\} + \{\bigcup_A (0, \inf g_\sigma(A)] : A \in \sigma, A \supset B\}$$

$$\supset \{\bigcup_A (0, \inf h_\sigma(A)] : A\in\sigma,\ A\supset B\}.$$

Therefore we have

$$S(f,E) + S(g,E) = \lim_{\sigma\in\mathcal{V}} S(f_\sigma,E) + \lim_{\sigma\in\mathcal{V}} S(g_\sigma,E)$$

$$\geq \lim_{\sigma\in\mathcal{V}} S(h_\sigma,E) = S(h,E)$$

and

$$I(f,E) + I(g,E) = \lim_{\sigma\in\mathcal{V}} I(f_\sigma,E) + \lim_{\sigma\in\mathcal{V}} I(g_\sigma,E)$$

$$\geq \lim_{\sigma\in\mathcal{V}} I(h_\sigma,E) = I(h,E)$$

This gives us the desired result

$$\int_E f dm + \int_E g dm = \oint \{I(f,E),\ S(f,E)\}$$

$$+ \oint \{I(g,E),\ S(g,E)\}$$

$$\geq \oint \{I(h,E),\ S(h,E)\} = \int_E h dm.$$

<u>Demonstration of (i)</u>. By lemma B, its proof and the fact that σ' is disjoint we have the following equations

$$S((kf)_\sigma,E) = m\{\bigcup_A (A \times \overline{kf_\sigma(A)}) \cap (E\times R) : A\in\sigma\}$$

$$= m\{\bigcup_B (B \times \overline{kf_{\sigma'}(B)}) \cap (E\times R)\} : B\in\sigma'\}$$

$$= \sum_B m\ \{(B\times\overline{kf_{\sigma'}}(B)) \cap (E\times R)\},\ B\in\sigma'$$

$$= \sum_B m(B \cap E)\ \lambda\ (\overline{kf}_{\sigma'}(B)),\ B \in \sigma'$$

$$= \sum_B km(B \cap E)\ \lambda\ (\bar{f}_{\sigma'}(B)),\ B \in \sigma'$$

$$= k \sum_B m\{(B x \bar{f}_{\sigma'}(B)) \cap (ExR) : B \in \sigma'\}$$

$$= km\{\bigcup_B (B x \bar{f}_{\sigma'}(B)) \cap (ExR) : B \in \sigma'\}$$

$$= km\{\bigcup_A (A \times \overline{f_\sigma(A)}) \cap (ExR) : A \in \sigma\}$$

$$= k\ S(f_\sigma, E).$$

In an analogous manner it can be shown that

$$I((kf)_\sigma, E) = kI(f_\sigma, E).$$

Therefore

$$S(kf,E) = kS(f,E)$$

$$I(kf,E) = kI(f,E)$$

and

$$\int_E kfdm = k \int_E fdm$$

We remark that it can be shown that

$$\int_E fdm + \int_E gdm \not\geq \int_E (f+g)dm$$

unless we place further restrictions on $\mathcal{V}$. We recall that $\mathcal{V}$ is assumed to be countable, directed and locally finite. None of these conditions require the coverings of $\mathcal{V}$ to become "small" relative to the measure m. We conjecture that if we require this, in some appropriate sense, then

$$\int_E f dm + \int_E g dm \supset \int_E (f + g)\, dm.$$

<u>Lemma C</u>: Let $S \subset X \times R$ and $E_1, E_2 \subset X$. Then

$$S \cap [(E_1 \cup E_2) \times R] = [S \cap (E_1 \times R)] \cup [S \cap (E_2 \times R)]$$

<u>Proof</u>: Let $x = (a,b)$ be an element of the left-hand side of the above equation. So

$$x \in S, \quad (E_1 \cup E_2) \times R.$$

This implies $a \in E_1$ or E_2, say E_1. Then $(a,b) \in A$, $E_1 x R$, and therefore x is an element of the right-hand side.

Let x be an element of the right-hand side. So $(a,b) \in S$, $E_1 \times R$ or $(a,b) \in E_2 \times R$, say $E_1 \times R$. This means $a \in E_1$, $b \in R$ and therefore $(a,b) \in (E_1 \cup E_2) \times R$. So x is element of the left-hand side, and we are done.

<u>Theorem 3</u>: Let $E_1, E_2 \subset X$ be disjoint m-measurable sets and f a non-negative real-valued function n-function.

$$f: (X, \mathcal{V}) \longrightarrow [R^+, \mathcal{V}_{R^+}]$$

Then

$$\int_{E_1} fdm + \int_{E_2} fdm = \int_{E_1 \cup E_2} fdm$$

where it is assumed the integrals on the left-hand side exist.

<u>Proof</u>: By definition

$$S(f_\sigma, E_1 \cup E_2) = m(Sf_\sigma \cap ((E_1 \cup E_2) \times R))$$

which equals

$$m((Sf_\sigma \cap (E_1 xR)) \cup (Sf_\sigma \cap (E_2 xR))$$

by lemma C. Since E_1 and E_2 are disjoint

$$S(f_\sigma, E_1 \cup E_2) = m(Sf_\sigma \cap (E_1 xR)) + m(Sf_\sigma \cap (E_2 xR))$$

$$= S(f_\sigma, E_1) + S(f_\sigma, E_2).$$

This implies

$$S(f, E_1 \cup E_2) = \lim_{\sigma \in \mathcal{V}} S(f_\sigma, E_1) + \lim_{\sigma \in \mathcal{V}} S(f_\sigma, E_2)$$

$$= S(f, E_1) + S(f, E_2).$$

Likewise we have

$$I(f, E_1 \cup E_2) = I(f, E_1) + I(f, E_2),$$

which gives

$$\int_{E_1 \cup E_2} f dm = \$\{I(f,E_1) + I(f,E_2),\ S(f,E_1) + S(f,E_2)\}$$

$$= \$\{I(f,E_1),\ S(f,E_1)\} + \$\{I(f,E_2),\ S(f,E_2)\}$$

$$= \int_{E_1} f dm + \int_{E_2} f dm.$$

<u>Definition VI:</u> We say a set A is <u>less then or equal</u> to a set B, $A \leq B$, where $A,B \subset R$ if

$$\sup A \leq \sup B$$

$$\inf A \leq \inf B.$$

Then we say <u>$f \leq g$</u>, f,g two real-valued n-functions

$$f,g: (X,\mathcal{V}) \longrightarrow [R,\mathcal{V}_R]$$

if

$$f_\sigma(A) \leq g_\sigma(A)$$

for all $A \in \sigma$, for all $\sigma \in \mathcal{V}$.

<u>Definition VII:</u> Let f and g be two n-functions

$$f,g: (X,\mathcal{V}) \longrightarrow (Y,\mathcal{V}')$$

We say

$$f \subset g$$

if

$$f_\sigma(A) \subset g_\sigma(A)$$

for all $A \in \sigma$, for all $\sigma \in \mathcal{V}$.

Theorem 4: Let f and g be two n-differentiable real-valued non-negative n-functions.

$$f,g: \quad (X,\mathcal{V}) \longrightarrow [R^+, \mathcal{V}_{R^+}]$$

such that

$$\int_E f dm \quad \text{and} \quad \int_E g dm$$

exist. Then

(i) $f \subset g \implies \int_E f dm \subset \int_E g dm$ and $Df \leq Dg$

ii) $f \leq g \implies \int_E f dm \leq \int_E g dm$

Proof of (i): First $Df \leq Dg$ follows immediately from the definition of n-derivative.

Now

$$f_\sigma(A) \subset g_\sigma(A)$$

implies

$$Sf_\sigma(A) = A \times \overline{f_\sigma(A)} \subset A \times \overline{g_\sigma(A)} = Sg_\sigma(A)$$

and

$$If_\sigma(A) = A \times \underline{f_\sigma(A)} \supset A \times \underline{g_\sigma(A)} = Ig_\sigma(A)$$

This in turn implies

$$Sf_\sigma = \{\underset{A}{\cup} Sf_\sigma(A) : A\epsilon\sigma\} \subset \{\underset{A}{\cup} Sg_\sigma(A) : A\epsilon\sigma\} = Sg_\sigma$$

and

$$If_\sigma \supset Ig_\sigma.$$

Hence

$$S(f_\sigma,E) = m\{Sf_\sigma \cap (ExR)\} \leq m\{Sg_\sigma \cap (ExR)\} = S(g_\sigma,E)$$

and also

$$I(f_\sigma,E) \geq I(g_\sigma,E).$$

So we have

$$I(g_\sigma,E) \leq I(f_\sigma,E) \leq S(f_\sigma,E) \leq S(f_\sigma,E)$$

and therefore

$$I(g,E) \leq I(f,E) \leq S(f,E) \leq S(g,E)$$

This gives us the desired result

$$\int_E fdm = \S\{I(f,E), S(f,E)\} \subset \S\{I(g,E), S(g,E)\} = \int_E gdm$$

Proof of (ii): $f \leq g$ implies

$$Sf_\sigma(A) = A \times \overline{f_\sigma(A)} \subset A \times \overline{g_\sigma(A)} = Sg_\sigma(A)$$

and

$$If_\sigma(A) = A \times \underline{f_\sigma}(A) \subset A \times \underline{g_\sigma}(A) = Ig_\sigma(A),$$

which gives

$$Sf_\sigma = \{\bigcup_A Sf_\sigma(A) : A\in\sigma\} \subset \{\bigcup_A g_\sigma(A) : A\in\sigma\} = Sg_\sigma$$

and

$$If_\sigma \subset Ig_\sigma.$$

Hence

$$S(f\ ,E) = m\{Sf_\sigma \cap (E\times R)\}$$

$$\leq m\{Sg_\sigma \cap (E\times R)\} = S(g_\sigma,E)$$

and also

$$I(f_\sigma,E) \leq I(g_\sigma,E).$$

Therefore we have

$$S(f,E) = \lim_{\sigma\in\mathcal{V}} S(f_\sigma,E) \leq \lim_{\sigma\in\mathcal{V}} S(g_\sigma,E) = S(g,E)$$

$$I(f,E) \leq I(g,E),$$

which gives us the desired result

$$\int_E f\,dm = \$\{I(f,E),\ S(f,E)\} \leq \$\{I(g,E),\ S(g,E)\}$$

$$= \int_E g\,dm$$

The following theorem shows that the n-integral is invarient under translations for an appropriate $\mathcal{V}$.

Theorem 5: Let X have an additive group structure and

$$f: (X, \mathcal{V}) \longrightarrow [R^+, \mathcal{V}_{R^+}].$$

Suppose that for any $a \in X$ such that $\forall A \in \sigma \in \mathcal{V}$ and $\forall B \in \sigma'$, σ' the disjoint cover associated with σ, we have $A + a \in \sigma$ and $B + a \in \sigma'$. Also $\forall A, A' \in \sigma$ we have $m(A) = m(A')$ and $m(a) = 0$. Define

$$f_\sigma^a(A) = f_\sigma(a + A)$$

$\forall A \in \sigma$, $\forall \sigma \in \mathcal{V}$. Then

$$\int_X f^a dm = \int_X f\,dm$$

where we assume the integral on the right-hand side exists.

Proof: By definition

$$Sf_\sigma^a = \{\bigcup_A A \times \overline{f_\sigma^a(A)} : A \in \sigma\}$$

$$= \{\bigcup_A A \times \overline{f_\sigma(A + a)} : A \in \sigma\}$$

So

$$
\begin{aligned}
S(f^a_\sigma, X) &= m\{\bigcup_A A \times \overline{f_\sigma(A + a)} : A \in \sigma\} \\
&= m\{\bigcup_B B \times \bar{f}_{\sigma'}(B + a) : B \in \sigma'\} \text{ (by Lemma B)} \\
&= \sum_B m(B \times \bar{f}_{\sigma'}(B + a)), \quad B \in \sigma' \\
&= \sum_B m(B)\, \lambda\, (\bar{f}_{\sigma'}(B + a)), \ B \in \sigma' \\
&= \sum_B m(B + a)\, \lambda\, (\bar{f}_{\sigma'}(B + a)), \ B \in \sigma'
\end{aligned}
$$

Since $B + a \in \sigma'$ by assumption and $m(a) = 0$

$$
\begin{aligned}
S(f^a_\sigma, X) &= \sum_B m(B)\, \lambda\, (\bar{f}_{\sigma'}(B)), \ B \in \sigma' \\
&= \sum_B m(B \times \bar{f}_{\sigma'}(B)), \ B \in \sigma' \\
&= m\{\bigcup_B (B \times \bar{f}_{\sigma'}(B)) : B \in \sigma'\} \\
&= m\{\bigcup_A A \times \overline{f_\sigma(A)} : A \in \sigma\} \\
&= S(f_\sigma, X).
\end{aligned}
$$

Likewise

$$I(f^a_\sigma, X) = I(f_\sigma, X).$$

and therefore

$$\int_X f^a dm = \int_X f dm.$$

<u>Definition VIII:</u> Let $E \subset X$, $\mathcal{V}$ a family of coverings of X and m a measure on X. We say <u>$\mathcal{V}$ becomes arbitrarily small relative to E and m</u> if $\forall \varepsilon > 0$ there exists a $\tau \in \mathcal{V}$ such that for all $\sigma \in \mathcal{V}$, $\sigma \geq \tau$ we have

$$m\{\bigcup_A A : A \in \sigma,\ A \cap E \neq \emptyset,\ A \not\subset E\} < \varepsilon$$

If $\mathcal{U}$ is a family of subsets of X, we say $\mathcal{V}$ <u>becomes arbitrarily small relative to $\mathcal{U}$ and m</u> when $\mathcal{V}$ becomes arbitrarily small for each $E \in \mathcal{U}$ and the same τ serves for all $E \in \mathcal{U}$.

<u>Definition IX:</u> Let $E \subset X$ and

$$f: \ (X, \mathcal{V}) \longrightarrow |R, \mathcal{V}_R|$$

We say f is bounded on E relative to $\mathcal{V}$ if for some constant $k \in R$ there exists a $\tau \in \mathcal{V}$ such that $\forall \sigma \geq \tau$

$$\sup\{f_\sigma(A)\} \leq k$$

$\forall A \in \sigma$, $A \cap E \neq \emptyset$.

<u>Theorem 6:</u> Let f be a continuous n-function

$$f: \ (X, \mathcal{V}) \longrightarrow |R^+, \mathcal{V}_{R^+}|$$

which is integrable on $E \subset X$ and bounded by k on E. Let $E \in \sigma_0$ some $\sigma_0 \in \mathcal{V}$, $m(E)$ finite and $\mathcal{V}$ become arbitrarily small relative to E and m. Then

$$\int_E f\,dm \subset f_{\sigma_0}(E)^{-}\, m(E)$$

where $f_{\sigma_0}(E)^{-}$ is the closure of $f_{\sigma_0}(E)$.

<u>Proof</u>: Let

$$\sigma^* = \{A : A \in \sigma,\ A \subset E\}$$

$$\sigma^{**} = \{A : A \in \sigma,\ A \cap E \neq \emptyset,\ A \not\subset E\}$$

Then

$$I(f_\sigma, E) = m\,\{\bigcup_A (A \times \underline{f_\sigma(A)}) \cap (E \times R) : A \in \sigma\}$$

$$\geq m\,\{\bigcup_{A^*} (A^* \times \underline{f_\sigma(A^*)}) \cap (E \times R) : A^* \in \sigma^*\}$$

$\forall\ \sigma \geq \sigma_0$, $\sigma^* \subset \sigma$. Since

$$f_\sigma(A^*) \subset f_{\sigma_0}(E)$$

then

$$\underline{f_\sigma(A^*)} \supset \underline{f_{\sigma_0}(E)}$$

$\forall\ A^* \in \sigma^*$. So

$$I(f_\sigma, E) \geq m\{\bigcup_{A^*} ((A^* \cap E) \times \underline{f_{\sigma_0}(E)}) : A^* \in \sigma^*\}$$

$$= m\{(\bigcup_{A^*} A^* \cap E) \times \underline{f_{\sigma_0}(E)} : A^* \in \sigma^*\}$$

$$= m\,\{\bigcup_{A^*} A^* \cap E : A^* \in \sigma^*\} \inf f_{\sigma_0}(E)$$

We also have

$$m\{\bigcup_{A^*} A^* \cap E : A^* \in \sigma^*\} \leq m(E)$$

$$\leq m\{\bigcup_{A^{**}} A^{**} \cap E : A^{**} \in \sigma^{**}\} + m\{\bigcup_{A^*} A^* \cap E : A^* \in \sigma^*\}$$

$$\leq \quad \varepsilon \quad + \quad m\{\bigcup_{A^*} A^* \cap E : A^* \in \sigma^*\}$$

with $\sigma^* \subset \sigma$ and σ sufficiently "large" since $\mathcal{V}$ becomes arbitrarily small. So

$$m\{\bigcup A^* \cap E : A^* \in \sigma^*\} \geq m(E) - \varepsilon$$

and

$$\begin{aligned} I(f_\sigma,E) &\geq (m(E) - \varepsilon) \inf f_{\sigma_0}(E) \\ &\geq m(E) \inf f_{\sigma_0}(E) - k\,\varepsilon \end{aligned} \qquad (1)$$

for σ sufficiently large. In a similar manner we can show

$$S(f_\sigma, E) - k\,\varepsilon \leq m(E) \sup f_{\sigma_0}(E).$$

Combining (1) and (2) we get

$$m(E) \inf f_{\sigma_0}(E) - k\,\varepsilon \leq I(f_\sigma,E) \leq S(f_\sigma,E)$$

$$\leq m(E) \sup f_{\sigma_0}(E) + k\,\varepsilon.$$

Since ε is arbitrary and $f_{\sigma_0}(E)$ is connected, we have

$$I(f,E) \in m(E)\, f_{\sigma_0}(E)^-$$

$$S(f,E) \in m(E)\, f_{\sigma_0}(E)^-$$

which gives the desired result

$$\int_E f\,dm \subset m(E)\, f_{\sigma_0}(E)^- .$$

4. The n-Integral and the Usual Integral

In this part we give some results relating the integral of an n-function with that of the "Riemann integral" of a usual function when the n-function is constructed from the usual function in a certain natural way.

Let us assume that ϕ is a usual function

$$\phi\colon X \longrightarrow R^+$$

which is m-integrable on $E \subset X$ and f is the positive real-valued n-function

$$f\colon (X, \mathcal{V}) \longrightarrow [R^+, \mathcal{V}_{R^+}]$$

constructed from ϕ by the image method. That is

$$f_\sigma(A) = \natural\phi(A)$$

$\forall A \in \sigma, \forall \sigma \in \mathcal{V}$.

We think it reasonable to extend the definition of Riemann integral as follows.

Definition X: Define the "Riemann Integral" on $E \subset X$

$$\int_E \phi \, dm$$

for $\phi: X \longrightarrow R^+$ in the following manner. Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of partitions of E, a set of non-zero measure. $P_n \in \{P_n\}_{n=1}^{\infty}$ being a partition means that

$$E = \{ \bigcup_{i=1}^{\infty} H_{in} : H_{in} \in P_n\}$$

$$H_{in} \cap H_{jn} = \emptyset , \quad i \neq j$$

$$m(H_{in}) > 0, \quad \forall \; i,n \quad 1,2....$$

$$\lim_{n\to\infty} \sup_i \{m(H_{in}) : H_{in} \in P_n\} = 0, \quad \forall \; i,$$

Define

$$I_n\phi = \sum_{i=1}^{\infty} m(H_{in}) \inf \phi(H_{in}),$$

$$S_n\phi = \sum_{i=1}^{\infty} m(H_{in}) \sup \phi(H_{in}).$$

When the below limits exist, we define

$$I\phi = \lim_{n\to\infty} I_n\phi$$

$$S\phi = \lim_{n\to\infty} S_n\phi$$

In the case $I\phi = S\phi < \infty$ for all possible sequences of partitions, it is said that ϕ is "Riemann integrable" on E and

$$\int_E \phi \text{ dem} = I\phi = S\phi \text{ .}$$

<u>Theorem 7</u>: Let

$$f\colon (X, \vartheta) \longrightarrow [R^+, \vartheta_{R+}]$$

be the n-function constructed from $\phi\colon X \longrightarrow R^+$ by the image method, with ϕ "Riemann" integrable on $E \subset X$. That is

$$\int_E \phi \, dm < \infty$$

Then with the below restrictions

$$\int_E fdm = \int_E \phi dm.$$

Hypothesis:

(i) ϑ becomes arbitrarily small relative to any partition P_n and the measure m.

(ii) f is bounded by k on E relative to ϑ.

Proof: We proceed in steps.

(a) We can write

$$S_n\phi = m\{ \bigcup_{i=1}^{\infty} (H_{in} \times \overline{\phi(H_{in})} : H_{in} \in P_n\}$$

$$= m\{ \bigcup_{i=1}^{\infty} (H_{in} \times \phi(H_{in})) \bigcap (E\times R) : H_{in} \in P_n\} \qquad (1)$$

$$I_n\phi = m\{ \bigcup_{i=1}^{\infty} (H_{in} \times \underline{\phi(H_{in})}) : H_{in} \in P_n\}$$

$$= m\{ \bigcup_{i=1}^{\infty} (H_{in} \times \underline{\phi(H_{in})}) \bigcap (E\times R) : H_{in} \in P_n\} \qquad (2)$$

Proof: By definition

$$S_n\phi = \sum_{n=1}^{\infty} m(H_{in}) \sup \phi(H_{in}), H_{in} \in P_n$$

$$= \sum_{n=1}^{\infty} m(H_{in}) \lambda (\overline{\phi(H_{in})}), H_{in} \in P_n$$

$$= \sum_{n=1}^{\infty} m(H_{in} \times \overline{\phi(H_{in})}), H_{in} \in P_n$$

$$= m \{ \bigcup_{i=1}^{\infty} (H_{in} \times \overline{\phi(H_{in})}) : H_{in} \in P_n\}$$

$$= m \bigcup_{i=1}^{\infty} (H_{in} \times \overline{\phi(H_{in})}) \bigcap (E\times R) : H_{in} \in P_n\}$$

since the H_{in} are disjoint and are subsets of E. Equations (2) are demonstrated similarly.

(b)

$$\overline{f_\sigma(A)} = \overline{\phi(A)}$$

$$\underline{f_\sigma(A)} = \underline{\phi(A)}$$

Proof: This is obvious since

$$f_\sigma(A) = \$\phi(A) = (\inf \phi(A), \sup \phi(A))$$

and

$$\overline{f_\sigma(A)} = (0, \sup f_\sigma(A)]$$

Because of this result in the definition of the integral of f we will use the right-hand sides of Expressions (3) instead of the left-hand sides without stating so explicity.

(c) Some convenient notation.

$$\sigma_n^* = \{A^* : A^* \in \sigma_n, A \subset F, \text{ some } F \in P_n\}$$

$$\sigma_n^{**} = \{A^{**} : A^{**} \in \sigma_n, A \not\subset F, \forall F \in P_n, A \cap E \neq \emptyset\}$$

where σ_n is the cover associated with P_n by the meaning of Conditions (i) and (ii).

(d) Let ϵ be a real number greater than zero.

$$S(f_{\sigma_n}, E) \leq S_n\phi + K\epsilon$$

Proof: Let P_n be a partition of E and σ_n the cover associated with P_n by Conditions (i) and (ii).

$$Sf_{\sigma_n} = \{\bigcup_A (A \times \overline{f_\sigma(A)}) : A \in \sigma_n, A \cap E \neq \emptyset\}$$

$$= \{\bigcup_A (A \times \overline{\phi(A)}) : A \in \sigma_n, A \cap E \neq \emptyset\}$$

$$= \{\bigcup_{A^*} (A^* \times \overline{\phi(A^*)}) : A^* \in \sigma_n^*\} \cup \{\bigcup_{A^{**}} (A^{**} \times \overline{\phi(A^{**})}) : A^{**} \in \sigma_n^{**}\}$$

since

$$\sigma_n^{**} \cup \sigma_n^* = \{A : A \in \sigma_n, A \cap E \neq \emptyset\}$$

By Condition (ii)

$$\{\bigcup_{A^{**}}(A^{**} \times \overline{\phi(A^{**})}) : A^{**} \in \sigma_n^{**}\} \subset \{\bigcup_{A^{**}}(A^{**} \times (0,k]) : A^{**} \in \sigma_n^{**}\}$$

Then by Condition (i)

$$m\{\bigcup_{A^{**}}(A^{**} \times (0,k]) : A^{**} \in \sigma_n^{**}\} < k\epsilon$$

So

$$m\{\bigcup_A (A \times \overline{\phi(A)}) : A \in \sigma_n, A \cap E \neq \emptyset\}$$

$$\leq m\{\bigcup_A (A^* \times \overline{\phi(A^*)} : A^* \in \sigma_n^*\} + k\,\varepsilon \qquad (4)$$

Also since each $A^* \subset H$ some $H \in P_n$ and $\overline{\phi(A^*)} \subset \overline{\phi(H)}$

We have

$$\{\bigcup_{A^*} (A^* \times \overline{\phi(A^*)}) \cap (E \times R) : A^* \in \sigma_n^*\}$$

$$\subset \{\bigcup_H (H \times \overline{\phi(H)}) \cap (E \times R) : H \in P_n\} \qquad (5)$$

Equation (5) gives

$$m\{\bigcup_{A^*} (A^* \times \overline{\phi(A^*)} : A^* \in \sigma_n^*\}$$

$$\leq \{\bigcup_H (H \times \overline{\phi(H)}) : H \in P_n\} \qquad (6)$$

Equations (4) and (6) then give

$$S(f_{\sigma_n}, E) = m\{\bigcup_A (A \times \overline{\phi(A)}) \cap (E \times R) : A \in \sigma_n\}$$

$$\leq m\{\bigcup_H (H \times \overline{\phi(H)}) \cap (E \times R) : H \in P_n\} + k\,\varepsilon$$

$$= S_n\phi + k\,\varepsilon$$

by (a).

(e) $$I(f_{\sigma_n}, E) \geq I_n - k\,\varepsilon$$

Proof: First we observe that

$$\{\bigcup_A (A \times \underline{\phi(A)}) \cap (E\times R) : A \in \sigma_n\} \supset \{\bigcup_{A^*} (A^* \times \underline{\phi(A^*)}) \cap (E\times R) : A^* \in \sigma_n^*\}$$

by the meaning of σ_n^*. So

$$m\{\bigcup_A (A \times \underline{\phi(A)}) \cap (E\times R) : A \in \sigma_n\}$$

$$\geq m\{\bigcup_{A^*} (A^* \times \underline{\phi(A^*)}) \cap (E\times R) : A^* \in \sigma_n^*\} \qquad (7)$$

Let $H_{in} \in P_n$ and define the following symbols

$$E_n = \{\bigcup_{A^{**}} A^{**} : A^{**} \in \sigma_n^{**}\}$$

$$H'_{in} = H_{in} - E_n, \quad \forall\, H_{in} \in P_n.$$

The property we will need of H'_{in} is that

$$H'_{in} \subset \{\bigcup_{A^*} A^* : A^* \in \sigma_n^*\} .$$

So

$$\{\bigcup_{i=1}^{\infty} (H'_{in} \times \underline{\phi(H_{in})}) \cap (E\times R) : H_{in} \in P_n\} \cup \{E_n \times (0,k]\}$$

$$\supset \{\bigcup_{i=1}^{\infty} (H_{in} \times \underline{\phi(H_{in})}) \cap (E\times R) : H_{in} \in P_n\}$$

and therefore

$$m \left\{ \bigcup_{i=1}^{\infty} H'_{in} \times \underline{\phi(H_{in})} \right) \cap (E\times R) : H_{in} \in P_n \} + k\,\epsilon$$

$$\geq m\{ \bigcup_{i=1}^{\infty} (H_{in} \times \underline{\phi(H_{in})}) \cap (E\times R) : H_{in} \in P_n \} \qquad (8)$$

Since each $A^* \in \sigma_n^*$ is a subset of some $H_{in} \in P_n$ and

$$\underline{\phi(H_{in})} \subset \underline{\phi(A^*)},$$

it is not hard to see that

$$m \{ \bigcup_{i=1}^{\infty} (H'_{in} \times \underline{\phi(H_{in})}) \cap (E\times R) : H_{in} \in P_n \}$$

$$\leq m\{ \bigcup_{A^*} (A^* \times \underline{\phi(A^*)}) \cap (E\times R) : A^* \in \sigma_n^* \} \qquad (9)$$

Inequalities (7), (9) and (8) then give

$$I(f_\sigma, E) = m\{ \bigcup_{A} (A \times \underline{f_{\sigma_n}(A)}) \cap (E\times R) : A \in \sigma_n \}$$

$$\geq m\{ \bigcup_{A^*} (A^* \times \underline{\phi(A^*)}) \cap (E\times R) : A^* \in \sigma_n^* \}$$

$$\geq m\{ \bigcup_{i=1}^{\infty} (H'_{in} \times \underline{\phi(H_{in})}) \cap (E\times R) : H_{in} \in P_n \}$$

$$\geq m\{ \bigcup_{i=1}^{\infty} (H_{in} \times \underline{\phi(H_{in})}) \cap (E\times R) : H_{in} \in P_n \} - k\,\epsilon$$

$$= I_n\phi - k\,\varepsilon .$$

(f) $$\int_E f\,dm = \int_E \phi\, dm$$

Proof: (d) and (e) give us the inequality

$$I_n\phi - k\,\varepsilon < I(f_{\sigma_n}, E) \leq S(f_{\sigma_n}, E) \leq S_n\,\phi + k\,\varepsilon$$

After taking the limits we have

$$\int_E \phi\, dm - k\,\varepsilon = \lim_{n\to\infty} I_n\,\phi - k\,\varepsilon \leq \lim_{\sigma_n \varepsilon \mathcal{V}} I(f_{\sigma_n},E)$$

$$\leq \lim_{\sigma_n \varepsilon \mathcal{V}} S(f_{\sigma_n},E) \quad \lim_{n\leftarrow\infty} S_n\phi + k\,\varepsilon = \int_E \phi\, dm + k\,\varepsilon$$

Since ε is arbitrary we have

$$\int_E f\,dm = \int_E \phi\, dn$$

which completes the proof.

Corollary I: Let

$$\phi : R^n \longrightarrow R^+$$

be an usual function such that

$$\int_E \phi\, d\,\lambda < \infty$$

where λ is the Lebesque measure on R^n, $E \subset R^n$, and $\lambda(E)$ is finite. Let $\mathcal{V}$ be the canonical family of open coverings on R^n which is defined in the beginning of this section.

Now let

$$f : (R^n, \mathcal{V}) \longrightarrow [R^+, \mathcal{V}_{R^+}]$$

be the n-function constructed from by the image method. Then under the below restrictions.

$$\int_E f d\lambda = \int_E \phi \, d\lambda$$

(i) E is the finite union of connected sets of R^n.

(ii) ϕ is bounded on an open set, U, containing E^-. That is

$$\sup\{\phi(U)\} \leq k < \infty.$$

Proof: If $E \subset R^n$ is connected, then it is clearly possible to find a family of partitions $\{P_n\}_{n=1}^{\infty}$ satisfying the conditions of Theorem (7) relative to $\mathcal{V}$. The same is true for the define union of connected sets. Condition (ii) of the theorem is satisfied by Condition (ii) of this corollary.

Remark: With the Canónical Standard Family of open coverings, $\mathcal{V}$, of R^n and $E \subset R^n$, Condition (i) of the previous theorem is not satisfied. We feel that this corollary could be generalized in a number of different ways. For instance, define the integral

$$\int_E f d\lambda = \lim_{n\to\infty} \int_{E_m} f d\lambda$$

such that $\lambda(E_m) < \infty$ and $\bigcup_{m=1}^{\infty} E_m = R^n$ with $E_1 \subset E_2 \subset E_3 \ldots$.

We would like to emphasize strongly that we by no means feel that the previous two results, or any of the others, are the most general possible. As we mentioned earlier, it would be foolhardy to continually try to do everything in the most general context as possible during this stage of the development of NDA.

Counterexample I: The following example shows that using the image method of constructing an n-function, f, from a usual function, ϕ, we do not have

$$\int_E \phi \, d\lambda = \int_E f d\lambda$$

where $\int_E \phi \, d\lambda$ is the Lebesque integral. Let

$$\phi : R \longrightarrow R^+$$

be the Derichlet function. That is,

$$\phi(x) = \begin{cases} 1 & x \quad \text{irrational} \\ 0 & x \quad \text{rational} \end{cases}$$

Then

$$\int_0^1 \phi d\lambda = 1, \qquad E = [0,1]$$

But letting $\mathcal{V}$ be the canonical family of open coverings on E, we have

$$f_\sigma(A) = \sharp\{\phi(A)\} = (0,1).$$

So

$$If_\sigma(A) = A \times \underline{f_\sigma(A)} = A \times \emptyset$$

$$Sf_\sigma(A) = A \times \overline{f_\sigma(A)} = A \times [0,1]$$

which implies

$$I(f_\sigma,E) = 0, \quad E = [0,1]$$

$$S(f_\sigma,E) = m([0,1] \times [0,1])$$

$$= 1,$$

and therefore

$$I(f,E) = 0,$$

$$S(f,E) = 1,$$

and

$$\int_E f\, d\lambda = \sharp\{0,1\} = (0,1) \neq \int_0^1 \phi\, d\lambda.$$

We point out that the "problem" here may be in the method of constructing an f to associated with this pathelogical usual function. For instance, f does not generated ϕ in the sense defined in Reference[4] where the conditions under which an n-function generate a usual function is studied in detail.

5. The Fundamental Theorem of Calculus.

Definition XI: Let

$$f : (X, \mathcal{V}) \longrightarrow [R^+, \mathcal{V}_{R^+}]$$

be a positive real-valued n-function, which is m-integrable on all $A \varepsilon \sigma \varepsilon \mathcal{V}$, m a measure or X which may or may not be the Gauss measure constructed from $\mathcal{J}$, the standard family of coverings on X. Then define the integral n-function g of f by

$$g_\sigma(A) = \S\{ \int_B fdm : B \subset A, \ B \ \varepsilon \ \tau \quad \tau \geq \sigma\}$$

$\forall A \ \varepsilon \ \sigma, \forall \sigma \ \varepsilon \ \mathcal{V}$.

We remind the reader in the following that the restrictions on $\mathcal{J}$ and $\mathcal{V}$ given in the beginning of this section are still applicable.

The NDA analog of the Fundamental Theorem of Calculus is given by following theorem. Since derivation of an n-function is defined relative to a standard family of coverings, $\mathcal{J}$, on X, and since in integration one uses a measure m on X which does not necessarily have to have any connection with the Gauss structure, one needs to assume some relationship. This will be studied in detail in a future article, and here we content ourselves with the reasonably general theorem below.

Theorem 8: Let

$$g : (X, \mathcal{V}) \longrightarrow [R^+, \mathcal{V}_{R+}]$$

be the integral n-function of f

$$f : (X, \mathcal{V}) \longrightarrow [R^+, \mathcal{V}_{R+}]$$

which we assume to be continuous and integrable on all $A \in \sigma \in \mathcal{V}$. Then with the restrictions below

$$Dg_\sigma(A) \subset f_\sigma(A)^-$$

$\forall A \in \sigma \in \mathcal{V}$, which we also write as

$$Dg \subset f^- .$$

The connection between the Gauss structures $(X, \mathcal{J})$ and $(R, \mathcal{J}_R)$ with the measures m and λ on X and R respectively is

(i) $$\underline{\lim}_{\alpha\in\mathcal{J}} \frac{n(A',\alpha')}{n(A,\alpha)} = \overline{\lim}_{\alpha\in\mathcal{J}} \frac{n(A',\alpha')}{n(A,\alpha)} = \frac{\lambda(A')}{m(A)}$$

where $\alpha' = G(\alpha)$, G being the Gauss transformation between $\mathcal{J}$ and $\mathcal{J}_{R'}$ $A' \in \sigma' \in \mathcal{V}_{R'}$, $m(A) < \infty$, $\forall A \in \sigma \in \mathcal{V}$. $\mathcal{J}_R$ is the canonical standard family of coverings on R and λ is the Lebesque measure on R.

(ii) Let ε be a real number greater than zero. Then there exists a $\sigma \in \mathcal{V}$ such that $m(A) < \varepsilon$, $\forall A \in \sigma$.

(iii) $\mathcal{V}$ becomes arbitrarily small relative to any $A \in \sigma$, any $\sigma \in \mathcal{V}$. f is bounded on each $A \in \sigma \in \mathcal{V}$ relative to $\mathcal{V}$.

Proof: By the definition of the integral n-function g of f

$$g_\sigma(A) = \mathfrak{F}\{\int_B fdm: B \subset A,\ B \in \tau,\ \tau \geq \sigma,\ \tau \in \mathcal{V}\}$$

Since f is positive and by Condition (ii) the inf and sup of the above set is clearly given by

$$g_\sigma(A) = (0,\ \sup \int_A fdm)$$

$\forall A \in \sigma \in \mathcal{V}$. Then since

$$\lambda(g_\sigma(A)) = \sup \int_A fdm, \quad \forall A \in \sigma \in \mathcal{V},$$

we have

$$\underline{D}g_\tau(B) = \underline{\lim}_{\alpha \in \mathcal{F}} \frac{n(g_\tau(B),\alpha')}{n(B,\alpha)} = \frac{\lambda(g_\tau(B))}{m(B)}$$

$$= \frac{\sup \int_B f\,dm}{m(B)} = \overline{\lim}_{\alpha \in \mathcal{F}} \frac{n(g_\tau(B),\ \alpha')}{n(B,\alpha)} = \overline{D}g_\tau(B) \qquad (1)$$

by Condition (i). By definition

$$\sup \int_B fdm = S(f,B) = \lim_{\gamma \in \mathcal{V}} S(f_\gamma, B)$$

$$= \lim_{\gamma \in \mathcal{V}} m\{\bigcup_D (D \times \overline{f_\gamma(D)}) \bigcap (BxR) : D \in \gamma \geq \tau\} \qquad (2)$$

$$\leq \lim_{\gamma \in \mathcal{V}} m\{\bigcup_D (D \times \overline{f_\gamma(D)}) \bigcap (BxR) : D \subset B,\ D \in \gamma \geq \tau\}$$

$$+ k\, \varepsilon' m(B)$$

where ε' is any positive real number. This last inequality is true since Condition (iii) tells us that we can always find a $\gamma_0 \in \mathcal{V}$ such that for all $\gamma \geq \gamma_0$ the total measure of the sets which intersect B, but are not contained in B, is less than any given real number (in this case $\varepsilon' m(B)$). Also one needs the fact that f is bounded on A by some number k.

Continuing, with the observation that f is continuous, we have

$$\sup \int_B fdm \leq \lim_{\gamma \in \mathcal{V}} m\{\bigcup_D (D \times \overline{f_\tau(B)}) \bigcap (BxR) : D \subset B,\ D \in \gamma \geq \tau\}$$

$$+ k\, \varepsilon' m(B)$$

$$= \lim_{\gamma \in \mathcal{V}} m\{(\bigcup_D D \cap B) \times \overline{f_\tau(B)} : D \in \gamma \geq \tau,\ D \subset B\} + k\, \varepsilon' m(B)$$

$$= \lim_{\gamma \in \mathcal{V}} m\{(\bigcup_D D \cap B) : D \in \gamma \quad D \subset B\}\, \lambda\, (\overline{f_\tau(B)}) + k\, \varepsilon' m(B)$$

$$\leq \lim_{\gamma \in} m(B) \sup f_\tau(B) + k\, \varepsilon' m(B)$$

$$= m(B) \sup f_\tau(B) + k\, \varepsilon' m(B) \qquad (3)$$

So using (1) and (3) we obtain

$$\underline{Dg}_\tau(B) = \overline{Dg}_\tau(B) \leq \sup f_\tau(B) + k\, \varepsilon' \qquad (4)$$

Returning to Equation (2) we get

$$\sup_B \int f dm = S(f,B) = \lim_{\gamma \in \mathcal{V}} S(f_\gamma, B)$$

$$= \lim_{\gamma \in \mathcal{V}} m\{\bigcup_D (D \times \overline{f_\gamma(D)}) \cap (B \times R) : D \in \gamma \geq \tau\}$$

$$\geq \lim_{\gamma \in \mathcal{V}} m\{\bigcup_D (D \times \underline{f_\gamma}(D)) \cap (B \times R) : D \in \gamma \geq \tau\}$$

$$\geq \lim_{\gamma \in \mathcal{V}} m\{\bigcup_D (D \times \underline{f_\gamma}(D)) \cap (B \times R) : D \in \gamma \geq \tau, D \subset B\}$$

$$\geq \lim_{\gamma \in \mathcal{V}} m\{\bigcup_D (D \times \underline{f_\tau}(B)) \cap (B \times R) : D \in \gamma \geq \tau, D \subset B\}$$

$$= \lim_{\gamma \in \mathcal{V}} m\{(\bigcup_D D \cap B) \times \underline{f_\tau}(B) : D \in \gamma \geq \tau, D \subset B\}$$

$$= \lim_{\gamma \in \mathcal{V}} m\{\bigcup_D D \cap B) : D \in \gamma \geq \tau , D \subset B\}\ \lambda\ \underline{(f_\tau(B))}$$

$$= \lim_{\gamma \in \mathcal{V}} m(B) \inf f_\tau(B)$$

$$= m(B) \inf f_\tau(B)\,. \tag{5}$$

Then using (1) and (5)

$$\underline{Dg}_\tau(B) = \overline{Dg}_\tau(B) \geq \inf f_\tau(B) \tag{6}$$

This implies by the definition of derivative and by (4) and (6) that

$$\overline{\underline{Dg}}_\sigma(A) = \{\underline{Dg}_\tau(B), \overline{Dg}_\tau(B) : B \subset A,\ B \in \tau \geq \sigma\}$$

$$\subset \{\inf f_\tau(B),\ \sup f_\tau(B) : B \in \tau \geq \sigma,\ B \subset A\} \pm k\,\varepsilon'$$

Since f is continuous

$$\sup \underline{\overline{Dg}}_\sigma(A) \leq \sup f_\sigma(A) + \varepsilon' \qquad \varepsilon \quad \{f_\sigma(A) \quad + \varepsilon'\}$$

$$\inf \underline{\overline{Dg}}_\sigma(A) \geq \inf f_\sigma(A) \qquad \varepsilon \quad f_\sigma(A)$$

and thus

$$Dg_\sigma(A) = \$\ \{\underline{\overline{Dg}}_\sigma(A)\} \quad \subset \quad f_\sigma(A) \quad \pm \quad \varepsilon'$$

Since ε' is arbitrary we then have the desired result

$$Dg_\sigma(A) \subset f_\sigma(A)^-$$

Corollary II: If

$$f: (R, \mathcal{V}) \longrightarrow [R^+, \mathcal{V}_{R^+}]$$

is a continuous, positive, real-valued n-function which is integrable on all $A \in \sigma \in \mathcal{V}$ with $\mathcal{V}$ the canonical standard family of open coverings on R, $(R, \mathfrak{J}_R)$ the canonical Gauss space, and G the identity Gauss transformation, then

$$Dg \subset f^-.$$

Proof: Lemma III.1 in Reference [2] shows condition (i) of the theorem is satisfied. Also $\mathcal{V}$ clearly satisfies Conditions (ii) and (iii).

6. The n-Integral of a Non-Positive n-Function

Up to now we have only defined the integral of a positive real-valued n-function. We could proceed in various ways as with the case of defining the integral of a usual function which may take on negative values. For the time being the following seems to be the most convenient.

Definition XII: Assume $|f|$ is bounded by k on $E \subset X$, that is,

$$|\sup f_\sigma(A)| \leq k$$

$\forall\ A \in \sigma,\ A \cap E \neq \emptyset, \forall\ \sigma \in \mathcal{V}$. Then

$$\int_E f dm = \int_E f^k dm - \int_E k dm$$

where

$$f_\sigma^k(A) = f_\sigma(A) + k$$

$$k_\sigma(A) = \{k\}$$

$\forall\ A \in \sigma \in \mathcal{V}$.

It is clear that

$$\int_E k dm = km(E)$$

In a future article the theorems proved for a positive n-function in this paper will be studied for an arbitrary real-

-valued n-function. We content ourselves here with the following theorem.

Theorem 9: With the notation and condition of Theorems 2 and 3 we have

$$\text{(i)} \qquad \alpha \int_E f\,dm = \int_E \alpha\, f\,dm$$

$$\text{(ii)} \qquad \int_{E_1} f\,dm + \int_{E_2} f\,dm = \int_{E_1 \; E_2} f\,dm$$

where $\alpha \in R$ is a real constant and f is bounded by k on E.

Proof: (i):

$$\begin{aligned}
\alpha \int_E f\,dm &= \alpha\left[\int_E f^k dm - m(E)k\right] \\
&= \alpha \int_E f^k dm - m(E)\, k\alpha \\
&= \int_E \alpha f^k dm - m(E)\, k\alpha \\
&= \int_E f\,dm
\end{aligned}$$

(ii): By definition we have

$$\begin{aligned}
&\int_{E_1} f\,dm + \int_{E_2} f\,dm \\
&= \int_{E_1} f^k dm - km(E_1) + \int_{E_2} f^k dm - km(E_2) \\
&= \int_{E_1} f^k dm + \int_{E_2} f^k dm - km(E_1) - km(E_2)
\end{aligned}$$

$$= \int_{E_1 \cup E_2} f^k dm - km(E_1 \cup E_2)$$

$$= \int_{E_1 \cup E_2} f dm$$

§ IV

PHYSICAL INTERPRETATION OF NON-DETERMINISTIC MATHEMATICS

1. In this section we apply basic concepts of non-deterministic mathematics, like derivative and integral of n-functions, to describe fundamental notions in physics connected with motion of particles and fields which might be of some relevance for quantum mechanics. Same investigation in this direction has been initiated in [6] and here we recall and generalize some of those ideas introducing as well several new ones.

The fundamental idea of this approach is to accept that even though the notion of metric space is convenient for macroscopic phenomena it is not so for microscopic ones. In more detail, our picture of the would comes first through our senses and then our imagination works out a form of expression for it. This form changes from time to time and it is conditioned by the basic beliefs of our civilization, namely, for the greek civilization the picture of the cosmos was different from that of the arabian civilization, or our western civilization. So in the end physics will provide a picture of the world according to the basic symbols of each civilization, which in our case are energy, time, dynamic space. In this way if we analyse it closer we notice that the evolution of classical mechanics, for example, started with the concept of Newtonian force and finally, in the hands of Lagrange, reached the stage that we have a space - the euclidean 3-dimensional space-together with a dynamic structure given by the idea of motion conditioned to a cer tain distribution of energy in space, as we see from the concept of Lagrangian. So the internal beauty of classical mechanics relies on that harmony between a certain space where the motion of bodies takes place and the dynamical variables conditioning that motion. In modern days this spirit is still there for all microscopic phenomena although the space in question is not necessarily R^3. For instance, in dynamical systems the space is a differentiable manifold M^n of dimension n and the dynamics structure is given by the action of a Lie groups on M^n; for the generalized theory of relativity the space is a four dimensional Riemannian manifold and the dynamical structure is given by Einstein's equations expressed for convenience in tensor form.

Now the question arises: for microscopic phenomena what is the space where motion takes place? If we look back to the history of atomic theory we notice that the first idea was to suppose that the space in question still is our old and dear euclidean space R^3. So the atom of Rutherford is nothing else but a statement in this belief. But soon it was found that such a model was in clear contradiction with eletromagnetic theory, because an atom with such structure would collapse immediately due to the loss of radiating energy. Then came N. Bohr with his hypothesis of orbits of stability and basically the same model of Rutherford was maintained together with that additional hypothesis taking care of the stability of the atom. Then a nice accordance of this model of the atom with spectroscopic evidence was reached and the theory was quite satisfactory, at least for the hydrogen atom. But for atoms with many electrons things did not work so well and then it was the begining of the tragedy: more hypothesis have to be introduced to take care of this and that and finally, due to Heisenberg's uncertainty principle, the conclusion was reached that the electron, for instance, was not a particle after all, in the sense we always understood the word, but it was instead "something" that does not have a path in the classical sense, but any how this something still lives in R^3! Now, how is it possible to have any kind of compatibility of R^3 with such concept of particle and its motion! For instance, does the electron have a volume?

If so we can consider parts of it, and all sorts of paradoxes will arrise in the line of well known and old philosophical considerations. Then it was thought : well let us forget about space and consider physical entities as pure mathematical symbols and then we get Schödinger's approach where the basic element is the wave function and the electron or any other particle will be represented by such a function satisfying Schrödinger's equation. All this is very nice and worked out beatifully as for as the agreement with experiment was concerned. However, the old question still remains involved namely, if there is any physical reality associated to a wave function and by God we want such a thing! - then such physical entity must be "somewhere in space". But what space? If the world in the small is still euclidian there is no possible compatibility between the geometric structure of the space and the dynamical structure given by Schrödinger's or Heisenberg's equations, because the concept of wave function or uncertainty principle are both incompatible with metric structure. Certainly we speak about "radius" of the nucleus, "distance" from electros to protons etc, but these can be only words, because if we take then seriously all kind of paradoxes take place and to overcome them more and more artificial hypothesis must be introduced.

Considering all that, why we just don't do the following: let us abandon the idea that the space at microscopic level is euclidian and instead assume that it is a topological space without relying on the concept of distance.

The deep philosophical question would be then: we must assume the existence in the real world of a structure in the small which cannot be grasped by our intuition. Well, as hard as it is to believe we must get use of it and learn to live with it, because there is no logical reason to conclude that it cannot be like that. The only reason to deny it is the one based on the usual meaning of the term intuition. But who ever said that reality must be necessarily only that which is accessible to our intuition? We have to get rid of that kind of prejudice and accept the existence of entities which can be detected indirectly by our senses but whose essence is not accessible to our intuition. If we agree on that, then the existence of a physical world, at least in the small, which is topological without notion of distance becomes acceptable.

Based on these consideration we discuss the possibility of building a mechanics in a topological space, In [6] a first attempt was done in that direction and here we try to generalize those ideas in particular by introducing the concept of energy, now possible because we have a theory of integration available in NDA.

2. We start by introducing some fundamental notions. Some of these have been discussed in [6].

<u>Definition I</u> - A germ in a pair $(X,\mathcal{V})$ is a collection of open sets $p = \{A_\sigma\}_{\sigma \in \mathcal{V}}$ with $A_\sigma \in \sigma \in \mathcal{V}$ such that:

$$G_I) \quad \bigcap_{\sigma \in \mathcal{V}} A_\sigma \neq \emptyset \ ;$$

$$G_{II}) \text{ if } \tau > \sigma, \text{ then } A_\tau \subset A_\sigma .$$

Given a set $E \subset X$ and a given $p \in \{A_\sigma\}_{\sigma \in \mathcal{V}}$ we denote by $p \cap E$ the set

$$\bigcap_{\sigma \in \mathcal{V}} A_\sigma \cap E$$

<u>Definition II</u> - A particle in $(X,\mathcal{V})$ is a continuous n--function

$$f: (R,\mathcal{U}) \longrightarrow (X,\mathcal{V}),$$

where R is the real line, such that if $q = \{\tilde{A}_{\tilde{\sigma}}\}_{\tilde{\sigma} \in \mathcal{U}}$ is a germ in $(R,\mathcal{U})$ then $\{f_{\tilde{\sigma}}(\tilde{A})\}$ belongs to a germ in $(X,\mathcal{V})$.

We make the important convention of do not distinguishing two particles if they agree on a cofinal subset of $\mathcal{U}$.

The set

$$C(f) = \bigcap_{\tilde{\sigma} \in \mathcal{U}} \overline{\bigcup_{\tilde{A} \in \tilde{\sigma}} f_{\tilde{\sigma}}(\tilde{A})}$$

is called the trajectory of f and due to Definition II is always non empty.

<u>Definition III</u> - A non-deterministic field, abbreviated, n-field, in $(X, \mathcal{V})$ is a continuous real n-function

$$\phi: \quad (X, \mathcal{V}) \longrightarrow [R^+, \mathcal{V}_R]$$

where R^+ is the set of non negative real numbers.

In [6] it was proposed an equation of motion

$$\text{(I)} \qquad \mu . D^2 f = \phi \circ f$$

for a particle f under the action of a n-field Φ, where μ was taken to be the mass of f. We intend to generalize this equation in several ways to cover the case of variable fields, variable mass, etc, in non-deterministic sense, which we shall clarify in the following.

<u>Definition IV</u> - A variable n-field in $(X, \mathcal{V})$, with respect to $(R, \mathcal{U})$ is a n-function.

$$\phi^t: (X \times R, W) \longrightarrow [R, \mathcal{V}_R]$$

where $W \subset \mathcal{V} \times \mathcal{U}$ and R is the set of real numbers.

The exponent t recalls the time and $(R, \mathcal{U})$ is called the <u>time scale</u>. Sometimes a n-field as defined before is called a <u>stationary n-field</u>.

Next we want to introduce what we call <u>attributes of a particle</u>. So let us consider a particle in $(X, \mathcal{V})$

$$f: \quad (R, \mathcal{U}) \longrightarrow (X, \mathcal{V})$$

and for every $\tilde{\sigma} \in \mathcal{U}$ let us define a collection $\sigma(f)$ of open sets in X by

$$\sigma(f) = \{A \in \sigma = f_{\mathcal{U}}(\tilde{\sigma}) : A = f_{\tilde{\sigma}}(\tilde{A}), \quad \tilde{A} \in \tilde{\sigma}\}.$$

Call

$$\mathcal{V}(f) = \{\sigma(f)\} \subset \mathcal{V}$$

i.e., the family of all $\sigma(f)$ of all $\tilde{\sigma} \in \mathcal{U}$. Then an <u>attribute of f</u> is a continuous n-function from $(X, \mathcal{V}(f))$ into $[R, \mathcal{V}_R]$. For instance, the mass of a particle is an attribute usually given by a constant n-function or in relativistic situation a n-function

$$\mu : (X, \mathcal{V}(f)) \longrightarrow [R, \mathcal{V}_R]$$

In the same way we can define electric charge, kinetic energy, etc as attributes of f depending on the particular situation we have in mind. As we proceed the use of attributes will be made clear.

Let us now introduce the notion of <u>periodical n-field</u>.

Let

$$\Phi^t : (X \times R, W) \longrightarrow [R, \mathcal{V}_R]$$

be a variable n-field in $(X, \mathcal{V})$, with respect to $(R, \mathcal{U})$ and assume the following:

(P_I) there is a constant $K > 0$ such that for every $a \in R$ and every germ p in $(X, \mathcal{V})$ if we select any covering $\sigma_\omega \in W$

and any $A_\omega = A \times \tilde{A} \in \sigma_\omega$ with $A \in \sigma$, $A \in p$, $\tilde{A} \in \tilde{\sigma}$

we have

$$\Phi_{\sigma_\omega}(A \times \tilde{A}) = \Phi_{\sigma_\omega}(A \times \tilde{B}) \qquad \text{(I)}$$

for any $\tilde{B} \in \tilde{\sigma}$ with $a \in \tilde{A}$ and $a + K \in \tilde{B}$

(P_{II}) the infimum of all K satisfying P_I is a positive number T.

Under conditions (P_I) and (P_{II}), Φ^t is called a <u>periodical n-field</u>, with period T.

If (I) is true only for a particular germ p of $(X, \mathcal{V})$ we say that Φ^t is <u>periodic at p</u> and T is called the <u>period of Φ^t at p</u>, indicated by T_p.

The frequency of Φ^t is given by

$$\nu = \frac{1}{T}$$

and analogously for the frequency ν_p at p.

<u>Definition V</u> - A physical system is a Gauss space $(X, \mathcal{F})$ together with particles and fields both stationnary and variable defined on it.

As an illustration for this definition we introduce the hydrogen atom seen from the point of view of non-deterministic mathematics. Let $(X, \mathcal{F})$ be a Gauss space with an open covering Γ given by a particular system of countably many open sets $V_1, V_2 \ldots V_n \ldots$ such that

(H_1) $V_1 \subset V_2 \subset \cdots \subset V_n \subset V_{n+1} \subset \cdots$

(H_2) there is a n-field Φ

$$\Phi\colon (X,\mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$$

such that for any $\sigma \in \mathcal{V}$ and any $A \in \sigma$ with $A \cap \mathrm{bd}\, V_n \neq \emptyset$ ($\mathrm{bd}\, V_n$ is the boundary of V_n) implies that

$$\Phi_\sigma(A) = 0$$

(H_3) there is an open subset N of X with $N \subset V_1$ such that for any $\sigma \in \mathcal{V}$ and any $A \in \sigma$ with $A \cap N \neq \emptyset$ implies

$$\Phi_\sigma(A) = 0$$

and N is called the nucleus of $(X,\mathcal{F})$

In general we call <u>stationary orbit for a n-field</u> Φ a <u>sub</u>set S of X satisfying condition (H_2) above, i.e., if $A \in \sigma \in \mathcal{V}$ is such that $A \cap S \neq \emptyset$ then $\Phi_\sigma(A) = 0$.

The stationary orbits of $(X,\mathcal{F})$ are written as $S_1, S_2 \ldots S_n \ldots$ and the numbers $1, 2, \ldots n \ldots$ are called <u>orbital quantum numbers</u>.

A Gauss space $(X,\mathcal{F})$ satisfying conditions (H_1), (H_2), (H_3) is called an <u>hydrogen atom</u> if there is a particle

$$f\colon (R,\mathcal{U}) \longrightarrow (X,\mathcal{V})$$

whose trajectory contains some stationary orbit of $(X,\mathcal{F})$.

The particle f is called an <u>electron</u> and has several attributes which are considered depending on the conditions relevant for a certain experiment we might be interested in a certain moment.

Other kind of atoms can be defined in similar way by adding more electrons to $(X, \mathfrak{F})$.

Sometime we are interested in the nucleus of $(X, \mathfrak{F})$ and to describe the action of nuclear forces when we have several particles in N we have to introduce appropiate n-fields, but the dynamics of the whole thing depends on the equation of motion.

In summary the main problem of non-deterministic mechanics is the following: given a physical system to study the equation of motion of the several particles involved as well as interactions among then and the n-fields in questions. The meaning of this problem will be clarified as we proced.

3. To give a first step in the solution of the main problem above we shall study now the relation between particle and n-field.

<u>Definition VI</u> - A variable n-field in $(X, \mathcal{V})$

$$\phi^t : (X \times R, \mathcal{W}) \longrightarrow [R, \mathcal{V}_R]$$

with respect to $(R, \mathcal{U})$ and a particle

$$f: (R, \mathcal{U}) \longrightarrow (X, \mathcal{V})$$

are dual to each other if for any $\sigma_\omega = \sigma \times \tilde{\sigma} \in W$ and any $A_\omega \in \sigma_\omega$, $A_\omega = A \times \tilde{A}$ we have

$$\phi^t_{\sigma_\omega}(A \times \tilde{A}) = 0 \quad \text{if} \quad A \cap f^{\sim}_{\sigma}(\tilde{A}) = \emptyset .$$

We also say that ϕ^t generates f or f generates ϕ^t.

Theorem 1 - Let $(X, \mathfrak{F})$ be a measurable Gauss space and m the measure generated by $\mathfrak{F}$. Let

$$f: (R, \mathfrak{U}) \longrightarrow (X, \mathcal{V})$$

be a particle in $(X, \mathcal{V})$ with attribute

$$h: (X, \mathcal{V}(f)) \longrightarrow [R, \mathcal{V}_R]$$

such that all sets in the coverings of $\mathcal{V}$ are m-measurable.

Then f generates a variable n-field in $(X, \mathcal{V})$ with respect to $(R, \mathfrak{U})$.

Proof - Let us consider W the family of all coverings $\sigma_\omega = \sigma \times \tilde{\sigma}$ with $\sigma \in \mathcal{V}$ and $\tilde{\sigma} \in \mathfrak{U}$. Take one of these and consider $A_\omega = A \times \tilde{A} \in \sigma_\omega$, $A \in \sigma$, $\tilde{A} \in \tilde{\sigma}$

Let us define $\phi^t_{\sigma_\omega}(A_\omega)$ as follows;

if $A \cap f^{\sim}_{\sigma}(\tilde{A}) = \emptyset$ put

$$\phi^t_{\sigma_\omega}(A_\omega) = 0$$

Otherwise put

$\phi^t_{\sigma_\omega}(A_\omega)$ = the smallest open interval in R, symmetric relative to 0 and containing

$$m[A \cap f_{\sigma}^{\sim}(\tilde{A})] \cdot h_{\sigma_{(f)}} [f_{\sigma}^{\sim}(\tilde{A})]$$

So every $A_\omega \in \sigma_\omega$ will define either a point or an open interval in R hence to σ_ω it is associated a collection of subsets of R belonging to $\mathcal{V}_R$ which we define as

$$\phi^t_{\mathcal{W}}(\sigma_\omega)$$

and this gives a function

$$\phi^t_{\mathcal{W}} : \mathcal{W} \longrightarrow \mathcal{V}_R$$

together with a function

$$\phi^t_{\sigma_\omega} : \sigma_\omega \longrightarrow \sigma_R = \phi^t_{\mathcal{W}}(\sigma_\omega)$$

as defined above. Therefore we have a n-function.

$$\phi^t : (X \times R, \mathcal{W}) \longrightarrow [R, \mathcal{V}_R].$$

This n-function is continuous because if $\tau_\omega > \sigma_\omega$ we have for $B \times \tilde{B} \subset A \times \tilde{A}$, $B \times \tilde{B} \in \tau_\omega$ and $A \times \tilde{A} \in \sigma_\omega$,

$$\phi^t_{\tau_\omega}(B \times \tilde{B}) \subset \phi^t_{\sigma_\omega}(A \times \tilde{A})$$

in all possible cases, due to the continuity of h and f and the fact that for any two measurable sets E and F

$$E \subsetneq F \Longrightarrow m(E) \leq m(F)$$

Therefore ϕ^t is a variable n-field and it is generated by f as follows immediately from the definition of ϕ^t, this completes the proof.

The variable n-field generated by f in this theorem may have the peculiarity of being zero for a fixed germ p of $(X, \mathfrak{V})$ outside a certain interval I_p of length T_p of R and we shall call such ϕ^t a <u>radiation</u>.

We call T_p the <u>period of</u> ϕ^t at p and $\nu_f = 1/T_p$ its frequency at p, all these concepts being relative to the germ p in question. If for some particular case T_p does not depend on p we say that the particle f generates a <u>radiation of frequency</u> $\nu = \frac{1}{T}$.

Let us discuss in more detail the radiation generated by a particle f. If we fix $\sigma_\omega \in \mathfrak{W}$ and $A_\omega = A \times \tilde{A} \in \sigma \times \tilde{\sigma} = \sigma_\omega$ and let A be fixed and $\tilde{A}$ changing.

Let us look to the interval I in R of length T_p discussed above namely, p is a germ containing A and for all $\tilde{A} \in \tilde{\sigma}$, with $\tilde{A} \cap I = \emptyset$ we have

$$f_{\tilde{\sigma}}(\tilde{A}) \cap A = \emptyset$$

and

$$\phi^t_{\sigma_\omega}(A \times \tilde{A}) = 0$$

. Intuitively when time is running the particle f is moving and touches A for all times in I and does not touch A for times not in I. Therefore the period T_p will depend on the velocity of f, i.e., Df and we might ask how T_p can be expressed as a function of Df . To be in accordance with Einstein's relation, $E = h\nu$, in the case of a particle moving under the influence of any n-field we must assume

$$E = \frac{\mu}{2} (Df)^2$$

where μ is the mass of f and by are previous definition of μ, E is a n-function

$$E; (X, \mathcal{U}(f)) \longrightarrow [R, \mathcal{U}_R].$$

So Einstein's equation will give

$$\frac{\mu}{2} (Df)^2 = h\nu_p = h/T_p$$

In the same way we can define the wave length of Φ^t by De Broglies's equation $p = h/\lambda$, which gives,

$$\mu\ Df = h/_{\lambda_p}$$

or

$$\lambda_p = h/\ \mu .\ Df$$

and again λ_p is a n-function

$$\lambda_p: (X, \mathcal{V}(f)) \longrightarrow [R, \mathcal{V}_R].$$

As we see we have a much more general situation than in the case of usual considerations in physics where period and frequency are constant. Here the concept of particle and wave are much more rich and flexible and we believe that this approach has much more chances to explain larger number of phenomena then the usual one. In forthcoming papers we shall deal with these questions in more detail.

To finish this paragraph we analyse briefly the converse of theorem 1, i.e., when a n-field generates a particle.

Let ϕ^t be a variable n-field

$$\phi^t: (X \times R, \mathcal{W}) \longrightarrow [R, \mathcal{V}_R]$$

with respect to $(R, \mathcal{U})$.

We say that ϕ^t is <u>regulated</u> if there is a function

$$\theta: \mathcal{U} \longrightarrow \mathcal{W}$$

such that

$$\tilde{\sigma}, \tilde{\tau} \in \mathcal{U}, \quad \tilde{\sigma} < \tilde{\tau} \implies \theta(\tilde{\sigma}) < \theta(\tilde{\tau})$$

<u>Theorem 2</u>: If ϕ^t is a regulated radiation then ϕ^t generates a particle.

<u>Proof</u>: We have to build a particle

$$f: (R, \mathcal{U}) \longrightarrow (X, \mathcal{V})$$

satisfying Definition VI. As ϕ^t is a radiation, by definition, to every germ p in $(X, \mathcal{V})$ it is associated on interval I_p such that for every $\sigma_\omega \in \mathcal{W}$; $\sigma_\omega = \sigma \times \tilde{\sigma}$ and $A \in \sigma$, $A \in p$ we have

$$\phi^t_{\sigma_\omega} (A \times \tilde{A}) = 0$$

if $\tilde{A} \cap I_p = \emptyset$, where $\tilde{A} \in \tilde{\sigma}$. Take any $\tilde{\sigma} \in \mathcal{U}$ and look to $\theta(\tilde{\sigma}) = \sigma x \tilde{\sigma} \in \mathcal{W}$ and define

$$f_{\mathcal{U}}: \mathcal{U} \longrightarrow \mathcal{V}$$

by

$$f_{\mathcal{U}}(\tilde{\sigma}) = \sigma \in \mathcal{V}.$$

Now let $\tilde{A} \in \tilde{\sigma}$ arbitrary and consider all germs p in $(X, \mathcal{V})$ such that $\tilde{A} \cap I_p \neq \emptyset$. Each of these germs has an open set in $\sigma = f_{\mathcal{U}}(\tilde{\sigma})$ and call A the union of all these sets and define

$$f_{\tilde{\sigma}} : \tilde{\sigma} \longrightarrow \sigma$$

by

$$f_{\tilde{\sigma}} (\tilde{A}) = A$$

Hence we have defined a n-function

$$f: (R, \mathcal{U}) \longrightarrow (X, \mathcal{V})$$

We have to show that f is continuous. Take $\tilde{\sigma}$, $\tau \in \mathcal{U}$ with $\tilde{\sigma} < \tilde{\tau}$. By the definition of θ

$$\theta(\tilde{\sigma}) = \sigma \times \tilde{\sigma} < \tau \times \tilde{\tau} = \theta(\tilde{\tau})$$

what implies that $\sigma < \tau$, or

$$f_{\mu}(\tilde{\sigma}) < f_{\mu}(\tilde{\tau}).$$

Next let $\tilde{B} \in \tilde{\tau}$ and $\tilde{A} \in \tilde{\sigma}$, $\tilde{B} \subset \tilde{A}$. By the definition of germ we have

$$f_{\tilde{\sigma}}(\tilde{B}) \subset f_{\tilde{\sigma}}(\tilde{A})$$

and this proves that f is continuous completing the proof of the theorem.

Here again Einstein's equations can be used to define the attributes of f. For instance, f will carry energy equal to h_{ν} where ν is the frequency of ϕ^{t} as discussed before, etc.

4. Our aim now is to generalize the equation of motion and study an application to the problem of absorption and emission of radiation.

First we remark that according to our considerations we have a unified view of the concept of particle and field and there fore the same model is good for the study of interaction of particle x field, particle x particle, field x field, because all reduces to n-functions. The same wile apply to emission and absorption of radia tion, because we can look to radiation as a particle or as a variable n-field according to our convenience, as everything reduces to n-function all this will be made clear as we proceed.

In the second place, when we say that in (X, ϑ) there is a stationary n-field Φ, that is more a mathematical statement than a physical one, because what is observable is not Φ itself but rather its interaction with particles. We do not see any other way how to verify the existence, for example, of an electric field at a point P, but observing its action on a particle at P. In the summary we claim that <u>only particles or variable fields are observables</u> .

Therefore an apparatus is an object which by the action of particles or variable fields will produce a sign or collection of signs which can be detected by an observer and only then we can say that such particle or variable field does exist. So if there are in the Universe stationary fields which so far never interacted with particles, according to our point of view, they do not exist. It is very important to fix from the begining what we mean by existence in physics. It is a simple convention to make a statements to have a physical meaning. We do not deny that the concept of existence can also be studied and discussed from other points of view, phylosophical, theological, etc, but for our needs at this moment they are useless.

In [6] we introduced the equation of motion of a particle f under the action of a stationary n-field Φ as

$$\mu D_f^2 = \Phi \circ f \qquad (I)$$

To generalize this equation to cover the case of variable n-fields the first thing we have to be concerned with is the question of sign because a variable n-field can have also negative values. We take care of this by looking to μ as a n-function

$$\mu: \ (R, \mathcal{U}) \longrightarrow [R, \mathcal{V}_R]$$

as we did before. This been settled let us consider a variable n--field

$$\phi^t: \ (X \times R, \mathcal{W}) \longrightarrow [R, \mathcal{V}_R]$$

in $(X, \mathcal{V})$ with respect to $(R, \mathcal{U})$. If

$$f: \ (R, \mathcal{U}) \longrightarrow (X, \mathcal{V})$$

is a particle in $(X, \mathcal{V})$ we define a n-function

$$\hat{f}: \ (R, \mathcal{U}) \longrightarrow (X \times R, \mathcal{W})$$

as follows; define

$$f_{\mathcal{U}}: \mathcal{U} \longrightarrow \mathcal{W}$$

by

$$\hat{f}_{\mathcal{U}}(\tilde{\sigma}) = f_{\mathcal{U}}(\tilde{\sigma}) \times \tilde{\sigma}$$

for any $\tilde{\sigma} \in \mathcal{U}$. In general this does not need to be in $\mathcal{W}$ and so we need the following definition: the variable n-field ϕ^t <u>interacts with</u> f if all coverings of the type

$$f_{\mathcal{U}}(\tilde{\sigma}) \times \tilde{\sigma}$$

belong to $\mathcal{W}$ for all $\tilde{\sigma} \in \mathcal{U}$. Assuming this, $\hat{f}_{\mathcal{U}}$ is well defined.

Next take any $\tilde{\sigma} \in \mathcal{U}$ and any $\tilde{A} \in \tilde{\sigma}$. Define

$$\hat{f}_{\tilde{\sigma}}: \tilde{\sigma} \longrightarrow \sigma \times \tilde{\sigma} = f_{\mathcal{U}}(\tilde{\sigma})$$

by

$$\hat{f}_{\tilde{\sigma}}(\tilde{A}) = f_{\tilde{\sigma}}(\tilde{A}) \times \tilde{A}$$

Hence $\hat{f}$ is defined and it is continuous.

Now we define the equation of motion of f under the action of Φ^t by

$$(\mu o f)\ D_f^2 = \Phi^t \circ \hat{f} \qquad \text{(II)}$$

The case (I) is contained in (II) by introduction the n--function

$$q: (X \times R, \mathcal{W}) \longrightarrow (X, \mathcal{V})$$

defined by

$$q_\omega: \mathcal{W} \longrightarrow \mathcal{V}$$

is given by

$$q_\omega(\sigma \times \tilde{\sigma}) = \sigma$$

for all $\sigma_\omega = \sigma \times \tilde{\sigma} \in \mathcal{W}$ and if $A_\omega = A \times \tilde{A} \in \sigma_\omega$ is given, we define

$$q_{\sigma_\omega}: \sigma_\omega \longrightarrow \sigma$$

by

$$q_{\sigma_\omega}(A_\omega) = A \in \sigma \in \mathcal{V}$$

Then if Φ is a stationary n-field, to apply (II) we write

$$\Phi^t = \Phi \circ q \ ,$$

and since $g \circ \hat{f} = f$, equation (II) reduces to equation (I).

We this in mind we can write expressions like

$$\Phi + \Phi^t$$

where Φ really means $\Phi \circ q$ and Φ^t will be regarded as a pertubation of the n-field Φ. Also sometimes we write

$$\mu \, D_f^2 = (\Phi + \Phi^t) \circ f$$

to mean

$$(\mu \circ f) \, D_f^2 = (\Phi \circ g + \Phi^t) \circ f \ .$$

Clearly these simplifications will be only permitted when no possible confusion might arise.

Now we are going to see how equation (II) might explain particle interaction and absorption and emission of radiation by an atom.

Let $(X, \mathcal{V})$ be a Gauss space and a n-field.

$$\Phi : (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$$

Suppose we have two particles in $(X, \mathcal{V})$

$$f, g : (R, \mathcal{U}) \longrightarrow (X, \mathcal{V}) \text{ .}$$

Call ϕ^t a variable n-field generated by the particle g, depending on the attributes of g. We can regard ϕ^t as a pertubation of Φ and some original path of f, defined by

$$\mu_f D_f^2 = \Phi \circ f \text{ ,}$$

where μ_f is the mass of f,

will be now modified as to satisfy

$$\mu_f D_f^2 = (\Phi + \phi^t) \circ f \qquad (1)'$$

what express the interaction of f and g from the point of view of f. We could also analyse the question of the action of f over g and so if ψ^t is a n-field generated by f we have for g

$$\mu_g D_g^2 = (\Phi + \psi^t) \circ g$$

For instance, if $(X, \mathcal{V})$ is an atom with Φ representing the electric field created by the nücleus of $(X, \mathcal{V})$, then we can regard g as a particle hitting the atom and interacting with some electron f of $(X, \mathcal{V})$. A modification in the path of f will result in a "jump" of f from one orbit to another and g after a certain period of time, where its action over f is sensible, will "disapear" in the sense that all its attributes will reduce to n-functions contantly zero. This can be interpreted, from the physical point of view, by assuming that all kinetic energy represented by an attribute of g is whole "transmitted" to f, namely it is incorporated in some attrib ute of f which is changed accordingly. So in the end of the process we have f in another orbit with some of its attributes changed through the action of g which does not exist anymore, because a particle whose attributes are zero is physically non existent because it cannot interact with anything. This phenomenon is usually called absorption of radiation by the atom and really, in most cases, g is already from the beginning a radiating n-field ϕ^t, or in popular language, a "monocromatic wave of frequency ν". Of course, we can include in this discussion the question of probability of "g to hit f", namely the probability of the second member of (1) be non-zero. However, we do not discuss this question now leaving it for some other time or for those readers "probabilistic--maniacs".

The same model applies also for the case of one particle g hitting another in free space, namely, suppose f is a free electron, i.e., under the action of no n-field what is described by putting $\phi = 0$ in the equation

$$\mu_f\, D_f^2 = \phi \circ f$$

what gives

$$D_f^2 = 0 .$$

Suppose further that g is a photon generating a variable n-field ϕ^t, hitting f, described by

$$\mu_f\, D_f^2 = \phi^t \circ \hat{f} .$$

As attribute for g we might consider its energy $h\nu$, if Φ^t has frequency ν. It is good to call attention that when a particle f does not have mass we still can use equation (II) if Φ^t acts as f. In this case we substitute μ by some other attribute of f. For instance, in the case of a photon we can take as attribute its energy $E = h\nu$ if it is connected with a monochromatic wave of frequency ν. Hence we can write equation (II) as

$$\frac{h\nu}{c^2} D^2 f = \Phi^t \circ \hat{f}$$

where Φ^t is some variable n-field acting on the photon. Naturally, we are assuming that the photon is travelling with the speed of light.

Now let us consider the case of emission of radiation by an atom $(X, \mathcal{V})$. This is possible when an electron jumps from one orbit to another. Let us assume that the n-field created by the nucleus of $(X, \mathcal{V})$ is Φ and so as far as the electron f is in a orbit of stability $\Phi = 0$. Suppose that f jump from an orbit S_n to another one S_p, $n > p$ and we do not discuss now how this occur and accept as a fact that it happens spantaneously. Due to this jump for the equation

$$\mu D^2 f = \Phi \circ f$$

to hold there must appear a pertubation Φ^t for otherwise Φ could not pass from zero to a value different from zero. This pertubation is a variable n-field which is detected by on apparatus as a radiation and we say that the atom emitted a radiation of a certain frequency.

Let us explain this question of eminion of radiation in more detail. To simplify matters we do not mention explicitty the coverings σ, τ, etc to which an open set belongs and say, for instance; "the open set A in X is taken by Φ in an open set A' of R" and use the notation $\Phi(A) = A'$. With this convention in mind suppose that an electron f is in the orbit S_n of the atom $(X, \mathcal{V})$. When it jumps to the orbit S_p, $p < n$, there is a perturbation Φ^t of the field Φ and so if Φ was zero for an open set A, $A \cap S_n \neq \emptyset$ it becomes now $\Phi^t(A \times \tilde{A}) \neq 0$ for the interval of time $\tilde{A}$ and all $A \in p$.

Suppose that B is an open set in X with $B \cap A \neq \emptyset$. We claim that also in B, Φ^t is not zero. Indeed, suppose that for a convenient τ refining the one containing B and A we select an open set $E \in \tau$, $E \subset B \cap A$. Now as Φ^t is continuous we see that as $\Phi^t(E \times \tilde{B}) \neq 0$ for an interval of time $\tilde{B} \subset \tilde{A}$, we cannot have $\Phi^t(B \times \tilde{B}) = 0$ because by the continuity of Φ^t

$$\Phi^t(E \times \tilde{B}) \subset \Phi^t(B \times \tilde{B})$$

In conclusion if $\Phi^t \neq 0$ for a germ p of $(X, \mathcal{V})$ it is also non-zero for any other germ p' with $p \cap p' \neq \emptyset$.

In this sense the pertubation Φ^t "propagates" in $(X, \mathcal{V})$. Of course it will decrease as it propagates because even being different from zero might approach zero as the process goes on, but in general, at least theoretical, Φ^t would propagate to a whole component (connected subset) of X.

So the propagation of variable n-fields is a consequence of their continuity. In general it will propagate, as said above, to a whole component of X and the separation between two components can be interpreted physically as the existence of an insulator for Φ^t. For example, if there is another Universe whose union with ours is not connected, we could not detect its existence as no variable n-field would propagate from one to another.

5. Let us now study the possibility of introducing the concept of energy in our space $(X, \mathcal{V})$. For that consider in the real line R the measure m induced by the canonical Gauss structure. As seen before it coincides with the Lebesgue's measure.

Definition VII - Let $\Phi:(X,\mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$ be stationary n-field and let $f:(R,\mathcal{U}) \longrightarrow (X,\mathcal{V})$ be a particle in $(X,\mathcal{V})$. Then if I is an interval in R, the work done by Φ, or the energy spent by Φ to move the particle in the interval I is

$$E = \int_I (f.Df)\ d\Phi$$

where integration is to be considered in the sense developed in §III, namely

$$\int_I (f.Df)\ d\Phi = \int_I Df.(\Phi \circ f)\ dm$$

The reason for this definition is given by the analogy with the classical case of mechanics. Indeed, using equation I , 4 we have

$$E = \int_I Df.(\Phi \circ f)\,dm = \int_I \mu\ Df\ .\ D^2 f\ dm$$

and the last integral is similar to the one in the classical case, i.e.,

$$\int_I \mu \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt$$

where $\vec{v}$ is the velocity of the particle at instant t.

Let us now consider the case of a variable n-field

$$\phi^t : (X \times R, \mathcal{W}) \longrightarrow [R, \mathcal{V}_R]$$

with respect to $(R, \mathcal{U})$.

First we ask what would be the energy spent by ϕ^t to move a particle f in $(X, \mathcal{V})$.

<u>Definition VIII</u> - The energy spent by a variable n-field ϕ^t to move a particle f in $(X, \mathcal{V})$ during on interval of time I is given by

$$E = \int_I Df.(\phi^t \circ \hat{f})\, dm$$

or shortly

$$E = \int_I (Df.\hat{f})\, d\phi^t$$

Using equation (II) this gives as before

$$E = \int_I \mu\, D^2 f \,.\, Df\, dm$$

A second case to be considered is the question of the amount of energy contained in a variable n-field by analogy with the case of electromagnetic field studied by Maxwell. However, this would be too long to consider here and we prefer to treat this question in detail in forthcoming papers.

Of course many other problems can be considered in the same line as, for instance, the question of potential energy, principle of conservation of energy, etc. All this can be the start point of future developments and we have plans to study these questions in due time.

6. We close this paper with a few general comments. From the elements introduced before we saw that the general problem of non-deterministic mechanics is the study of the equation of motion of particles under the action of n-fields defined in a Gauss space $(X, \mathfrak{J})$. To simplify our exposition let us consider the case of one particle

$$f: (R, \mathfrak{U}) \longrightarrow (X, \mathcal{V})$$

and a n-field

$$\Phi: (X, \mathcal{V}) \longrightarrow [R, \mathcal{V}_R]$$

or eventually a variable n-field

$$\Phi^t: (X \times R, \mathcal{W}) \longrightarrow [R, \mathcal{V}_R]$$

So as seen before the equations of motion are

$$(I)\quad \begin{array}{ll} (\mu o f)\ Df^2 = \phi^t \ o\ \hat{f}, & \text{for variable n-fields} \\ (\mu o f)\ Df^2 = \phi\ \ o\ f, & \text{for stationary n-fields.} \end{array}$$

But an equation in itself is not sufficient to define precisely a physical situation: one needs what is usually called initial or boundary conditions. This questions has been considered in [6] an here we recall those ideas under a slight different approach based on the motion of germ.

Let $p = \{A_\sigma\}_{\sigma \in \mathcal{V}}$ be a germ in $(X, \mathcal{V})$ and $p_R = \{A_{\sigma_R}\}_{\sigma_R \in \mathcal{V}_R}$ a germ in $[R, \mathcal{V}_R]$. Then we consider the following non-deterministic Cauchy's problem: To find a solution f for (I) above such that:

$$\text{(i)}\quad f_\sigma^\sim (A_\sigma^\sim) = A_\sigma \in p$$

$$\text{(ii)}\quad Df_\sigma^\sim (A_\sigma^\sim) = A_{\sigma_R} \in p_R$$

In general we might have many different solutions as discussed in [6] and we can also relax condition (i), (ii) by using ≃ intead of = (see §I,) but these details are not our concern now.

In general these are hard problems to solve and we obtain usually many solutions satisfying (I) and given initial con-conditions (see [6]), which is a main difference from the classical Cauchy's problem. From the physical point of view we can give interpretations to this multitude of solutions and in the future we intend to deeper our analysis in this direction.

Another point to call attention is the case when we know the solution of equation (I) and we do not know the fields involved. Then we might try to find them and most often this occurs in practice. For instance, one might know the spectrum of frequency of radiations when an atom or a nucleus is excited and then one could be interested in finding the n-fields acting on them by using equation (I). Of course, we still miss a more complete theory, strictly mathematical for handling such problems and this might be an interesting line of research.

Finally we want to say that when in $(X, \mathcal{F})$ we have some linear structure available, classical theory of derivatives could be used, and to adjust our equation (I) we must study each component of the particles and n-fields involved, when X has a finite basis. In this way we do not loose the linear structures present. In the future we shall study this question in detail.

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