Differential Capacity of Gaussian Channels

Max H. M. Costa
FEEC - Unicamp

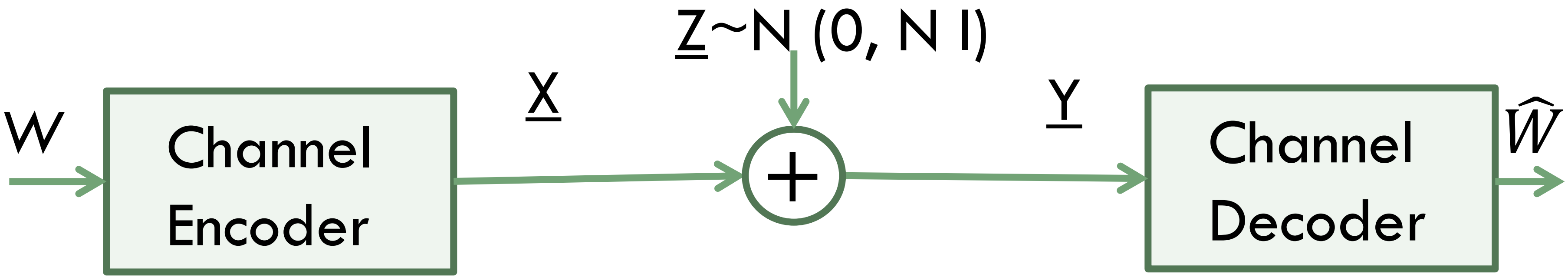
Based on joint work with Chandra Nair (CUHK)

Summary

- Gaussian Channel (Discrete time AWGN)
- Gaussian Channel (Continuous time, band limited)
- Gaussian Multiple Access Channel
- Gaussian Broadcast Channel
- Gaussian Interference Channel - standard form
- Model 1: Z-Interference Channel
- Model 2: Symmetric Interference Channel

The Gaussian Channel

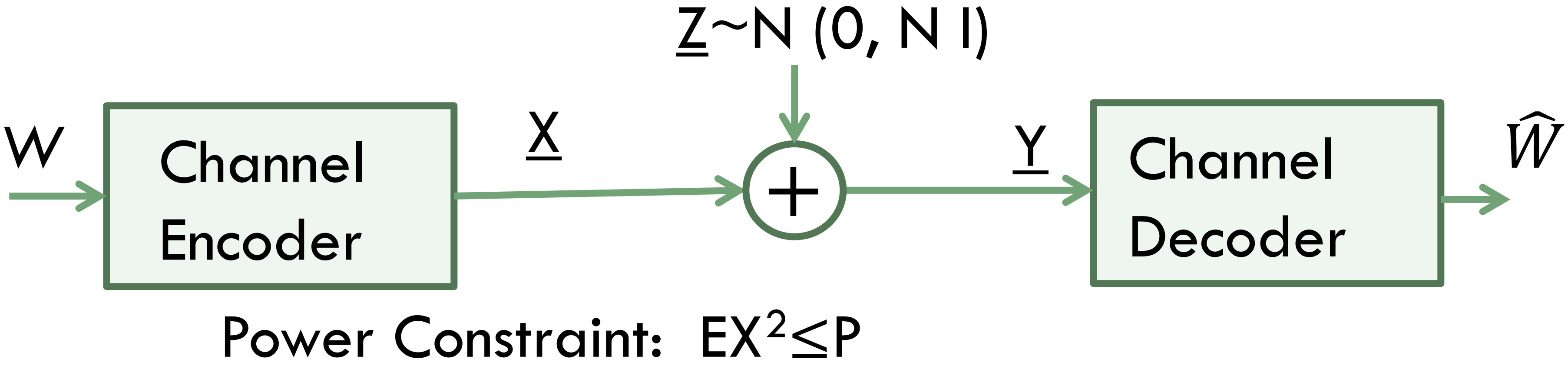
- The Gaussian Channel Problem:



Power constraint: $EX^2 \leq P$

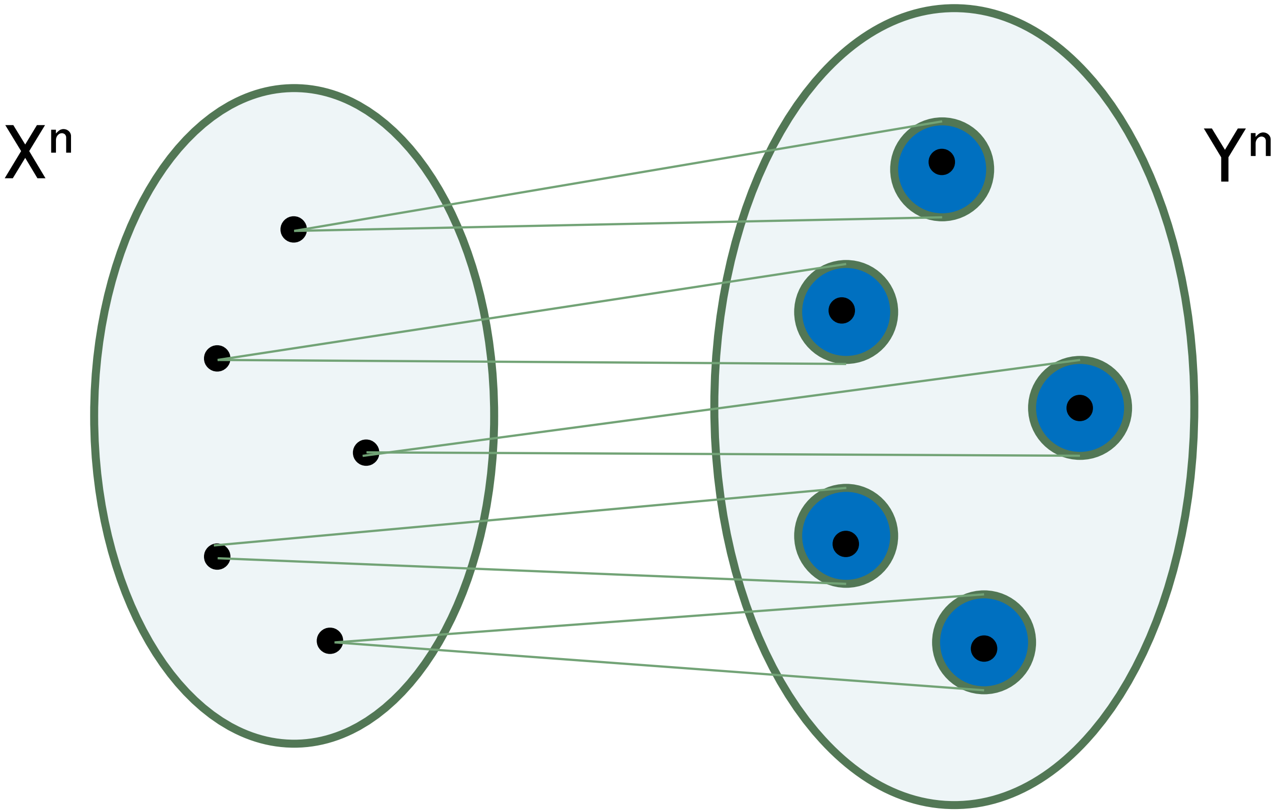
- $W \in \{1,2,\ldots,2^{nR}\}$ = message set of rate R
- $\underline{X} = (x_1\ x_2\ \ldots\ x_n)$ = codeword input to channel
- $\underline{Y} = (y_1\ y_2\ \ldots\ y_n)$ = codeword output from channel
- $\widehat{W}$ = decoded message P(error) = $P\{W \neq \widehat{W}\}$

The Gaussian Channel



The Gaussian Channel

- Using the channel n times:



The Gaussian Channel

-
- Capacity $C = \max_{f(x):\, EX^2 \leq P} I(X;Y)$
- $I(X;Y) = h(Y) - h(Y|X) = h(Y) - h(X+Z|X)$
- $= h(Y) - h(Z) \leq \frac{1}{2}\log 2\pi e(P+N) - \frac{1}{2}\log 2\pi e N$
- $= \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$ bits/transmission

The Gaussian Channel

- The capacity of the discrete time additive Gaussian channel:

$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right) \text{ bits/transmission}$$

- achieved with X ~ N(0 , P).

Bandlimited Gaussian Channel

- Consider the channel with continuous waveform inputs *x(t)* with power constraint $(\frac{1}{T}\int_0^T x^2(t)dt \le P)$ and Bandwidth limited to W. The channel has white Gaussian noise with power spectral density $N_0/2$ watt/Hz.
- In the interval *(0,T)* we can specify the code waveform by 2WT samples (Nyquist criterion). We can transmit these samples over discrete time Gaussian channels with noise variance $N_0/2$. This gives
- $$C = W\log(1+\frac{P}{N_0 W})$$ bit/second

Bandlimited Gaussian Channel

-
- $$C = W \log(1 + \frac{P}{N_0 W}) \text{ bit/second}$$
- Note: If W→ ∞
- we have $C = \frac{P}{N_0} log_2 e$ bits/second.

Bandlimited Gaussian Channel

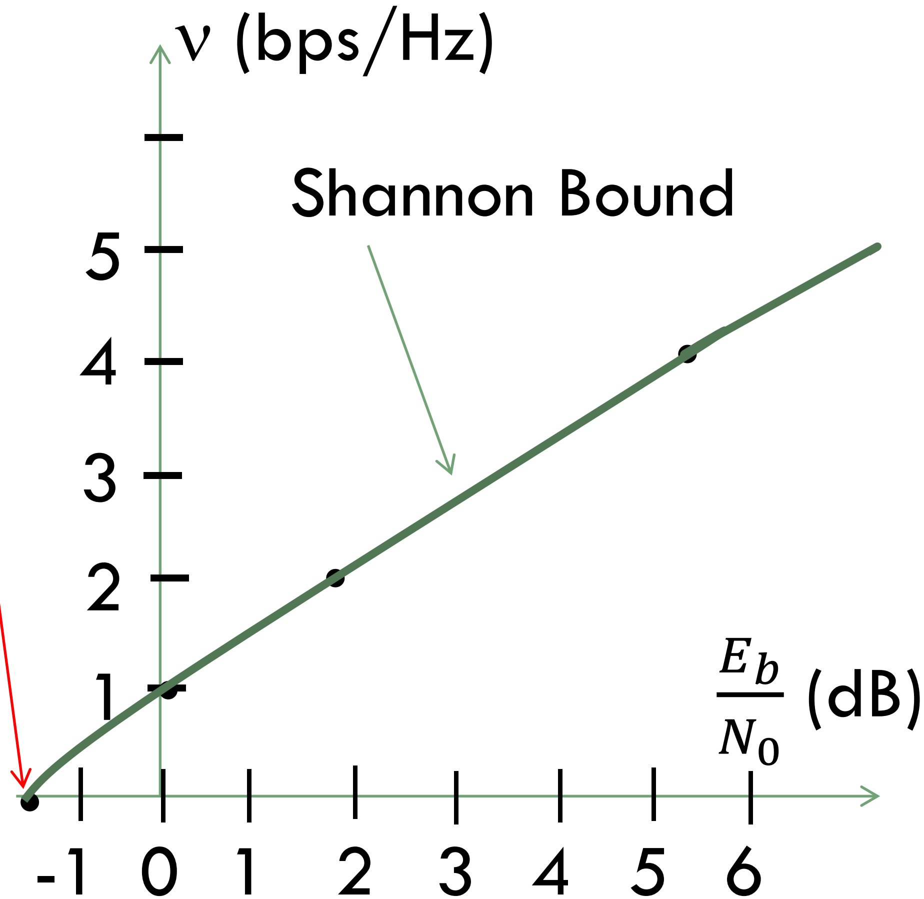
- Let $\frac{R}{W}$ be the spectral density ν in bits per second per Hertz. Also let $P = E_b R$ where E_b is the available energy per information bit.
- We get
- $\frac{R}{W} \leq \frac{C}{W} = \log(1 + \frac{E_b R}{N_0 W})$ bit/second.
- Thus
- $$\frac{E_b}{N_0} \geq \frac{2^{\nu} - 1}{\nu}$$

This relation defines the so called Shannon Bound.

The Shannon Bound

$$\frac{E_b}{N_0} \geq \frac{2^{\nu}-1}{\nu}$$

?	$\frac{E_b}{N_0}$	$\frac{E_b}{N_0}$ (dB)
→0	0.69	-1.59
0.1	0.718	-1.44
0.25	0.757	-1.21
0.5	0.828	-0.82
1	1	0
2	1.5	1.76
4	3.75	5.74
8	31.87	15.03



Shannon's Water Filling Solution

Parallel Gaussian Channels

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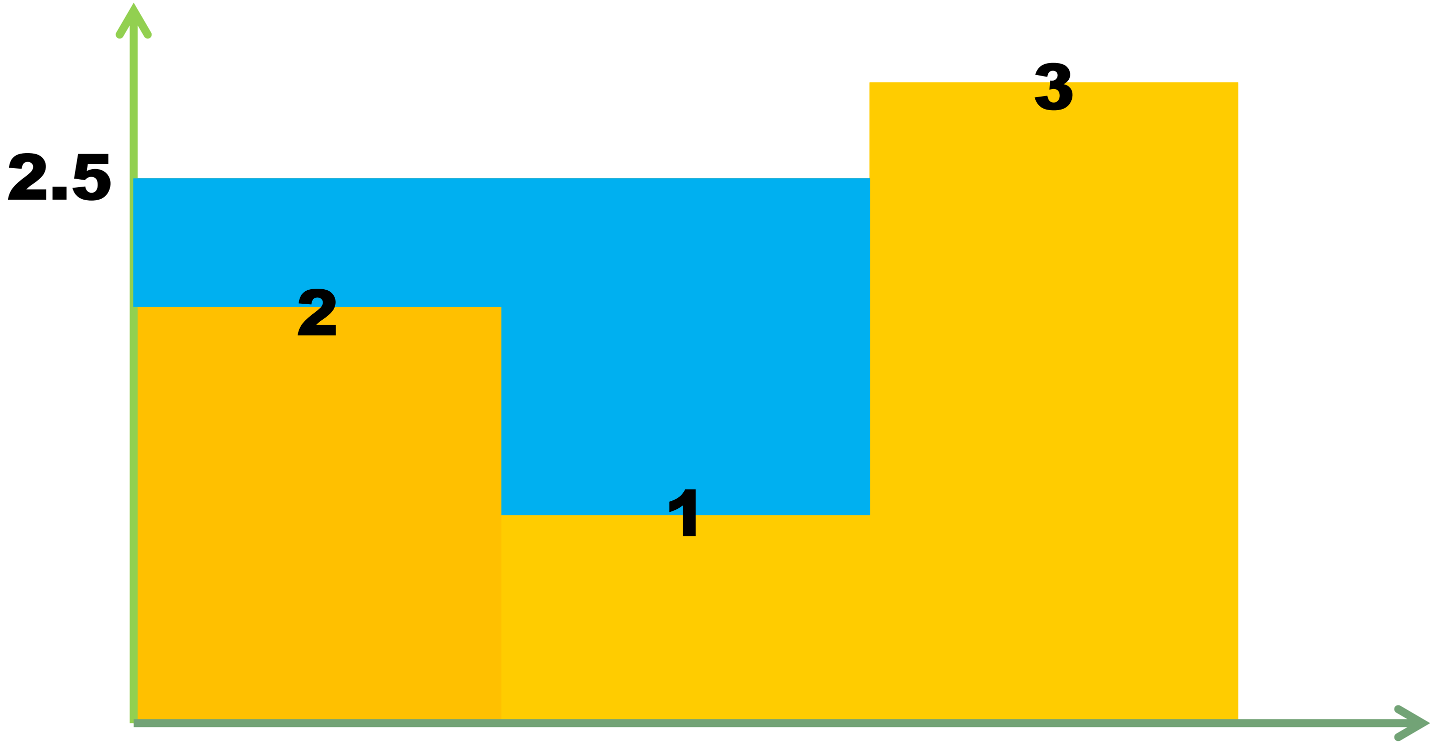

Example of Water Filling

- Channels with noise levels 2, 1 and 3.
- Available power = 2
- Capacity= $\frac{1}{2}\log\left(1+\frac{0.5}{2}\right)+\frac{1}{2}\log\left(1+\frac{1.5}{1}\right)+\frac{1}{2}\log\left(1+\frac{0}{3}\right)$
- Level of noise + signal power = 2.5
- No power allocated to the third channel.

Parallel Gaussian Channels

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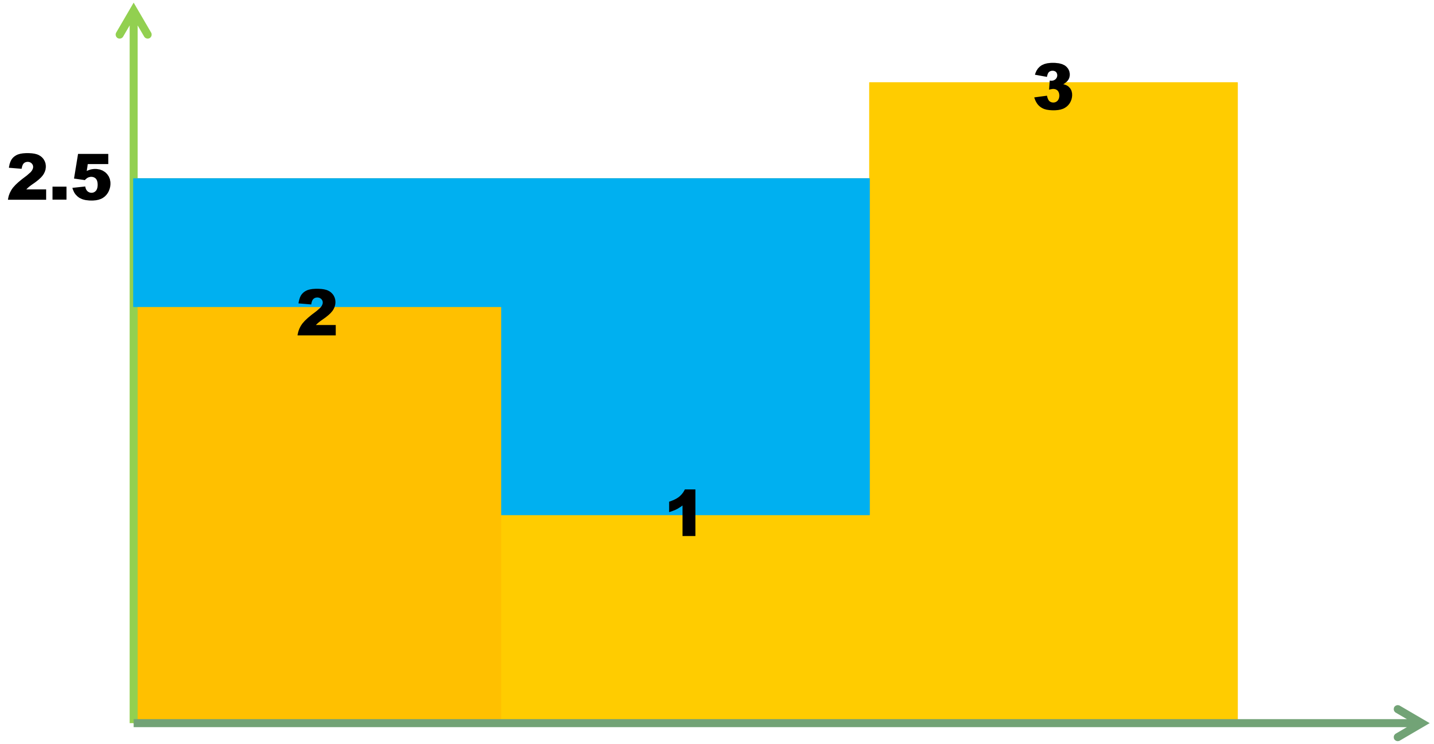
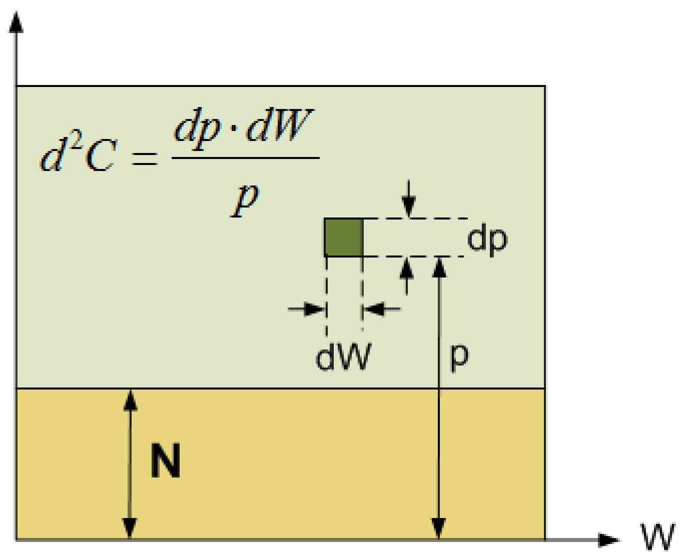
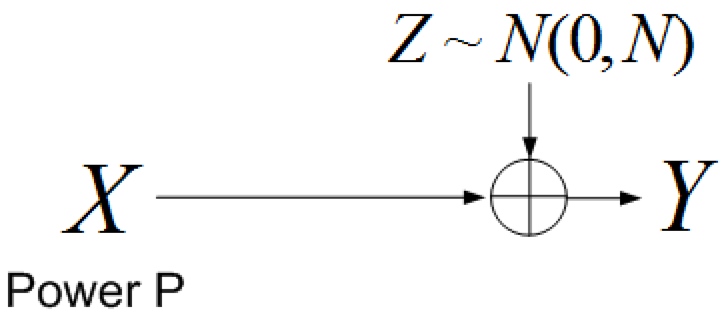


Differential capacity

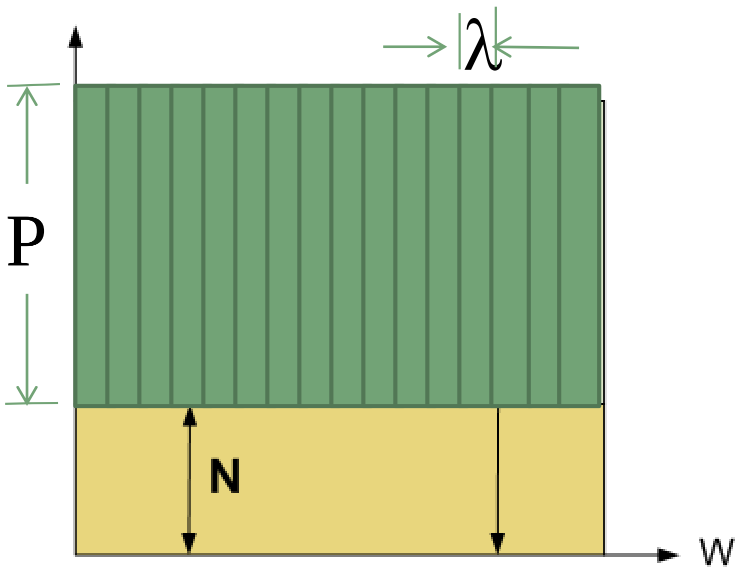


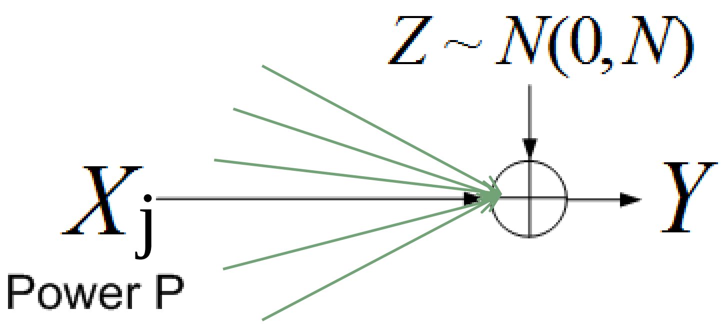


$$C = \iint d^2C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

Discrete time channel seen as a unit band contínuous time channel

Multiplex strategies (TDMA, FDMA)

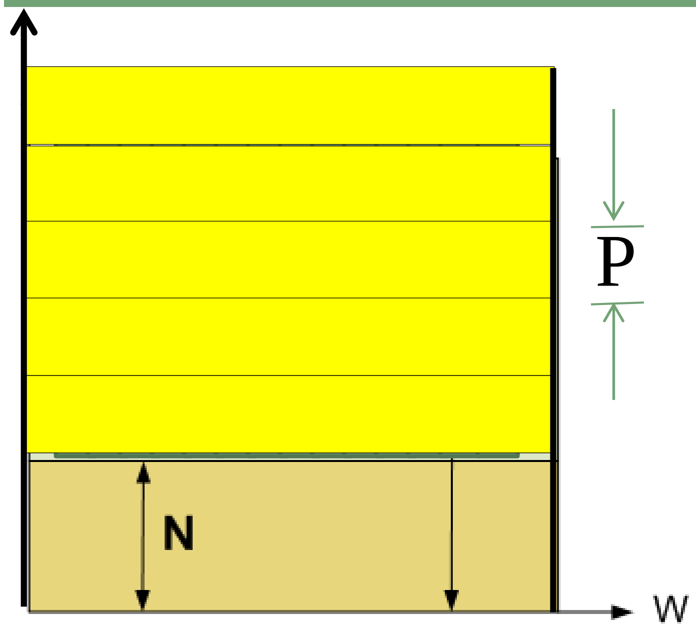


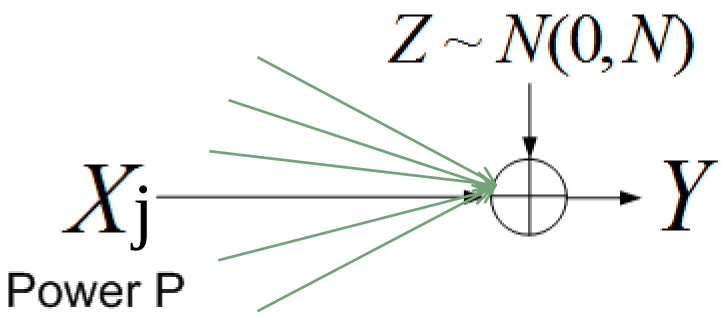


$$C_j = \iint d^2C = \frac{\lambda}{2}\log\left(1 + \frac{P}{N}\right)$$

Aggregate capacity:: $\sum C_j = \frac{1}{2}\log(1 + \frac{P}{N})$

Multiplex strategies (non-orthonal CDMA)



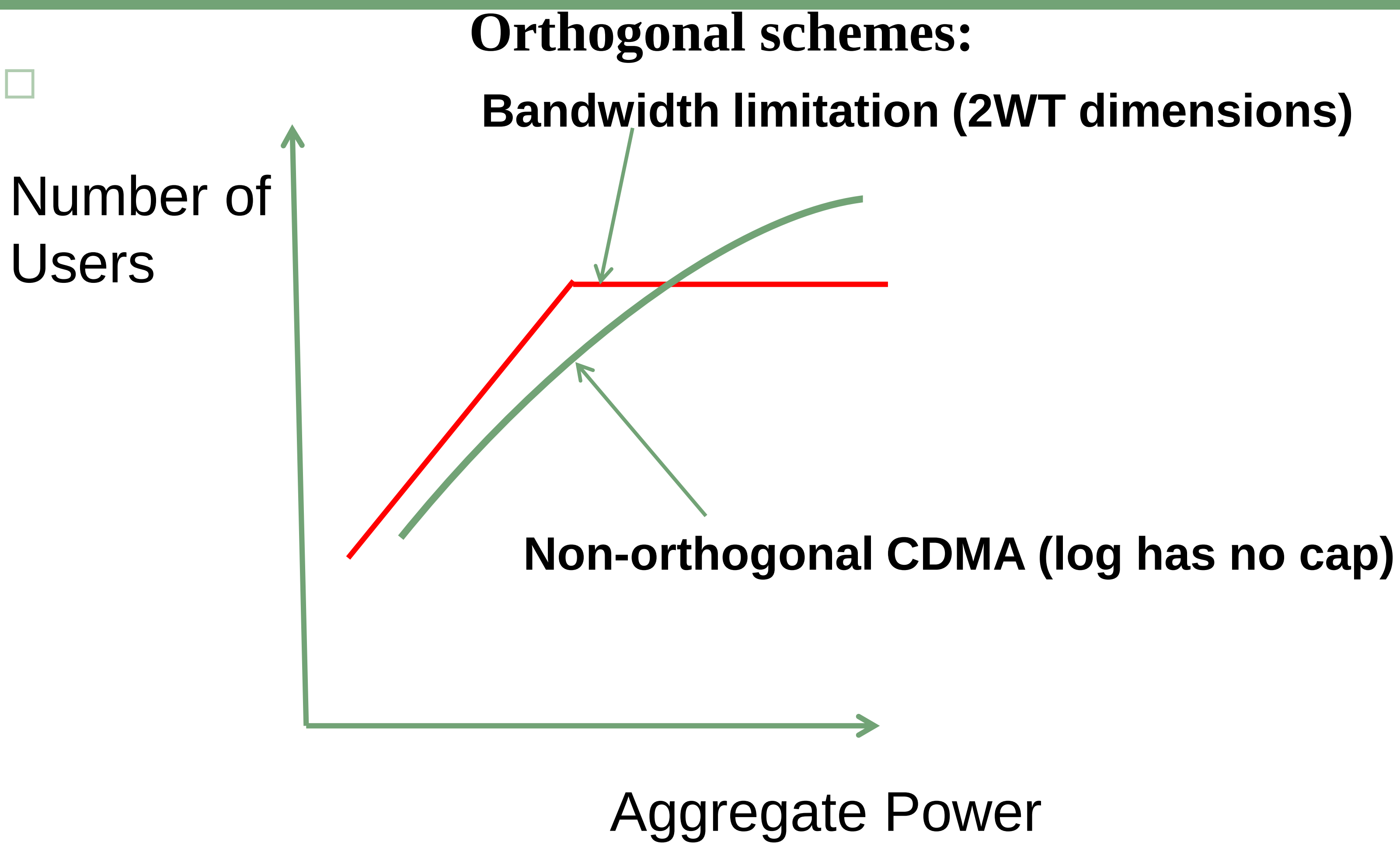


$$C_j = \frac{1}{2}\log\left(1 + \frac{P}{N+(j-1)P}\right)$$

Aggregate capacity:

$$\sum_{j=1}^{M} C_j = \frac{1}{2}\log\left(1 + \frac{MP}{N}\right)$$

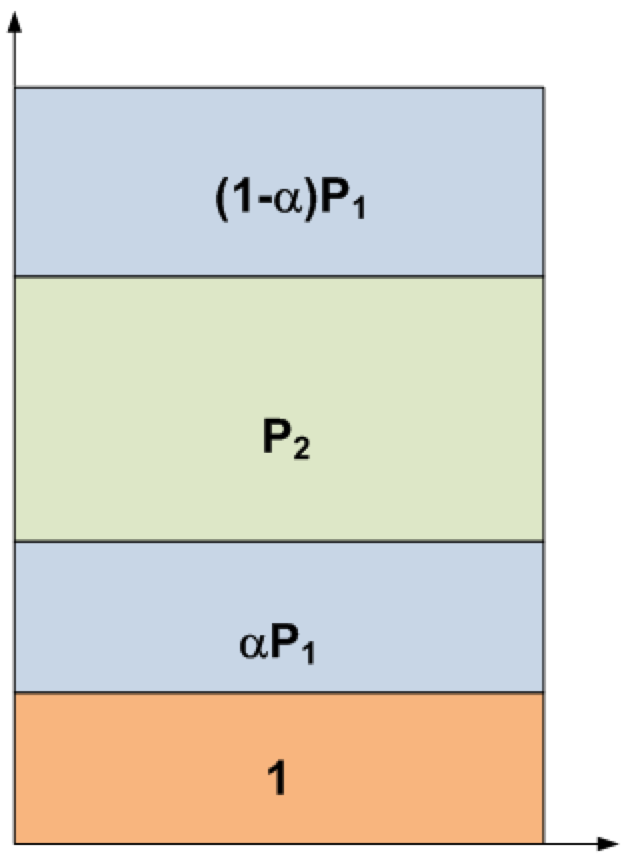
TDMA or FDMA versus CDMA



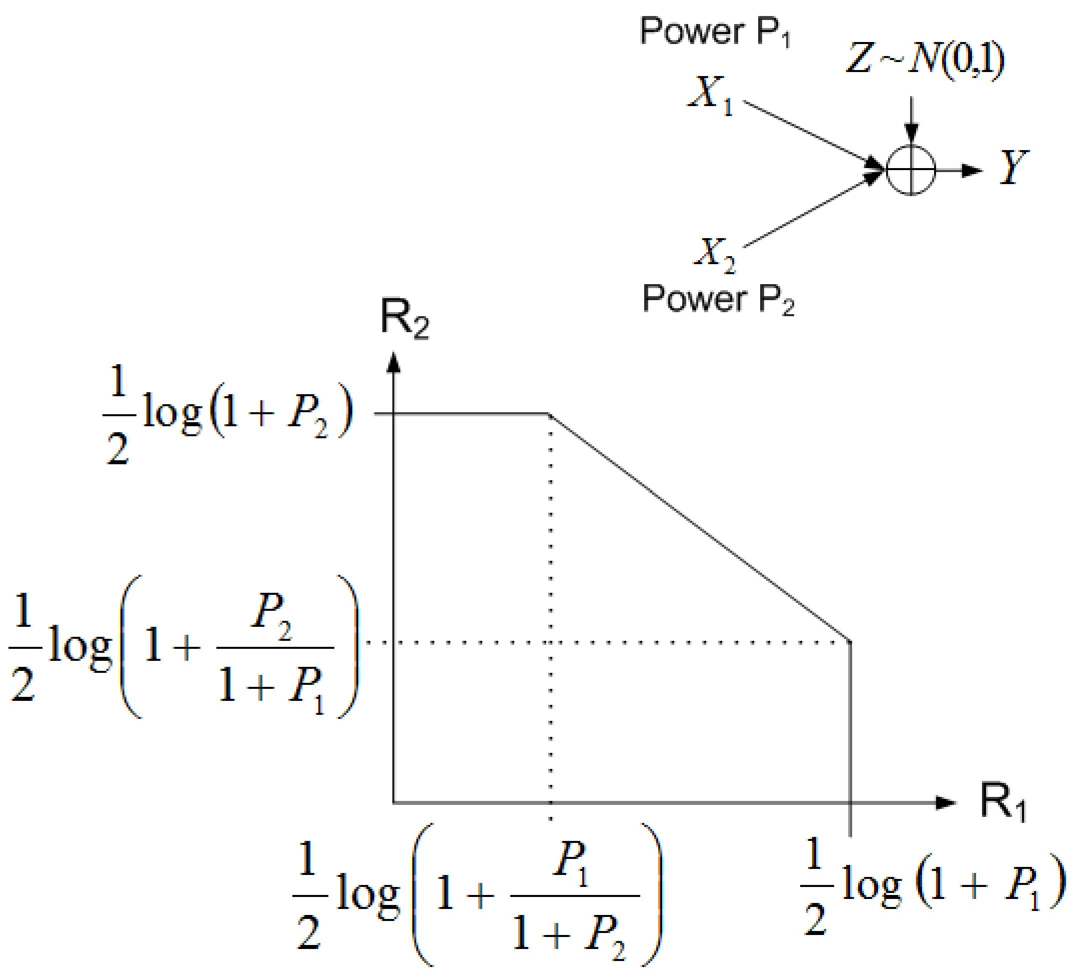
Multiple User Information Theory

- Some building blocks:
- Multiple Access Channels (MACs)
- Broadcast Channels (BCs)
- Interference Channels (IFCs)
- Relay Channels (RCs)
- Note: These channels have their discrete time and continuous time versions.

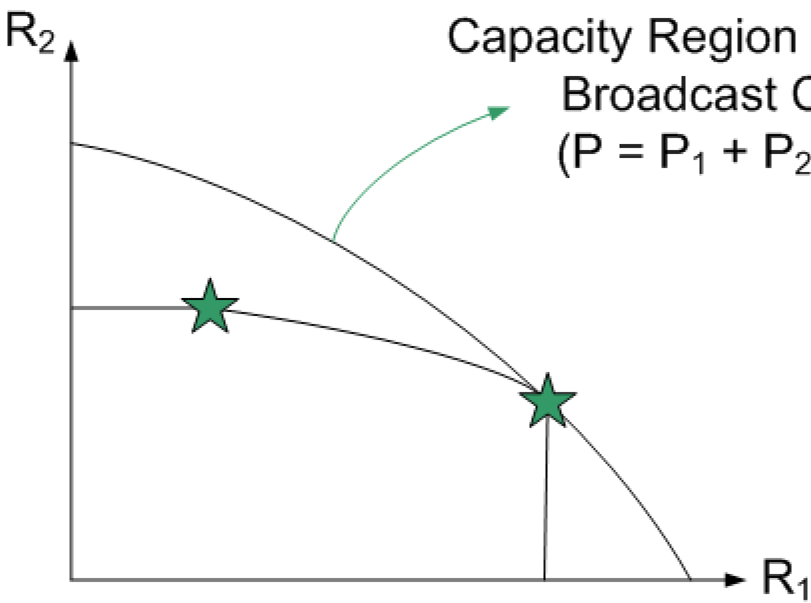
Multiple Access Channel

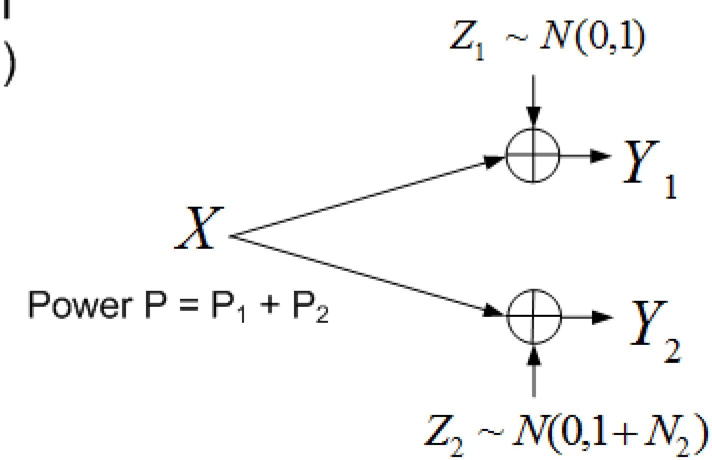


$0 \le \alpha \le 1$



Gaussian Broadcast Channel

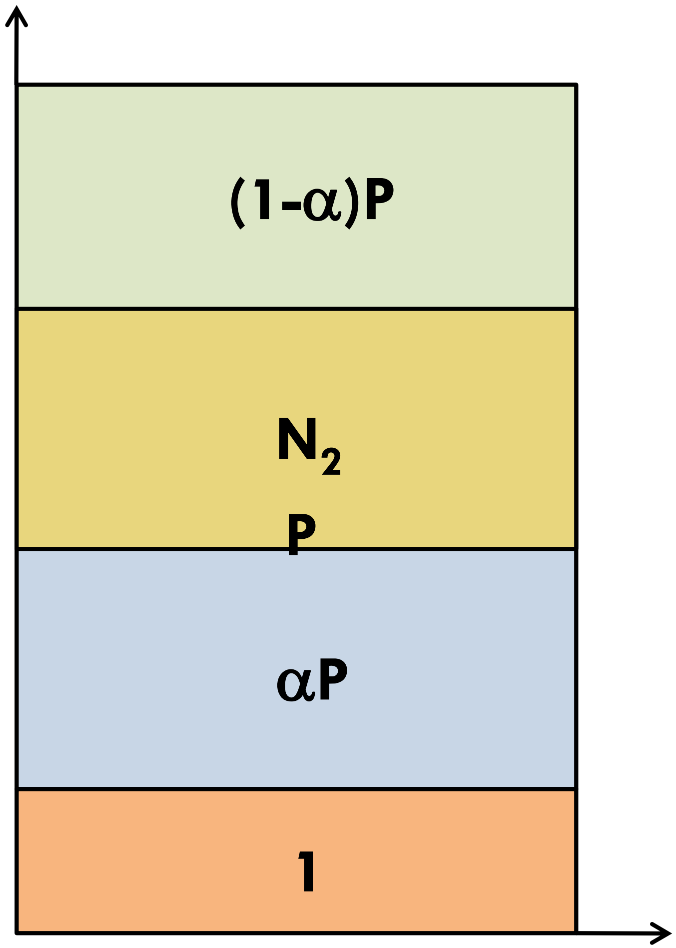


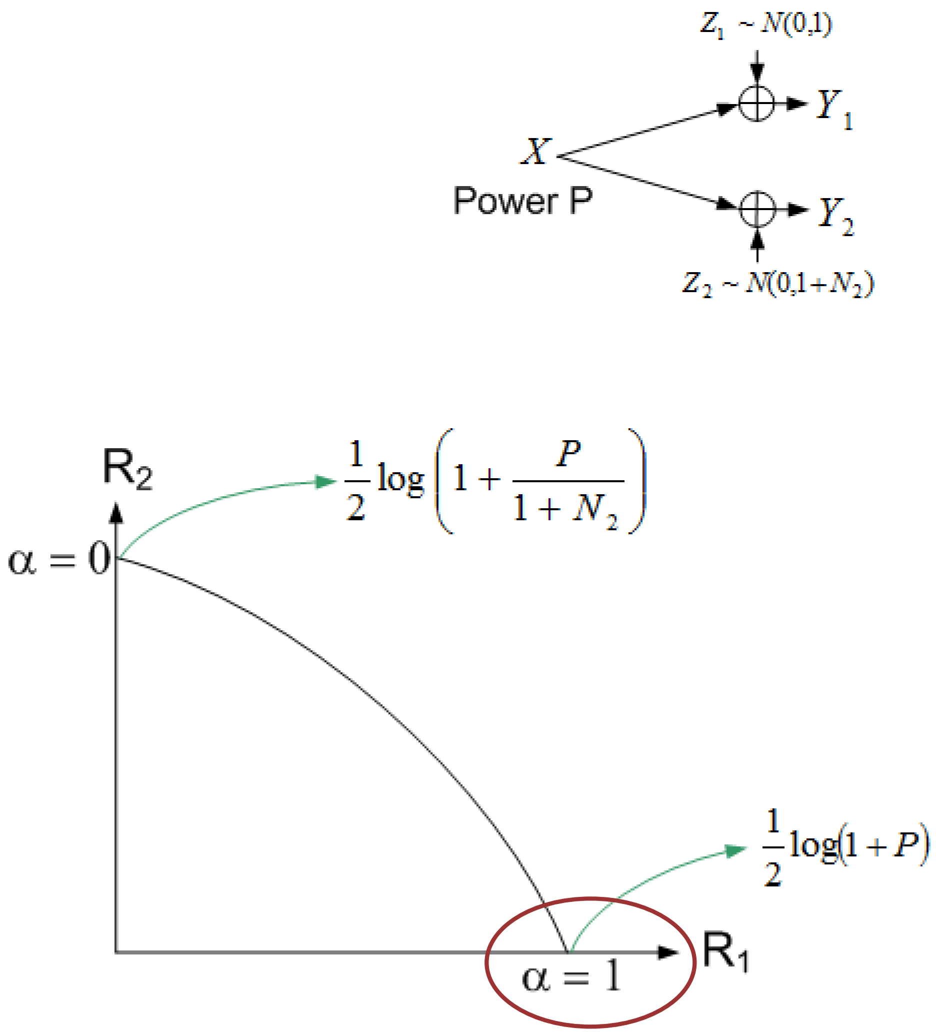


$$C_{BC}\{R_1, R_2\}: \quad 0 \le R_1 \le \frac{1}{2}\log(1+\alpha P)$$

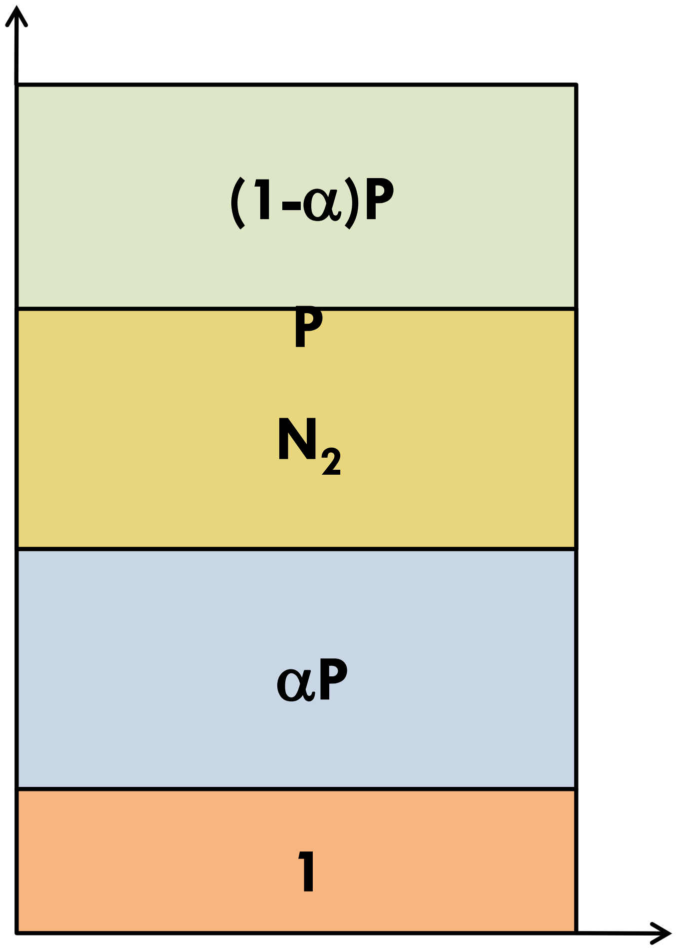
$$0 \le \alpha \le 1 \qquad 0 \le R_2 \le \frac{1}{2}\log\left(1+\frac{(1-\alpha)P}{1+N_2+\alpha P}\right)$$

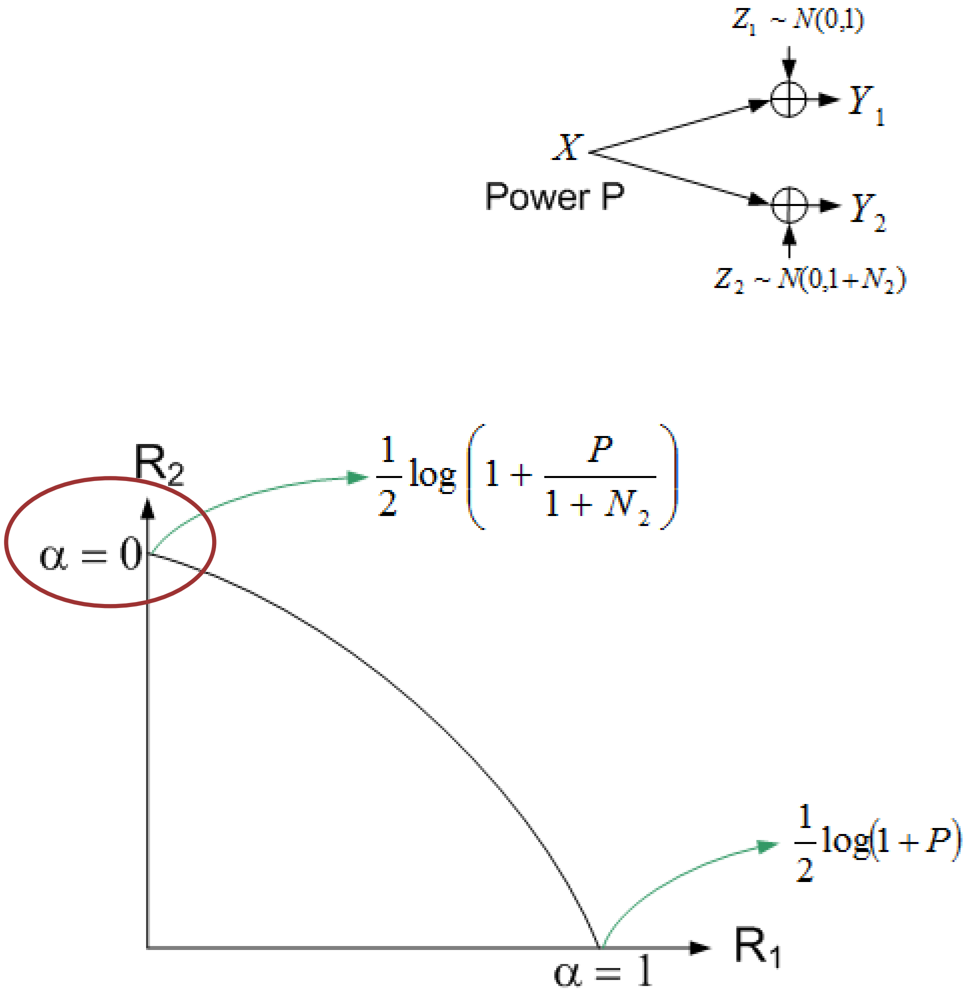
Superposition coding





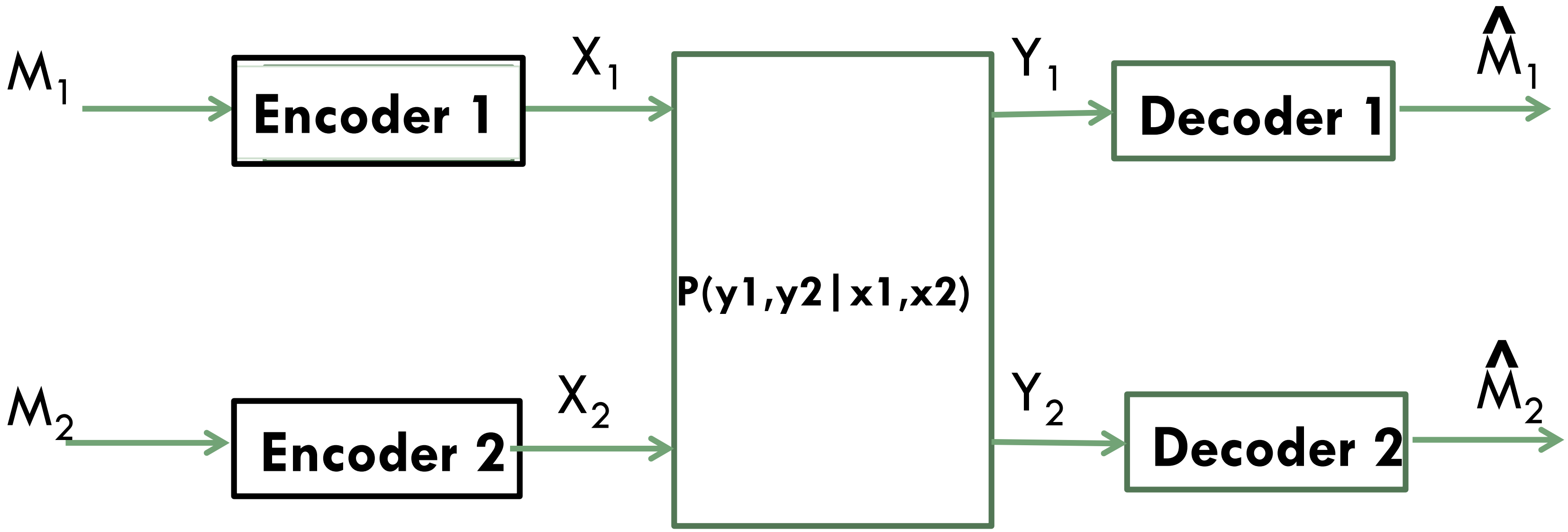
Superposition coding



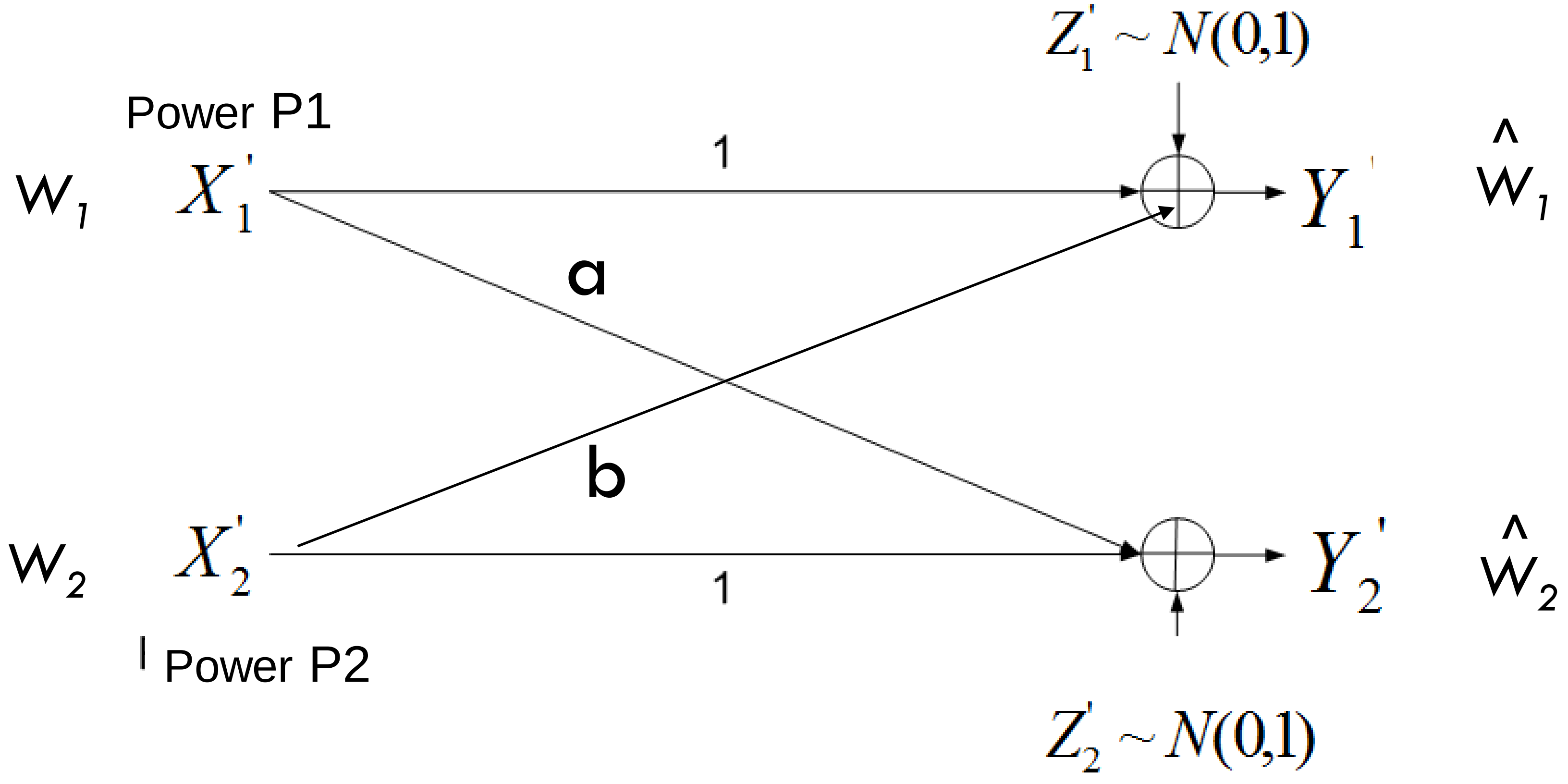


Discrete Memoryless Interference Channel

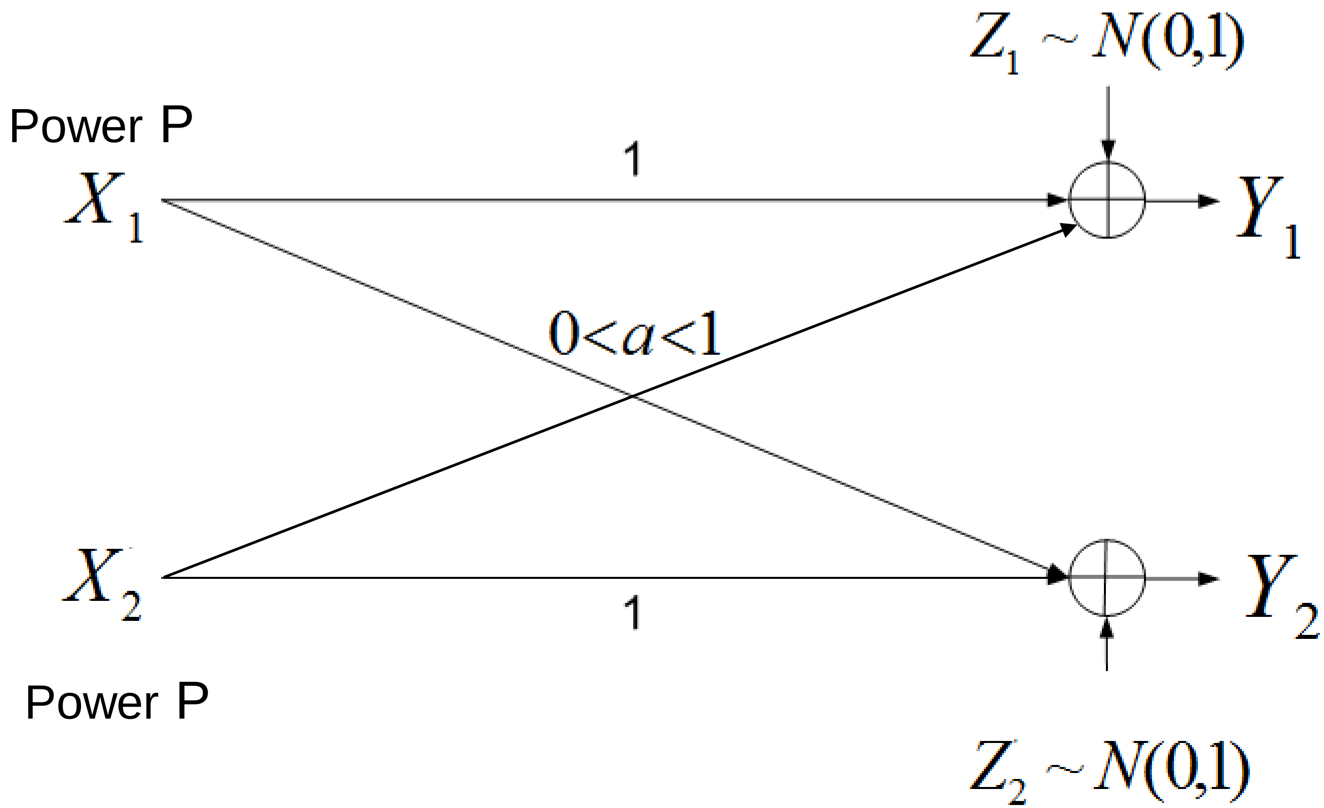
- Two-user case:



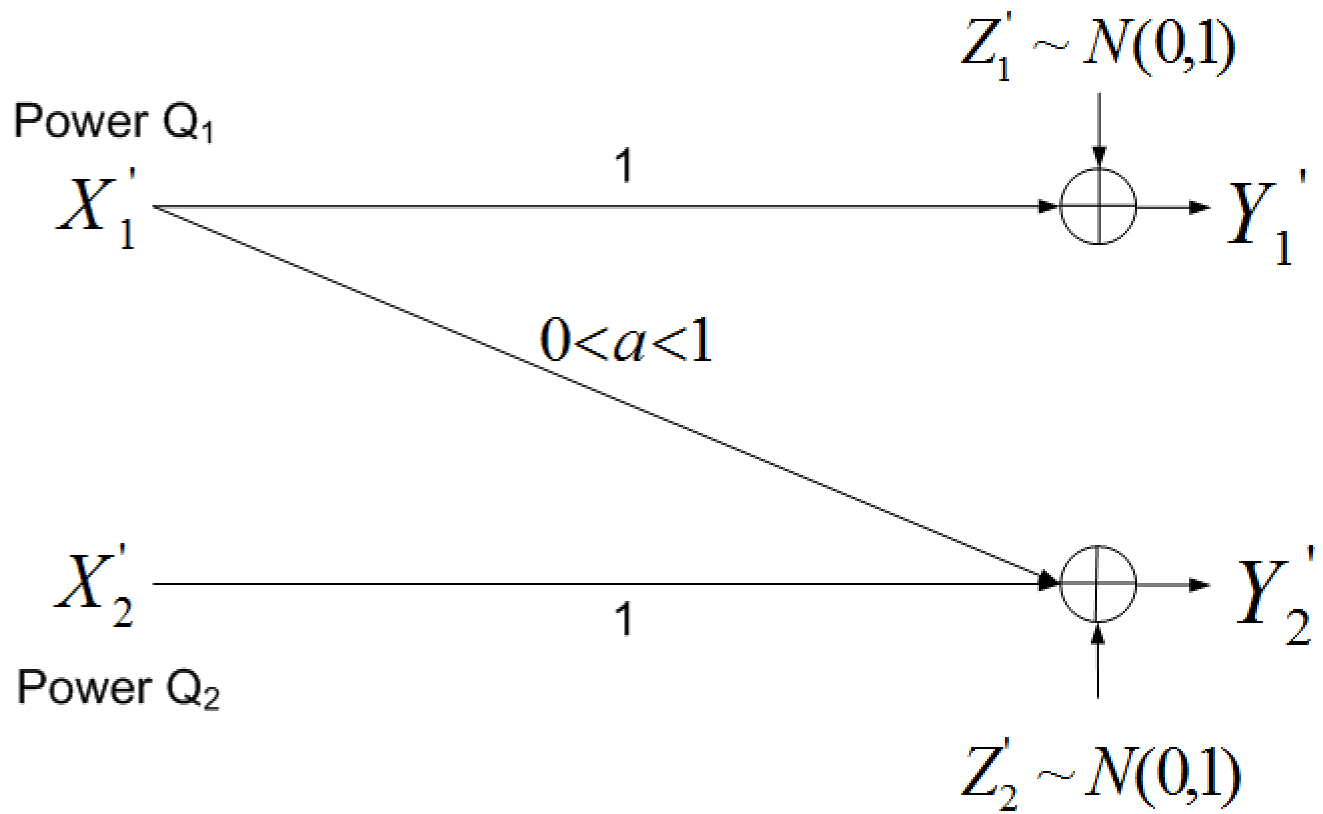
Standard Gaussian Interference Channel



Symmetric Gaussian Interference Channel



Z-Gaussian Interference Channel



The possibilities:

Things that we can do with interference:

1. Ignore (take interference as noise (IAN)
2. Avoid (divide the signal space (TDM/FDM))
3. Partially decode both interfering signals
4. Partially decode one, fully decode the other
5. Fully decode both (only good for strong interference, $a \geq 1$)

Brief history

- Carleial (1975): Very strong interference does not reduce capacity ($a^2 \geq 1+P$)

- Sato (1981), Han and Kobayashi (1981): Strong interference ($a^2 \geq 1$) : IFC behaves like 2 MACs

- Motahari, Khandani (2009), Shang, Kramer and Chen (2009), Annapureddy, Veeravalli (2009):
  Very weak interference ($2a(1+a^2P) \leq 1$) :
  ? Treat interference as noise (IAN)

History (continued)

- Sason (2004): Symmetrical superposition to beat TDM – found part of optimal choice for α
- Etkin, Tse, Wang (2008): capacity to within 1 bit, good heuristical choice of $\alpha P = 1/a^2$

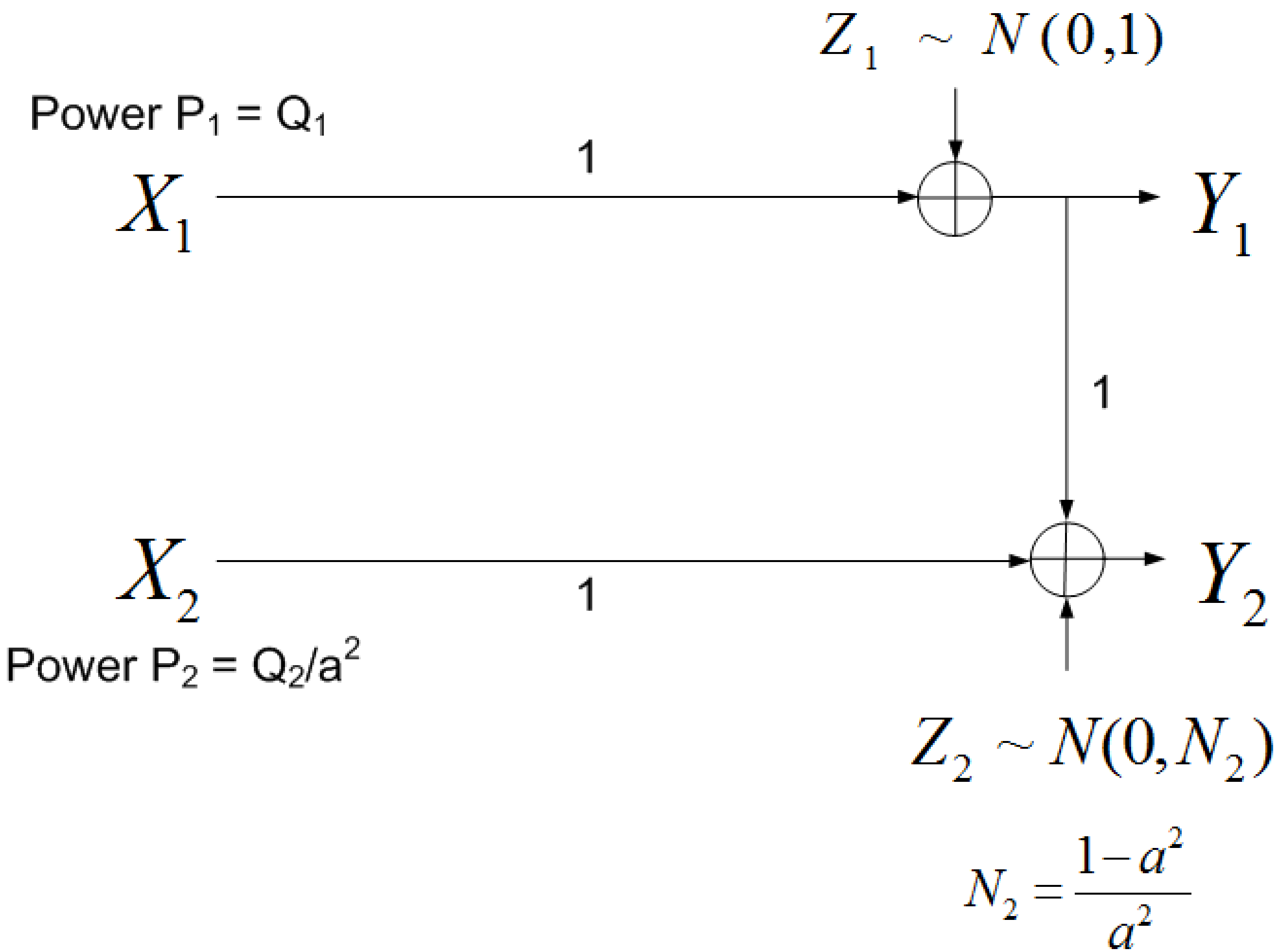
Han and Kobayashi Region

- $R1 < I(X1; Y1 \mid U2, Q)$
- $R2 < I(X2; Y2 \mid U1, Q)$
- $R1 + R2 < I(X1, U2; Y1 \mid Q) + I(X2; Y2 \mid U1, U2, Q)$
- $R1 + R2 < I(X2, U1; Y2 \mid Q) + I(X1; Y1 \mid U1, U2, Q)$
- $R1 + R2 < I(X1, U2; Y1 \mid U1, Q) + I(X2, U1; Y2 \mid U2, Q)$
- $2R1 + R2 < I(X1, U2; Y1 \mid Q) + I(X1; Y1 \mid U1, U2, Q) + I(X2, U1; Y2 \mid U2, Q)$
- $R1 + 2R2 < I(X2, U1; Y2 \mid Q) + I(X2; Y2 \mid U1, U2, Q) + I(X1, U2; Y1 \mid U1, Q)$
- for some $p(q)\, p(u1, x1 \mid q)\, p(u2, x2 \mid q)$

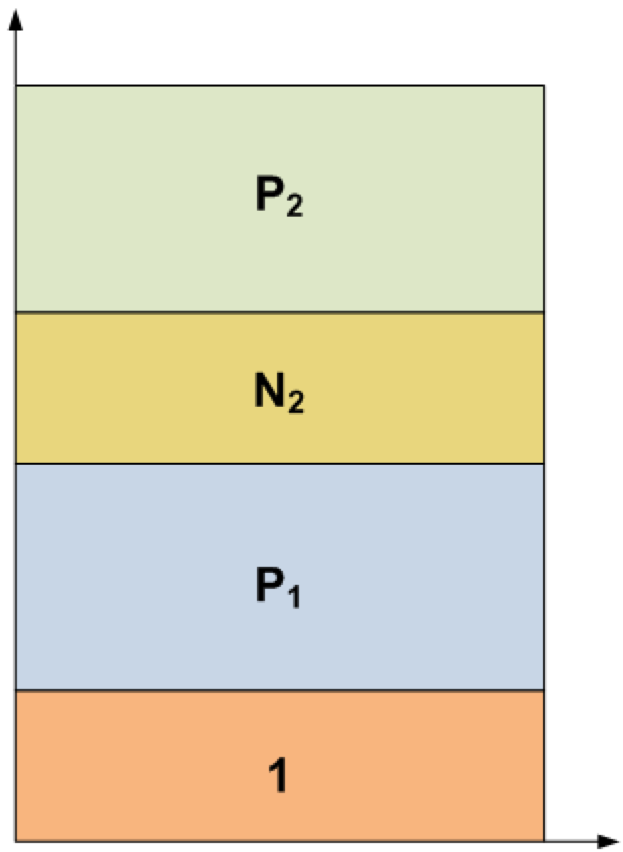
Model 1: Z interference Channels

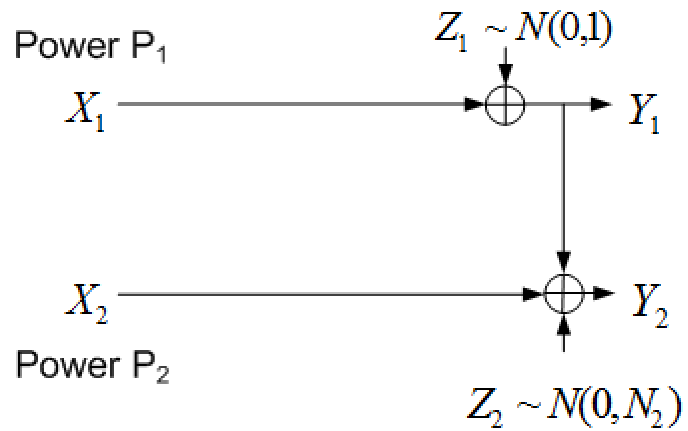
- Z-Gaussian Interference Channel as a degraded interference channel
- Discrete Time Channel as a band limited channel
- Multiplex Region: growing **Noisebergs**
- Overflow Region: back to superposition

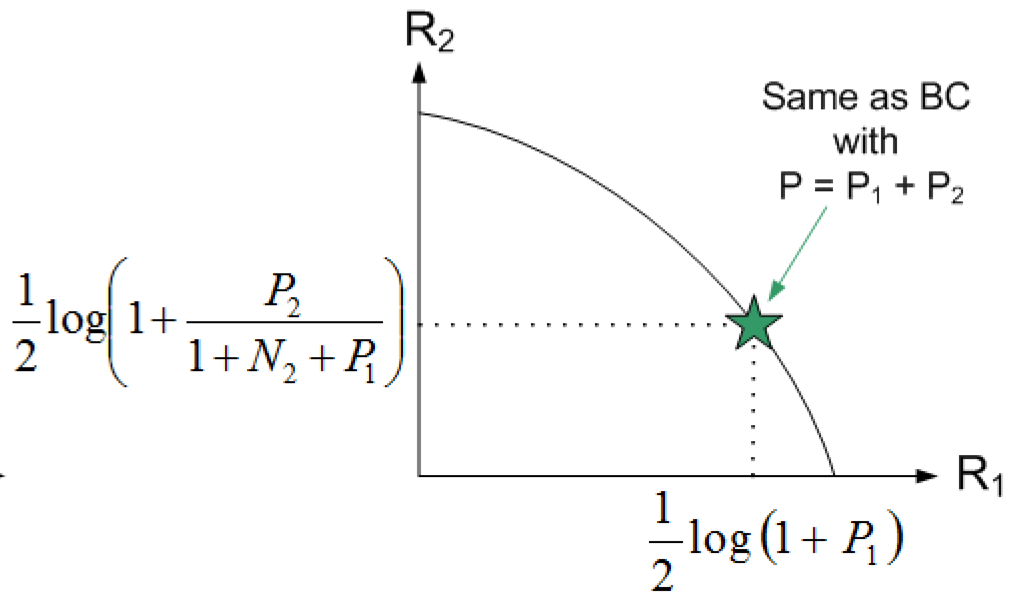
Degraded Gaussian Interference Channel



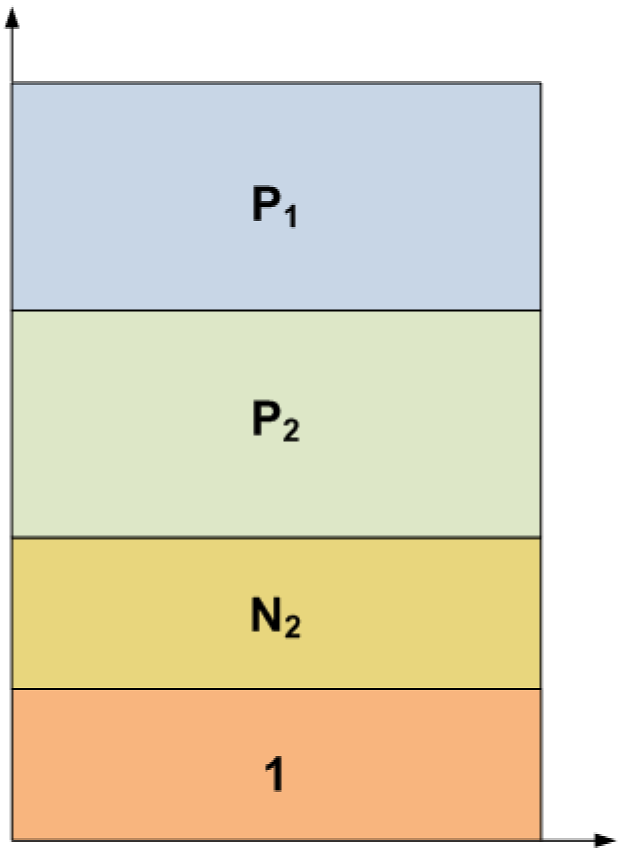
Degraded Interference Channel
- One Extreme Point (Sato's point)

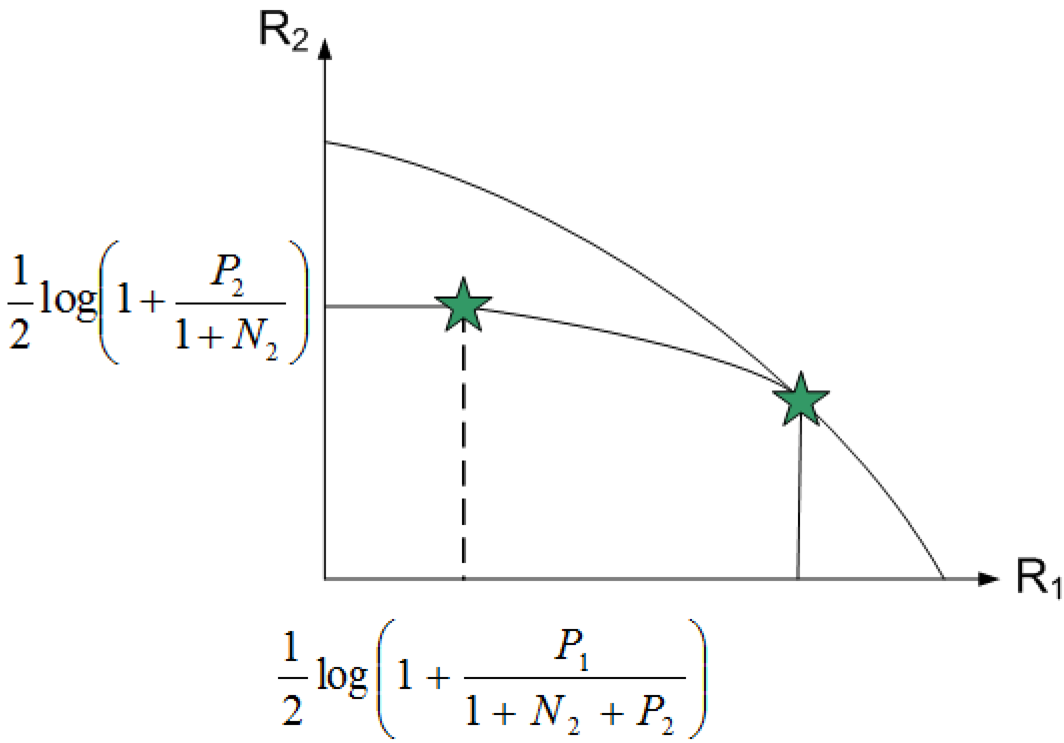




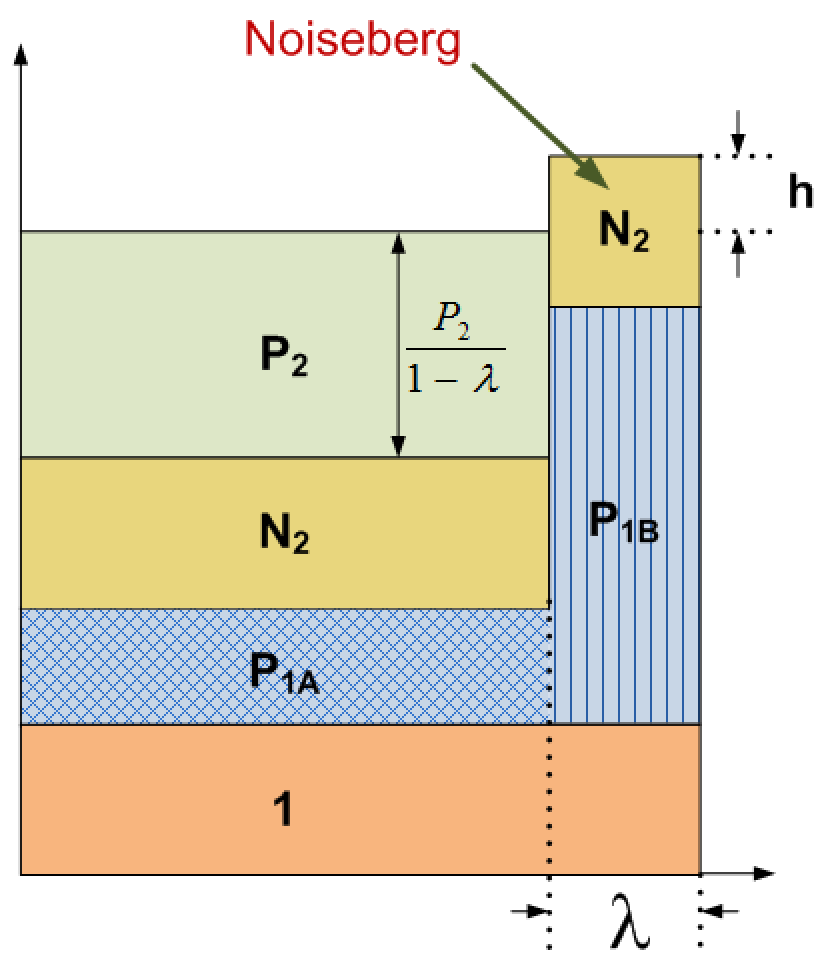


Degraded Interference Channel
- Another Extreme Point

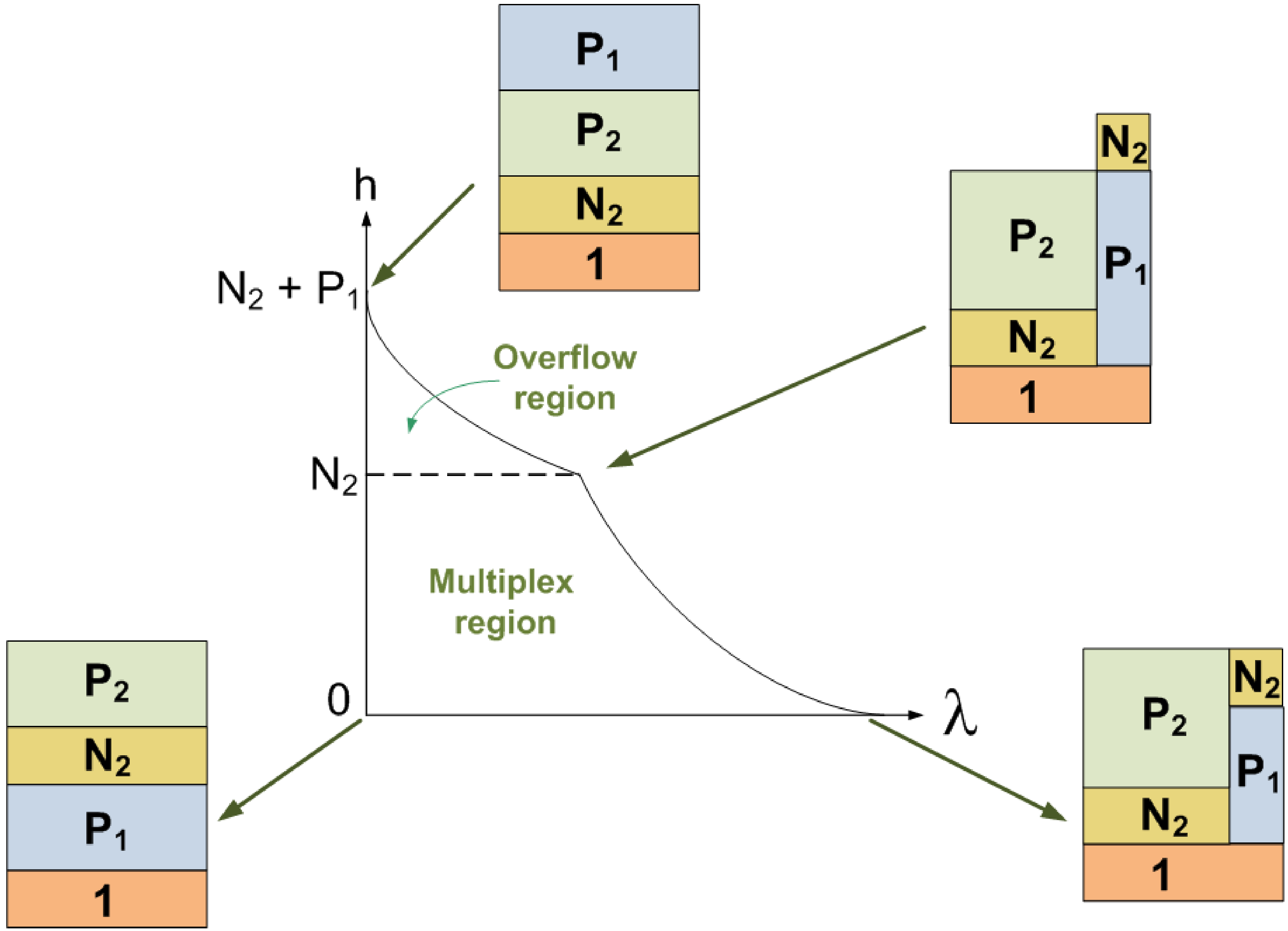




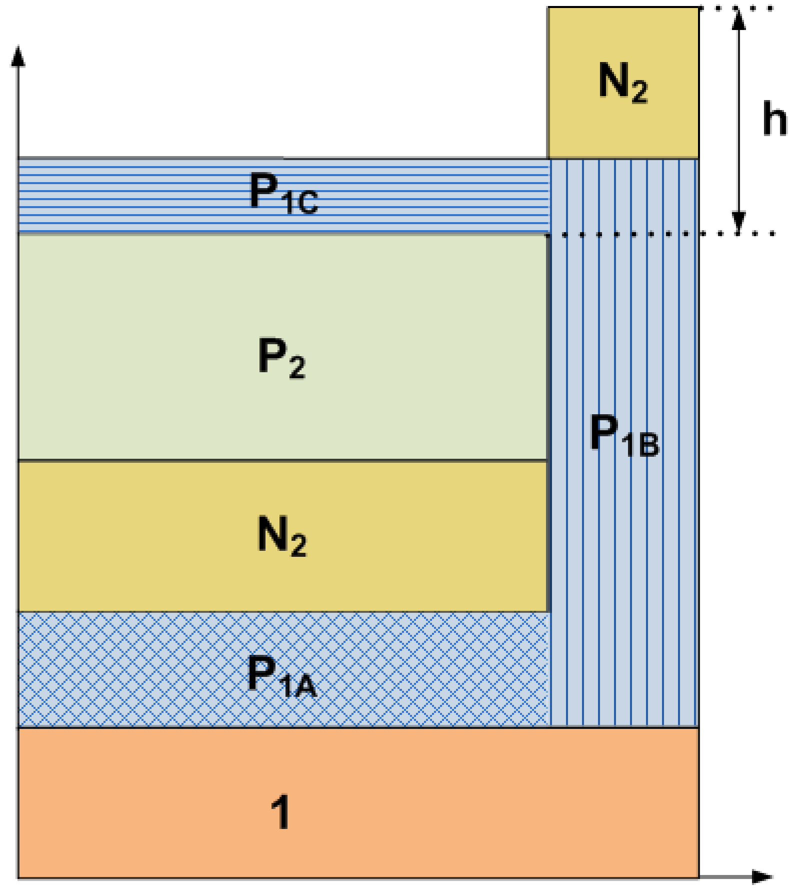
Intermediary Points (Multiplex Region)



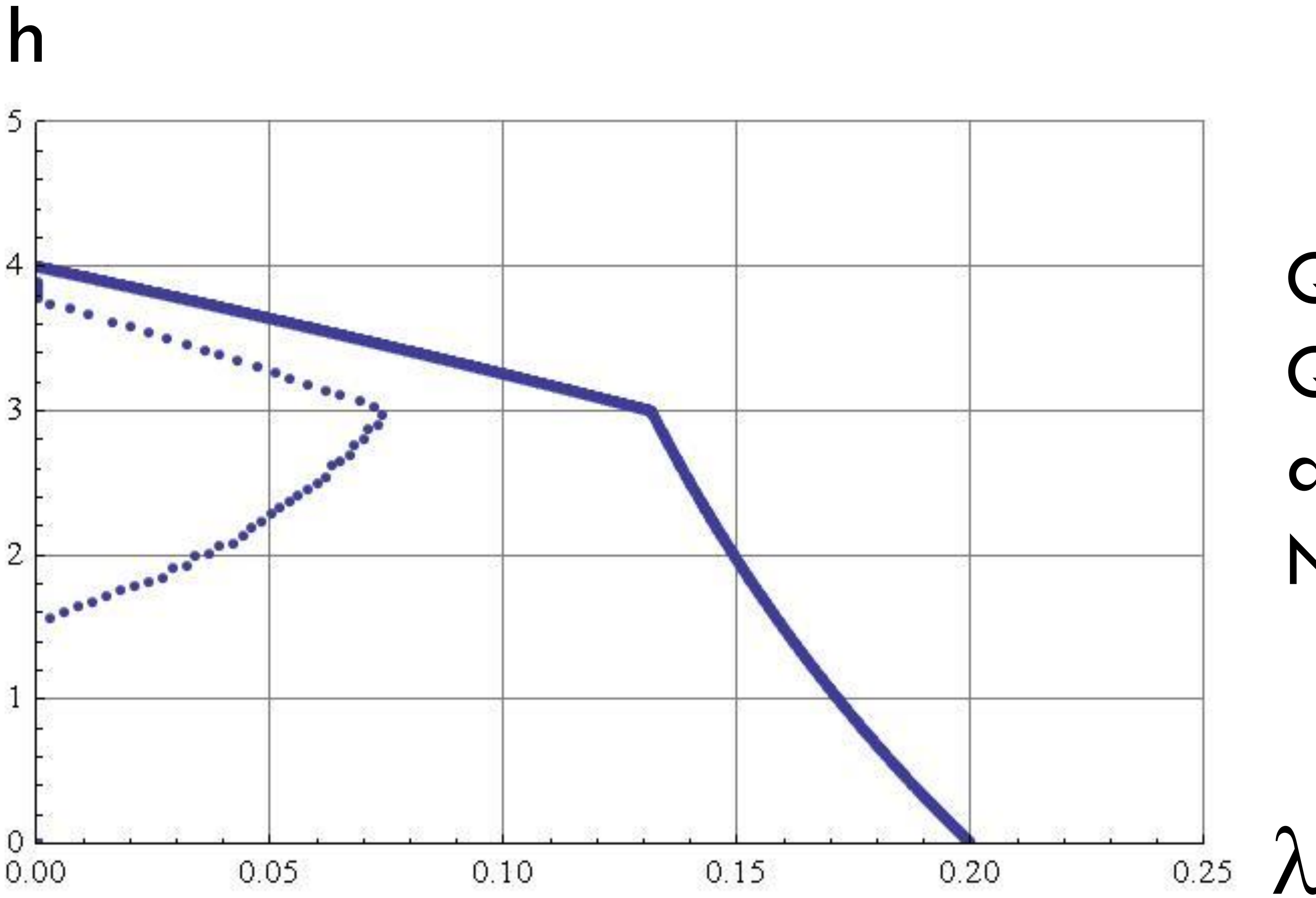
Admissible region for (λ, h)



Intermediary Point (Overflow Region)

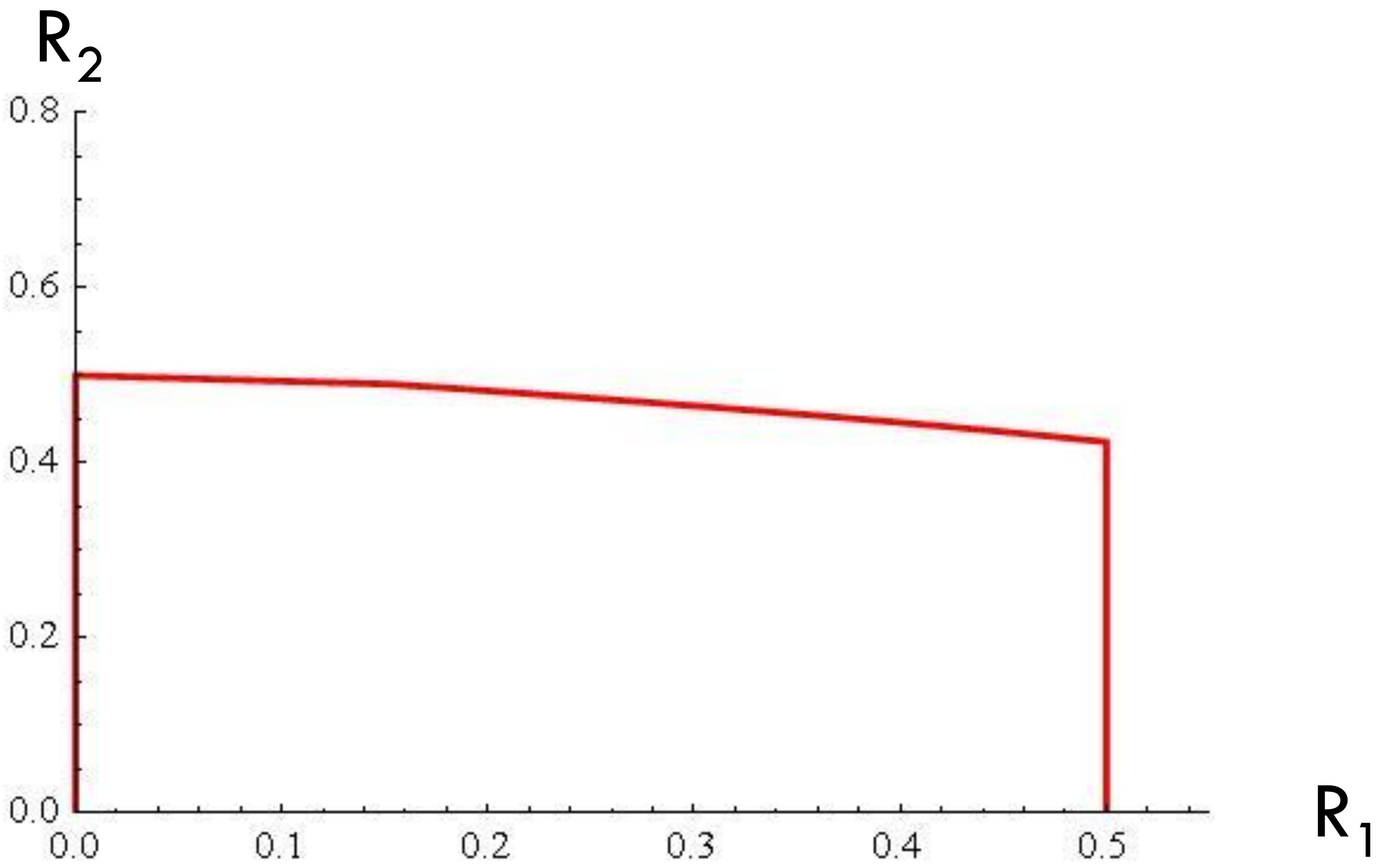


Admissible region



$Q_1=1$
$Q_2 = 1$
$a = 0.5$
$N_2 = 3$

The Z-Gaussian Interference Channel Rate Region



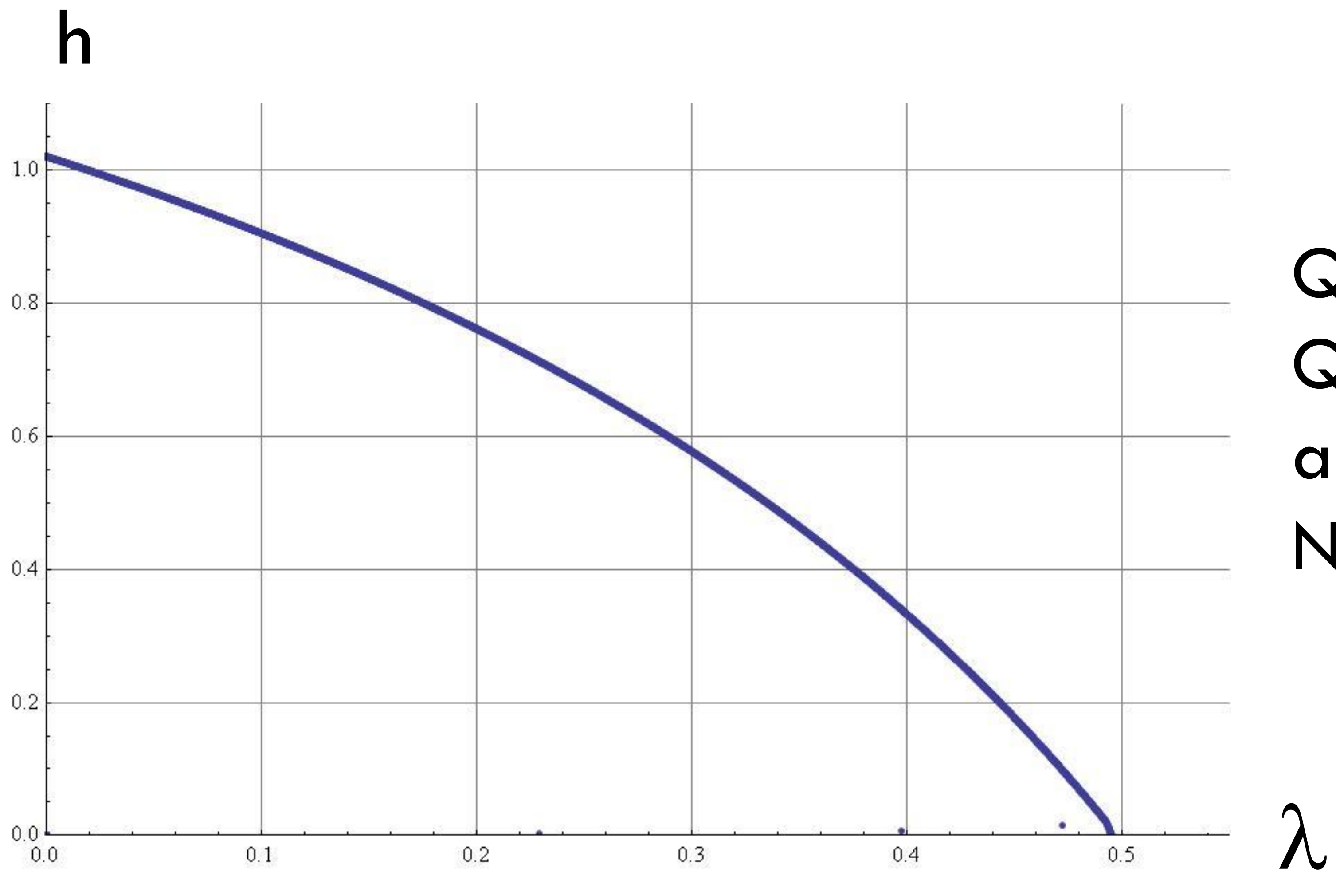
$Q_1 = 1$

$Q_2 = 1$

$a = 0.5$

$N_2 = 3$

Admissible region



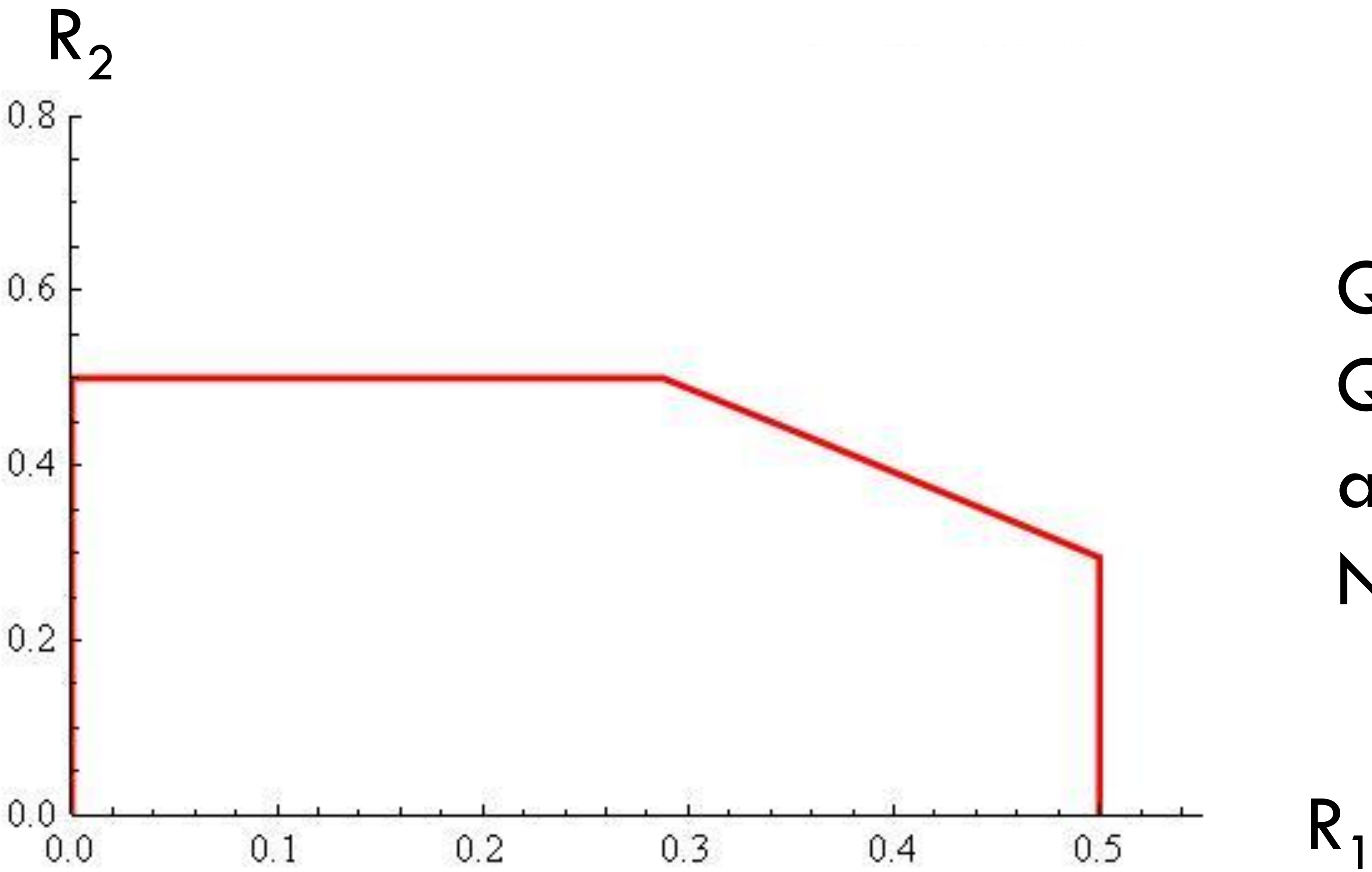
$Q_1 = 1$

$Q_2 = 1$

$a = 0.99$

$N_2 = 0.02$

The Z-Gaussian Interference Channel Rate Region



$Q_1 = 1$

$Q_2 = 1$

$a = 0.99$

$N_2 = 0.02$

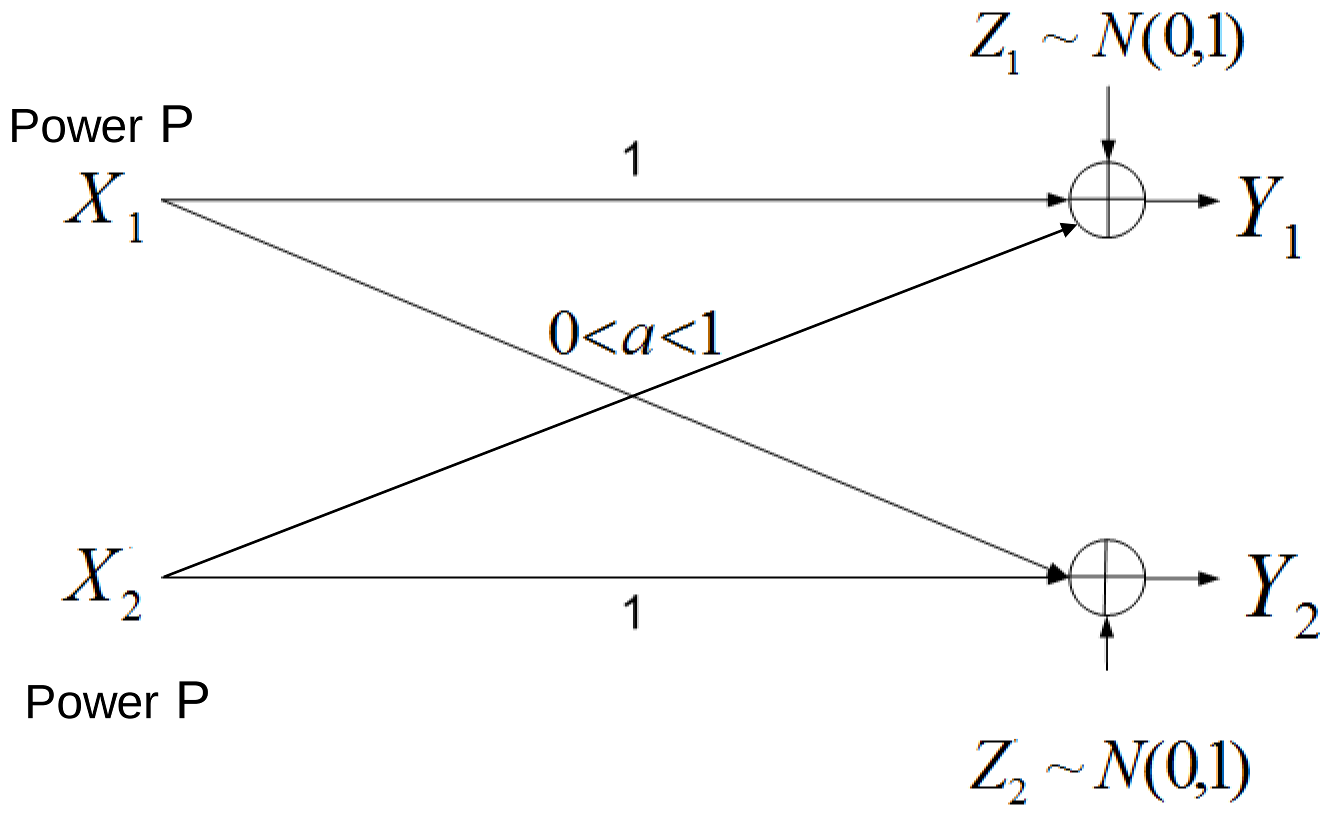
Remarks

- This is Han-Kobayashi region for Gaussian signaling (ISIT, 2023)

- Simple 2-D parameter space: (λ, h)

- Need entropy power-like inequality to establish capacity region

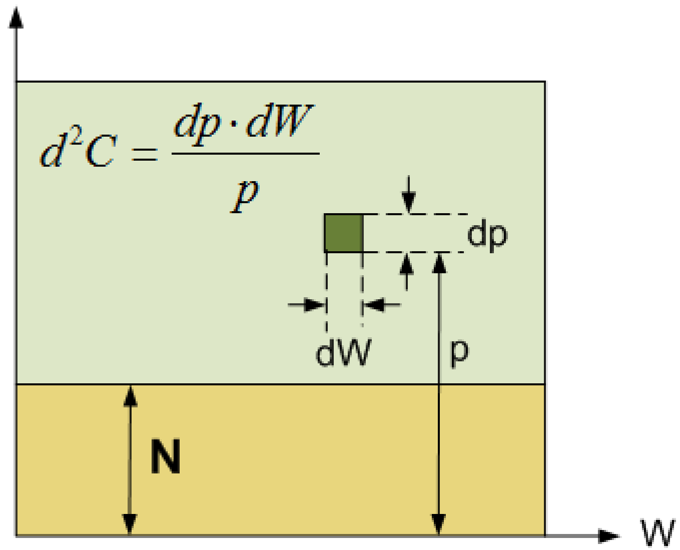
Model 2: Symmetric Gaussian Interference Channel

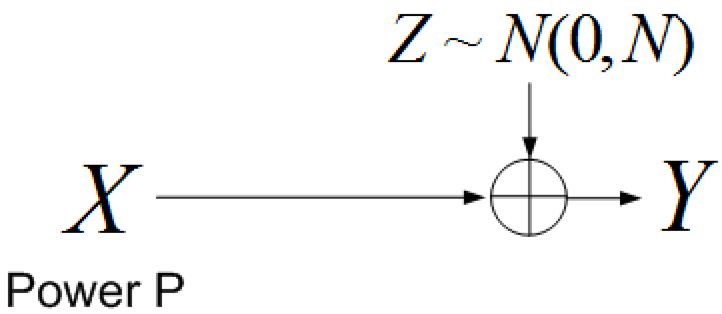


Symmetric Interference Channels

- Discrete time channel seen as a band limited channel – differential capacity
- Concave envelopes
- Symmetric and Asymmetric Superposition
- Phase transitions in parameter space

Differential capacity



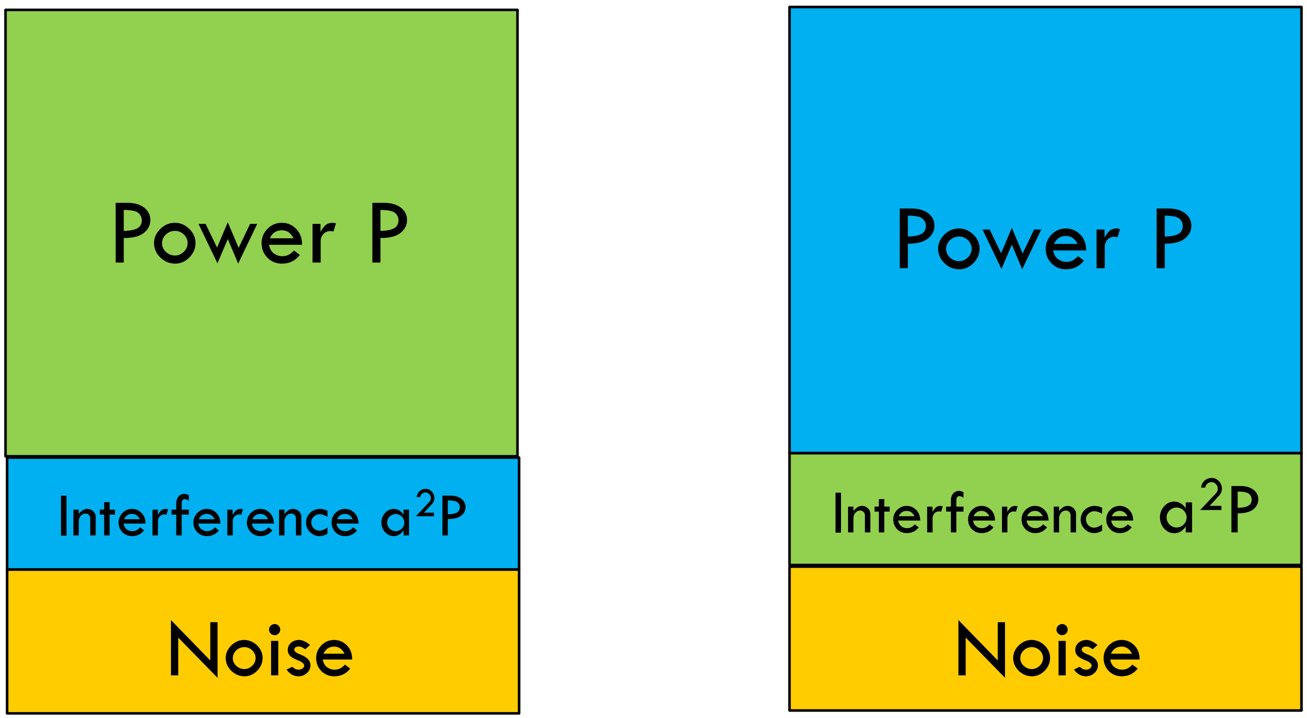


$$C = \iint d^2C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

Discrete time channel seen as a band limited channel

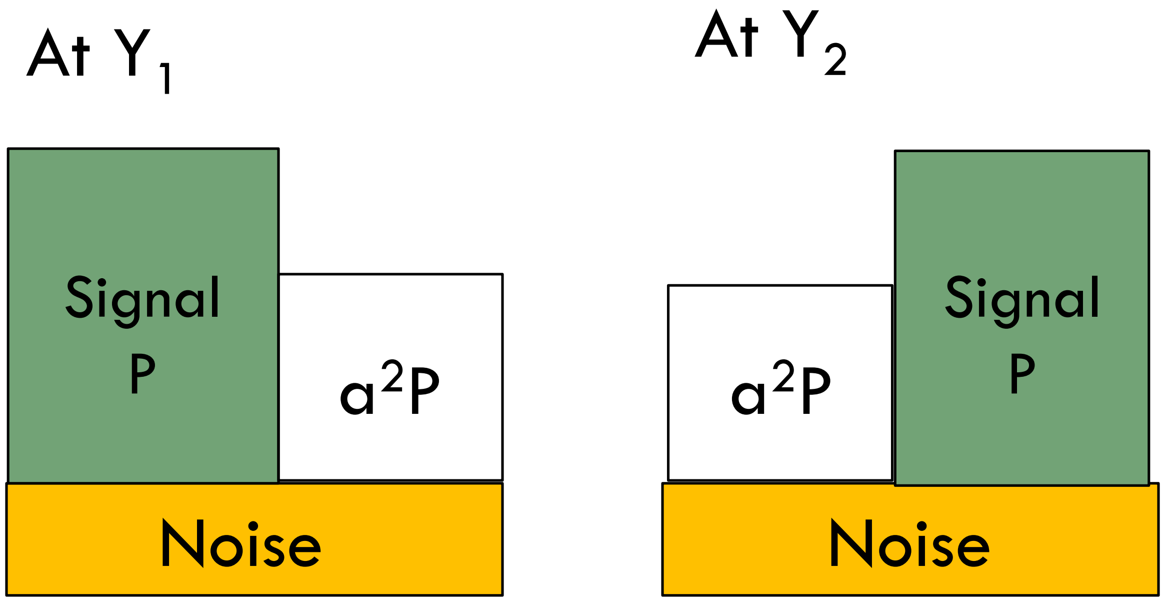
Interference channel: Spectra at Y_1 and Y_2

- At Y_1 / At Y_2



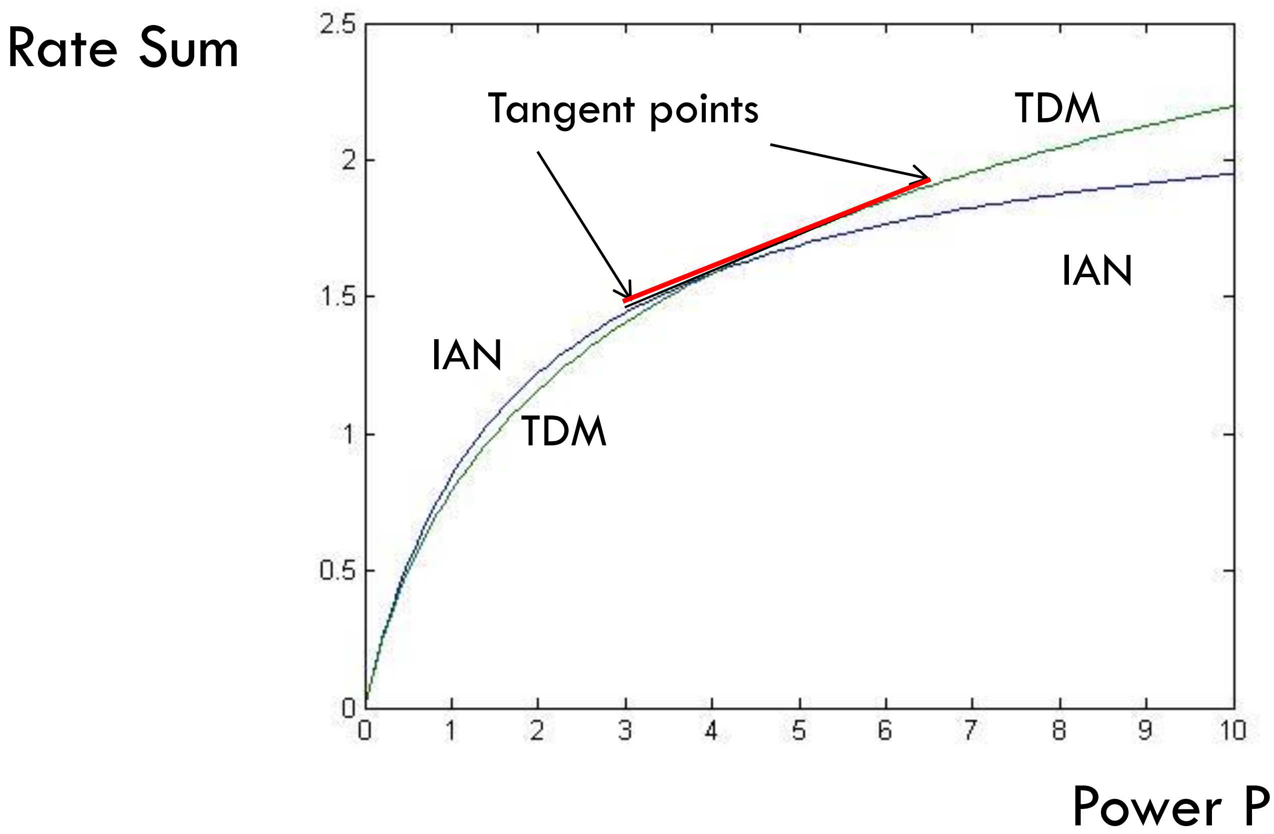
IAN: $R_1 + R_2 \leq \log\left(1 + \frac{P}{1+a^2P}\right)$

Interference Channel: TDM/FDM:



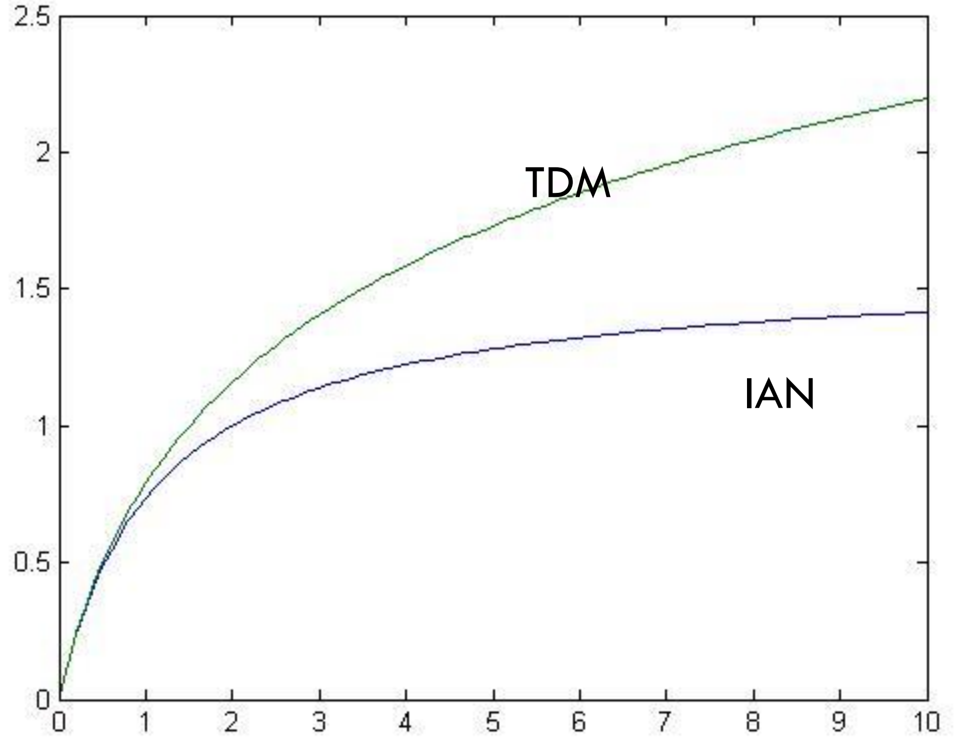
$$R_1 + R_2 \leq \frac{1}{2}\log(1 + 2P)$$

Concave Envelope
IAN vs TDM/FDM, a^2=0.25



Multiplex domination IAN vs TDM/FDM, $a^2=0.5$

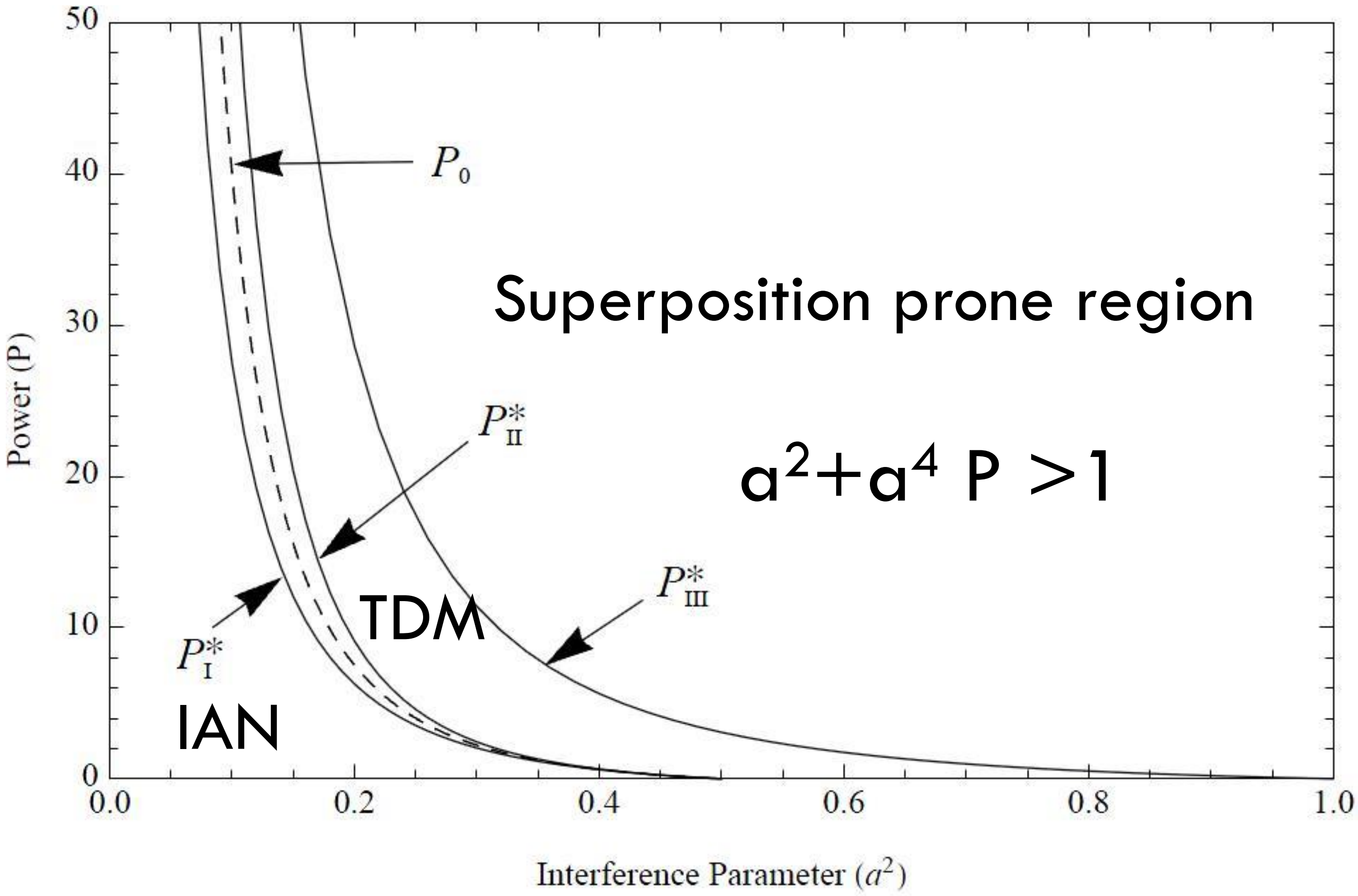
Rate Sum



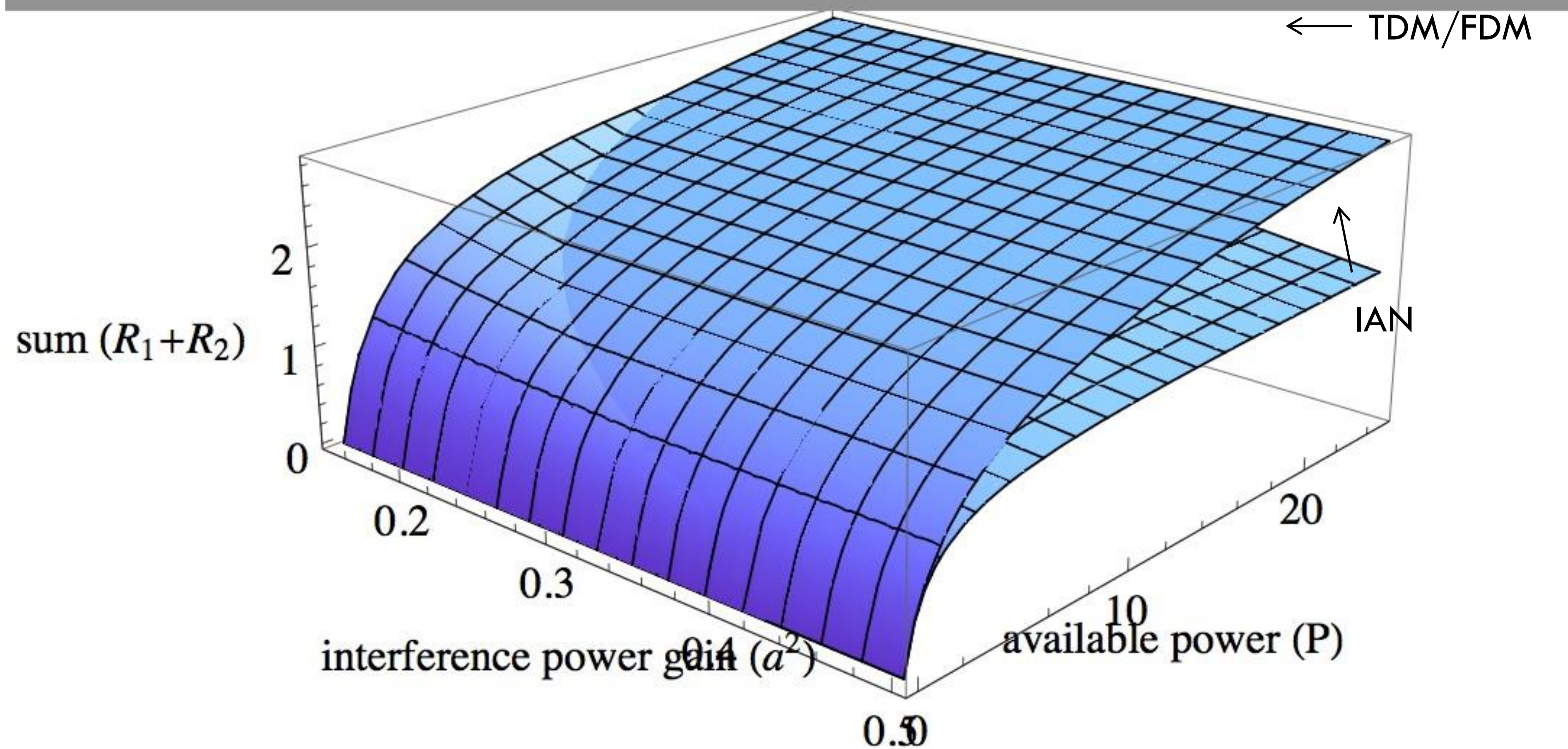
Power P

No intersection beyond $a^2=0.5$

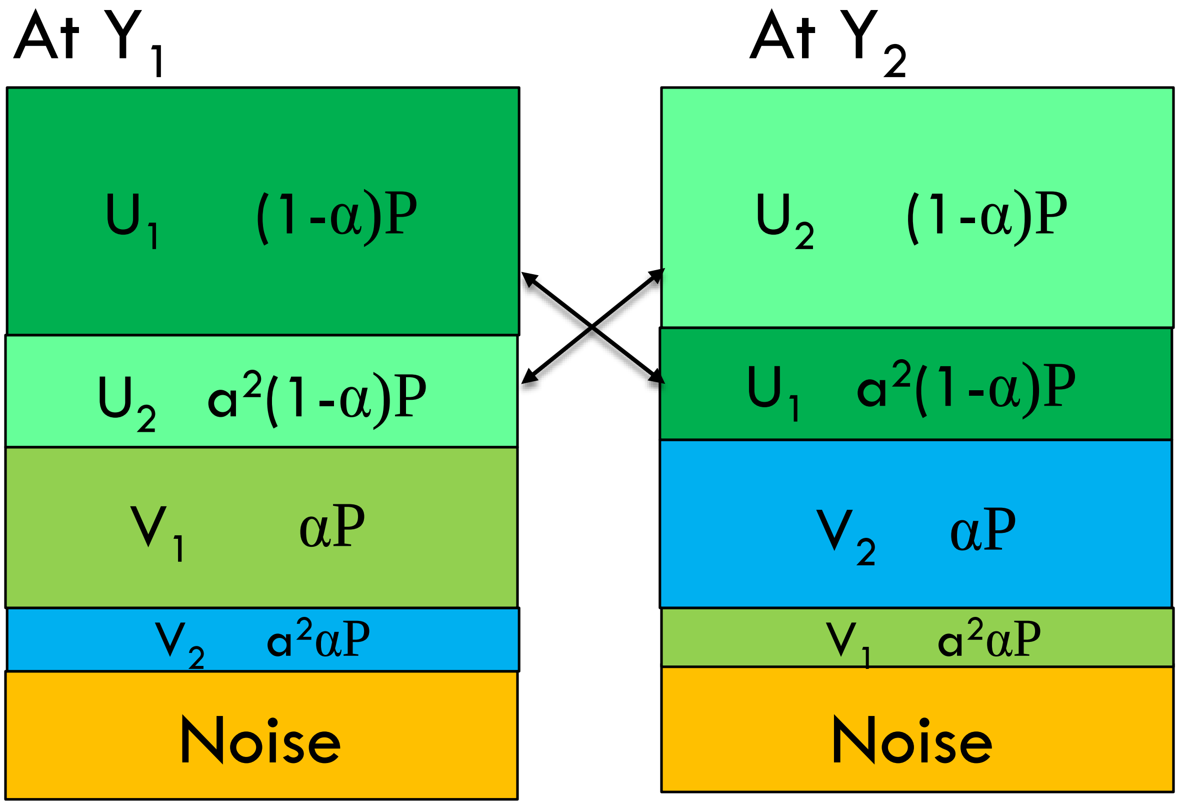
Interference as Noise and TDM/FDM



Rate Sum for IAN and TDM/FDM

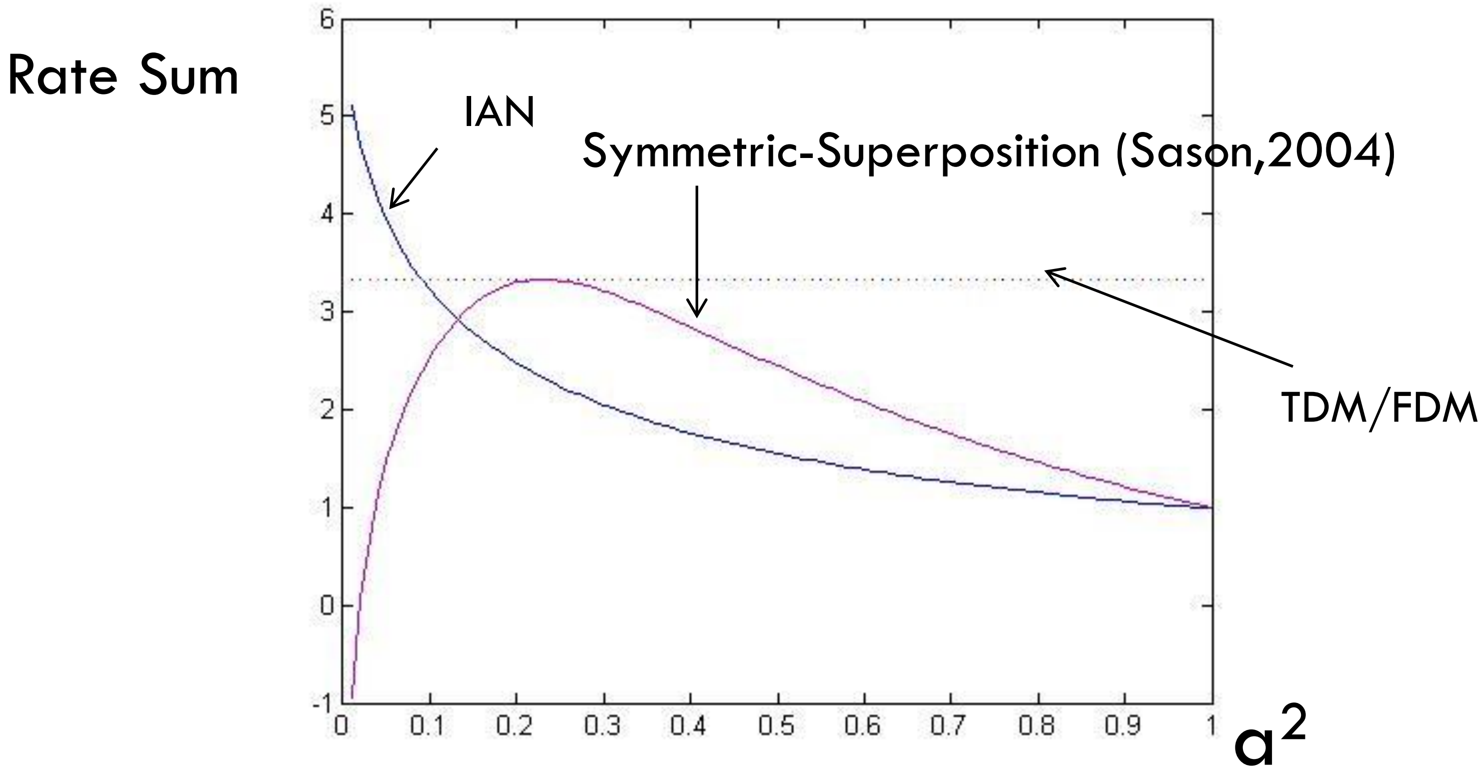


Superposition: partially decoding

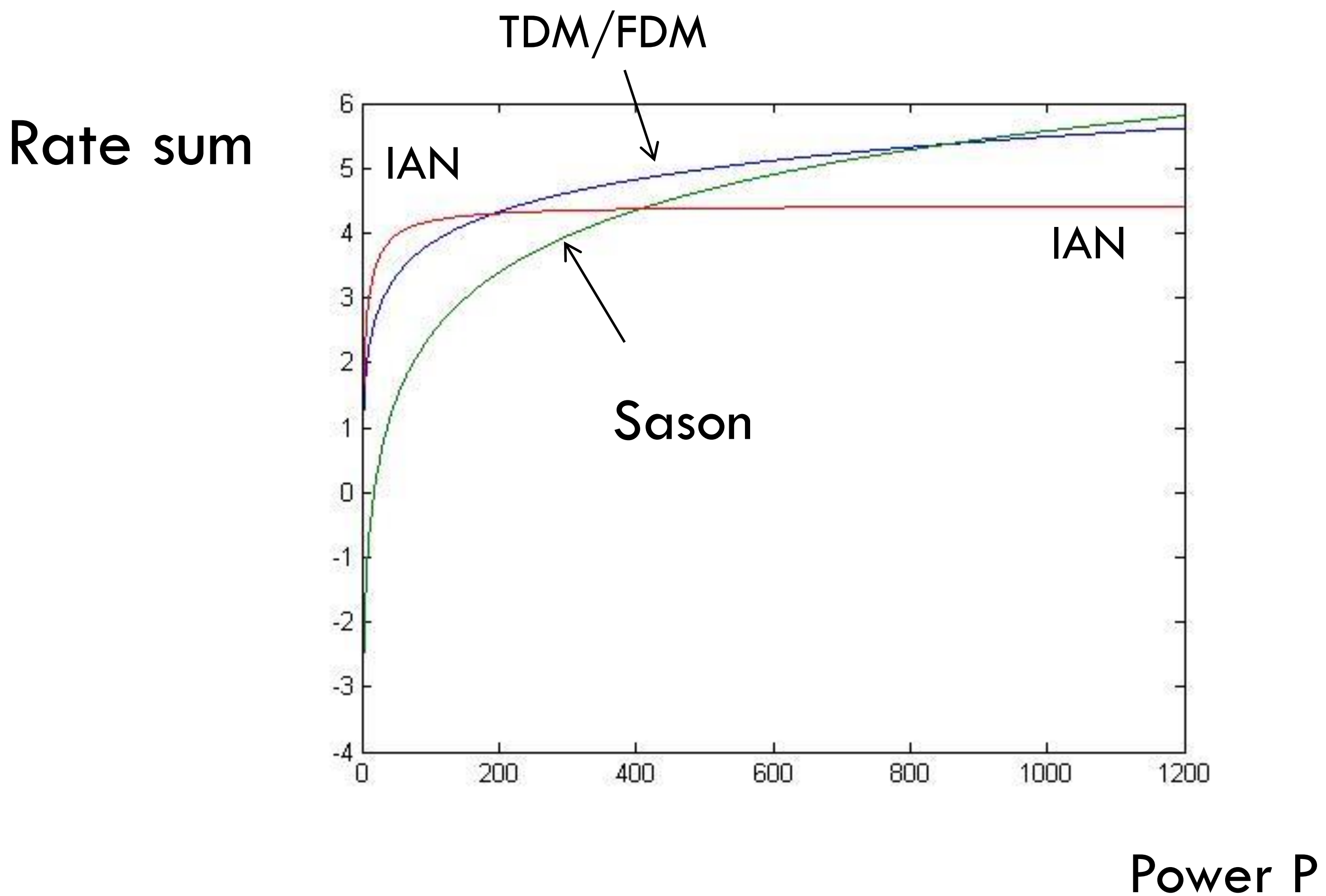


$$R_1 + R_2 \leq \log\left(\frac{1+P+a^2P}{\frac{1-a^2}{a^2}+a^2(1+a^2P)}\right)$$, Sason (2004)

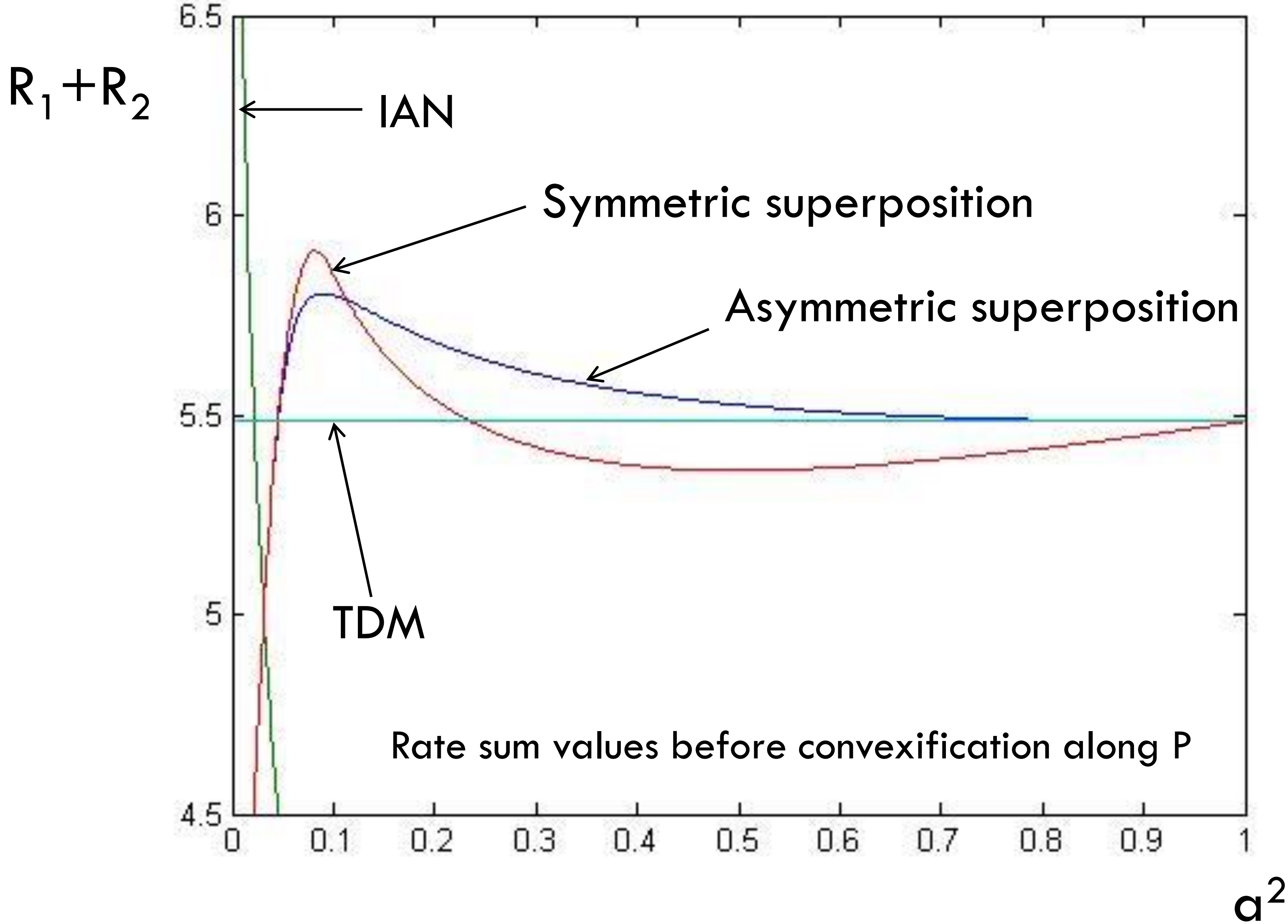
Point where Symmetric Superposition starts beating TDM/FDM
P=50



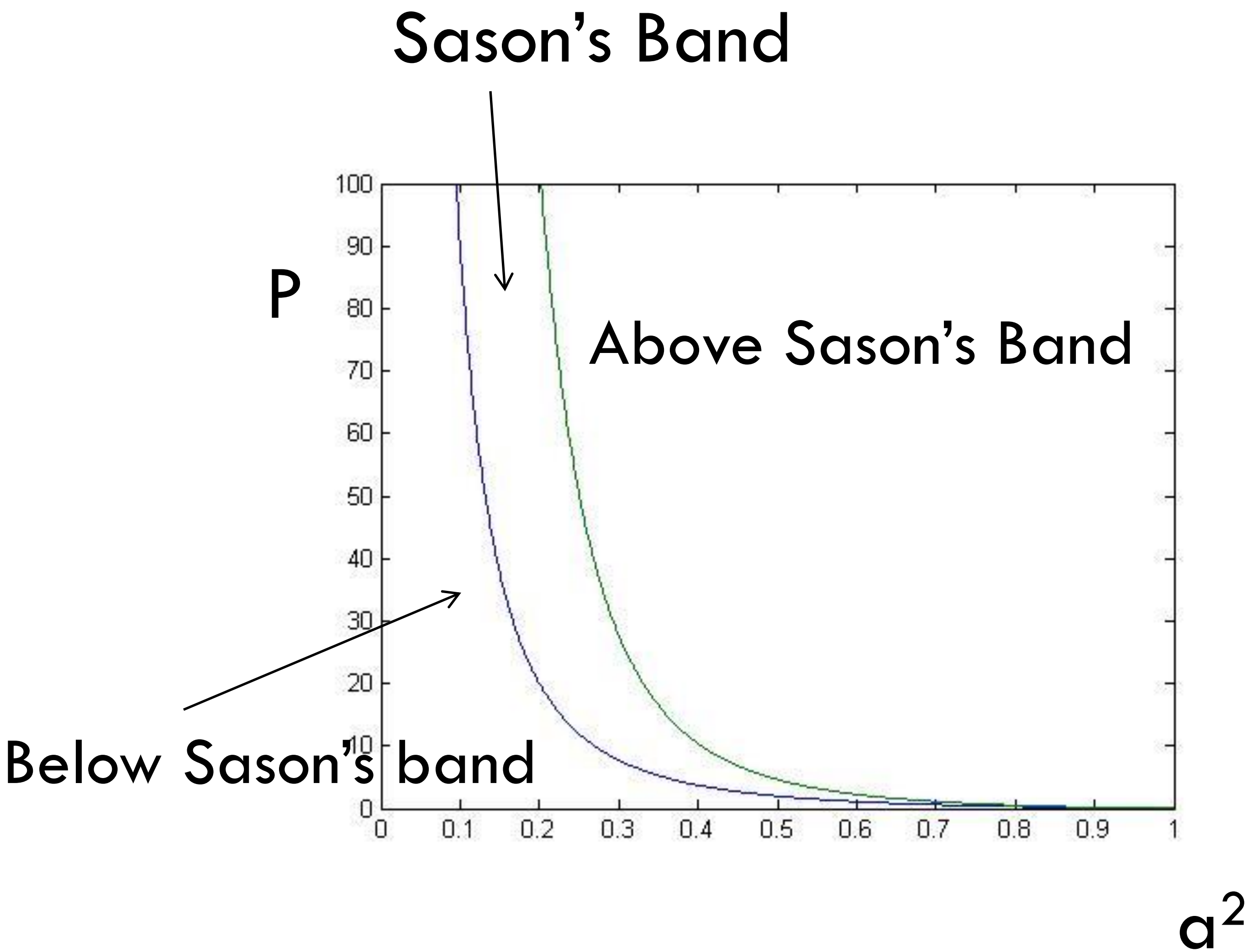
Rate Sum, a^2=0.05: Need convexification



Rate sum for P=1000, $0 \le a^2 \le 1$



Symmetric superposition:



Symmetric Superposition (continued):

- **Optimal choice for $\alpha = \alpha_1 = \alpha_2$:**

- Case 1:
- If $\frac{(1-a^2)}{a^4} \leq P \leq \frac{(1-a^6)}{a^6(1-a^2)}$ $(Sason's\ Band)$

  then set $\propto P = a^2(1 + a^2 P) - 1;$

- Case 2:
- If $P \geq \frac{(1-a^6)}{a^6(1-a^2)}$ $(Above\ Sason's\ Band)$

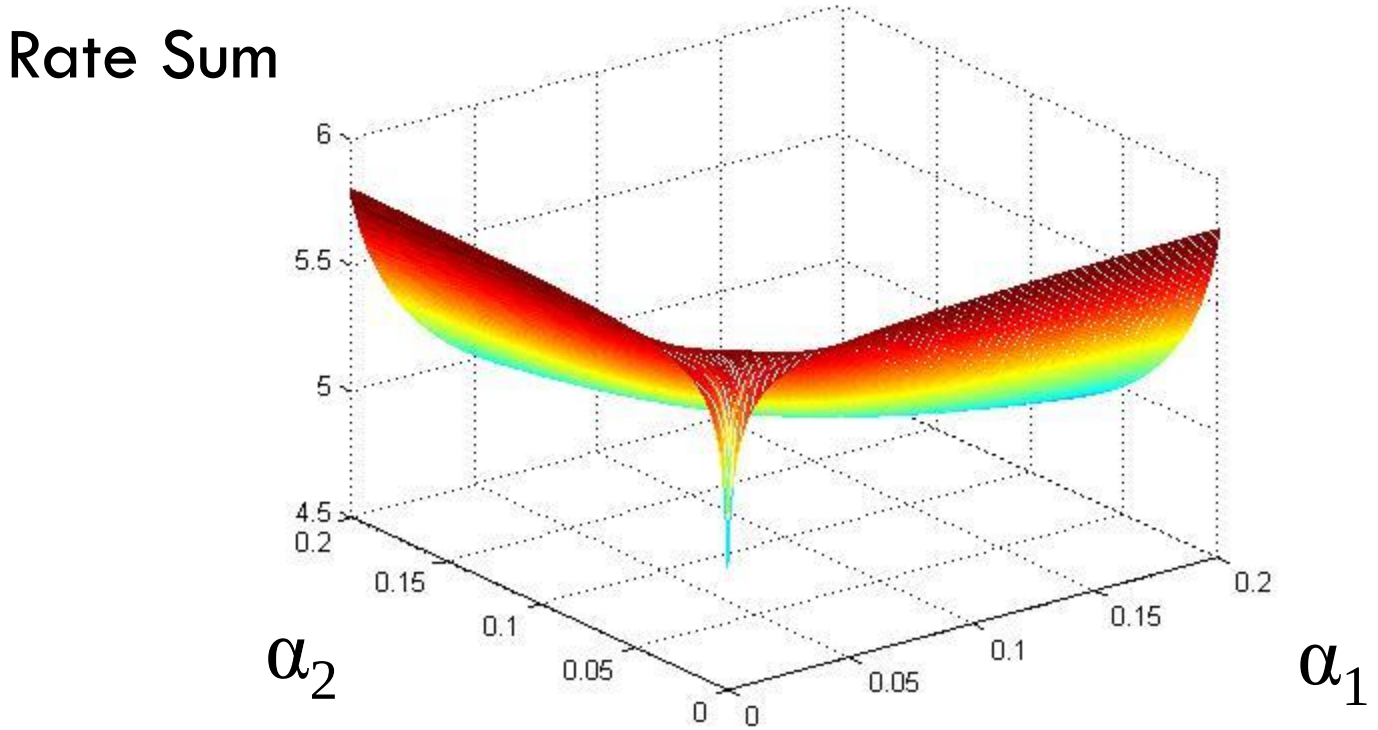
  then set $\propto P = \frac{(1-a^2)}{a^2(1+a^2)}.$

Note: Invariant with P

Symmetric Superposition (continued):

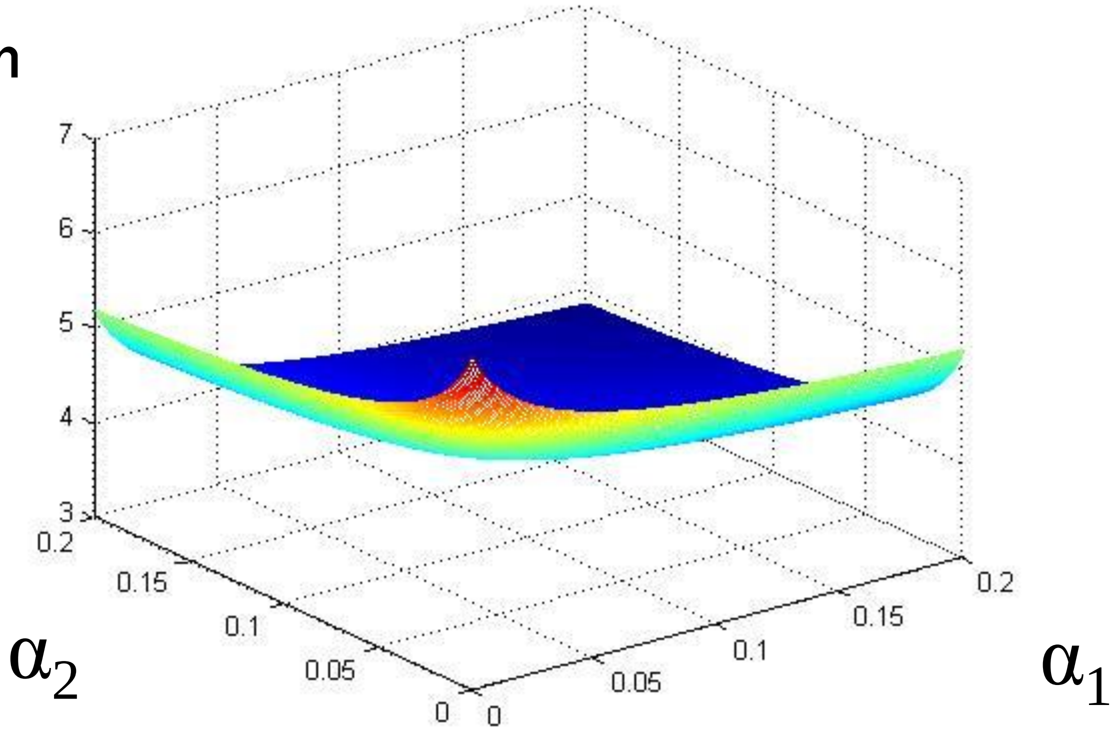
- In Sason's Band:
- $$R_1+R_2 \leq \log\left(\frac{a^2\left(1+P+a^2P\right)}{1-a^2+a^4(1+a^2P)}\right)$$
- Above Sason's Band:
- $$R_1+R_2 \leq \frac{1}{2}\log\left(\frac{\left(1+a^2\right)^2\left(1+P+a^2P\right)}{4a^2}\right)$$

The hummingbird function:

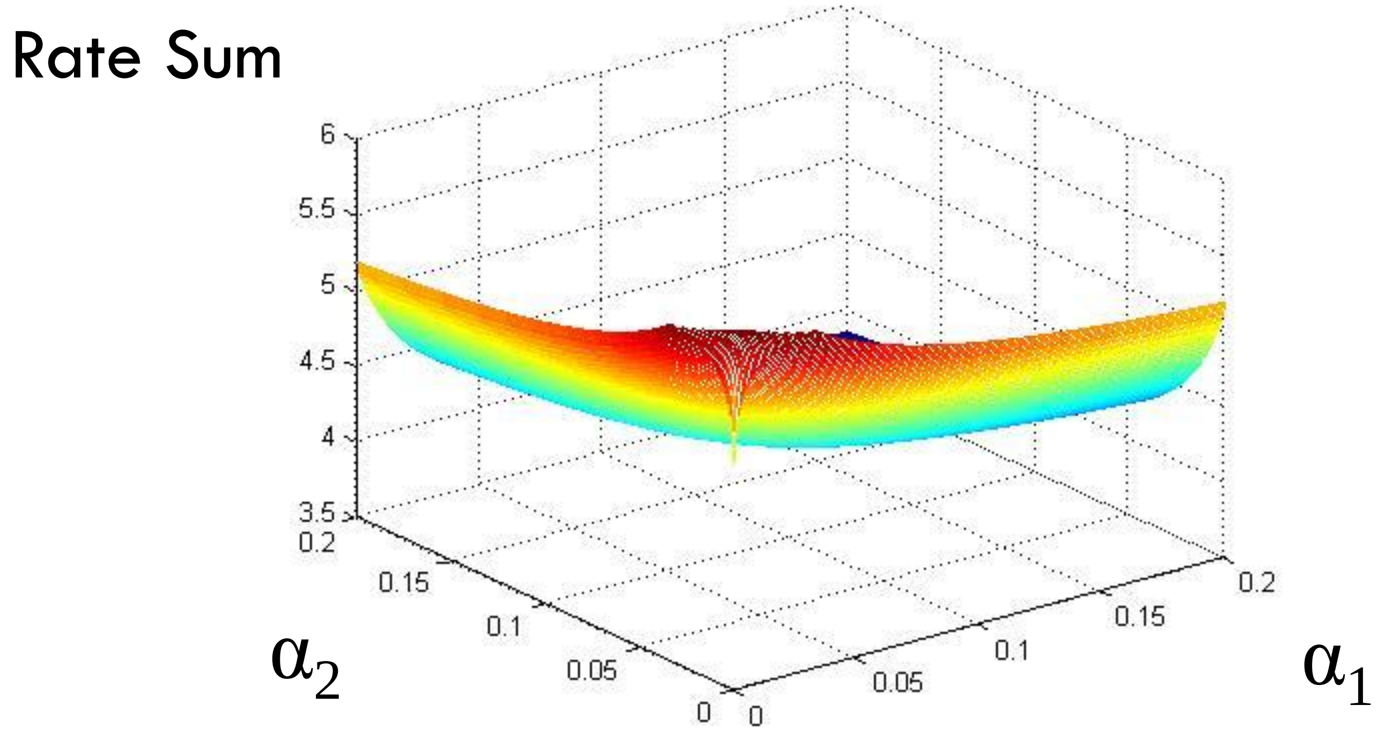


The shroud function

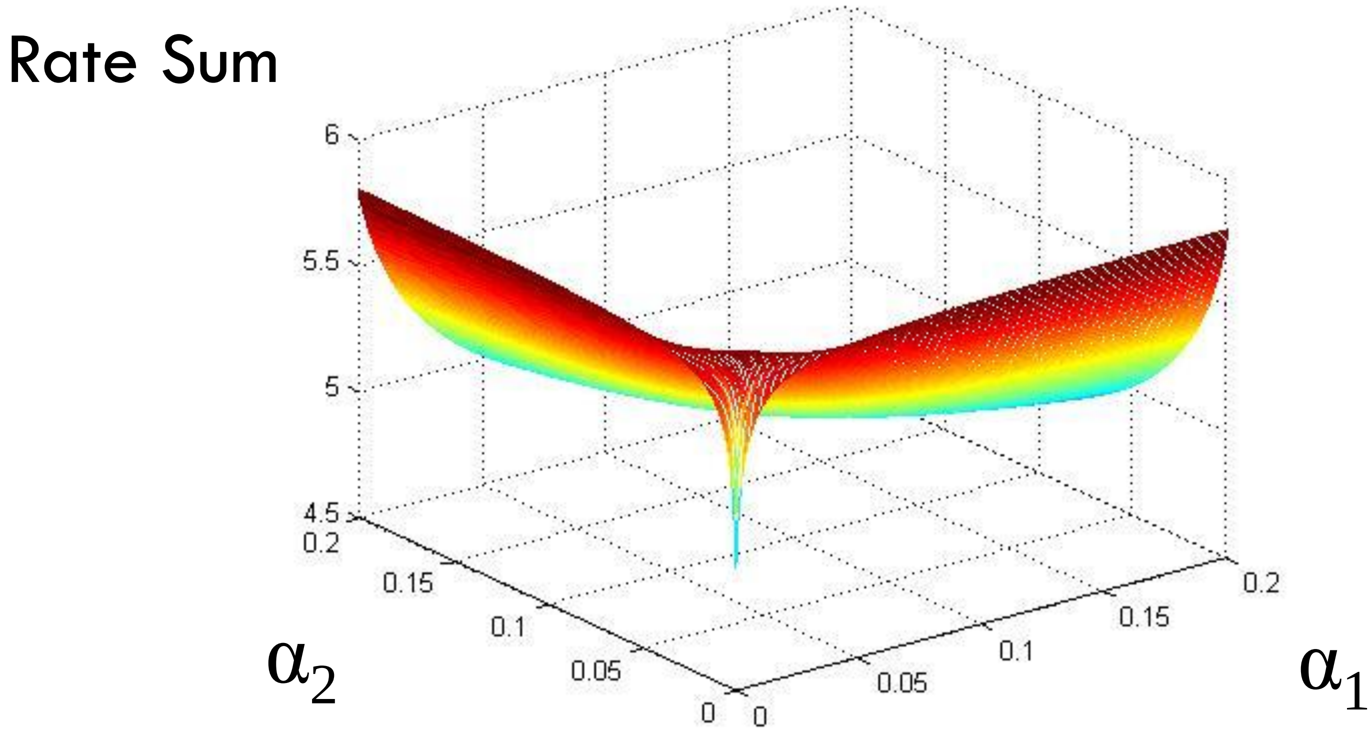
Rate Sum



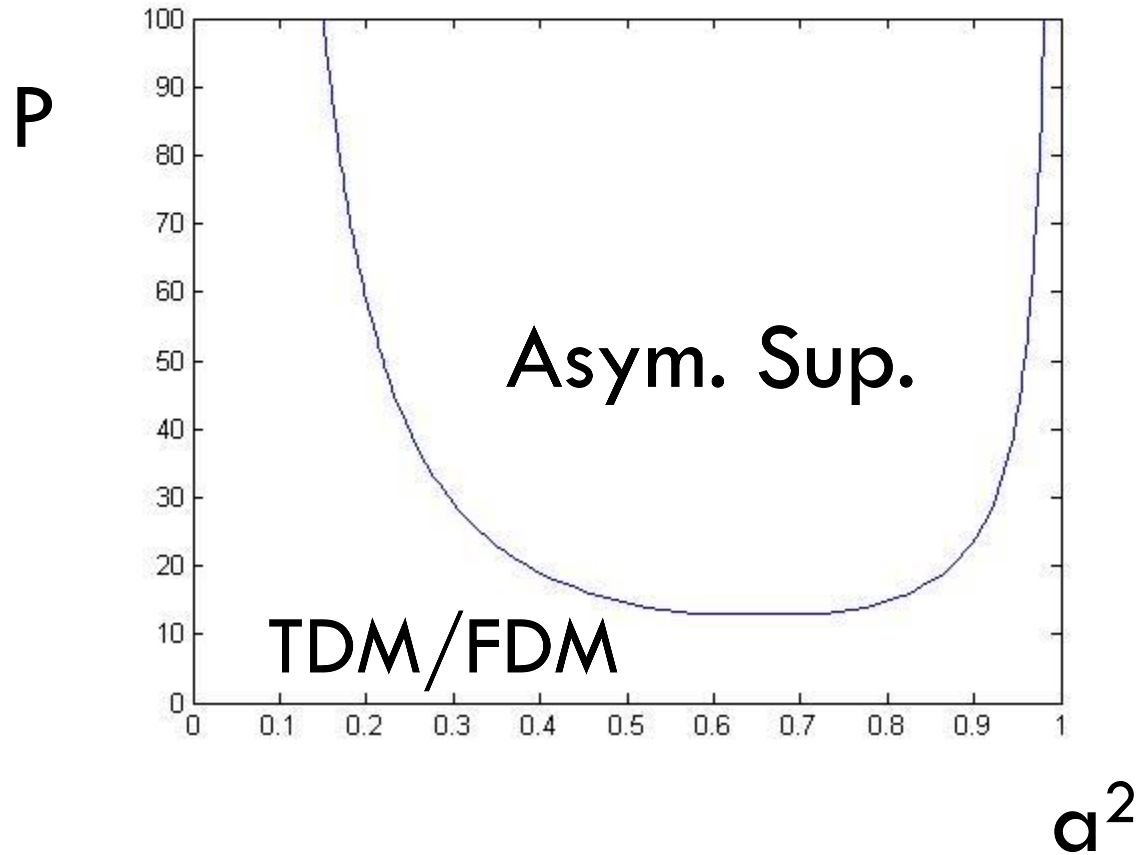
Min (hummingbird, shroud)



Flapping wings

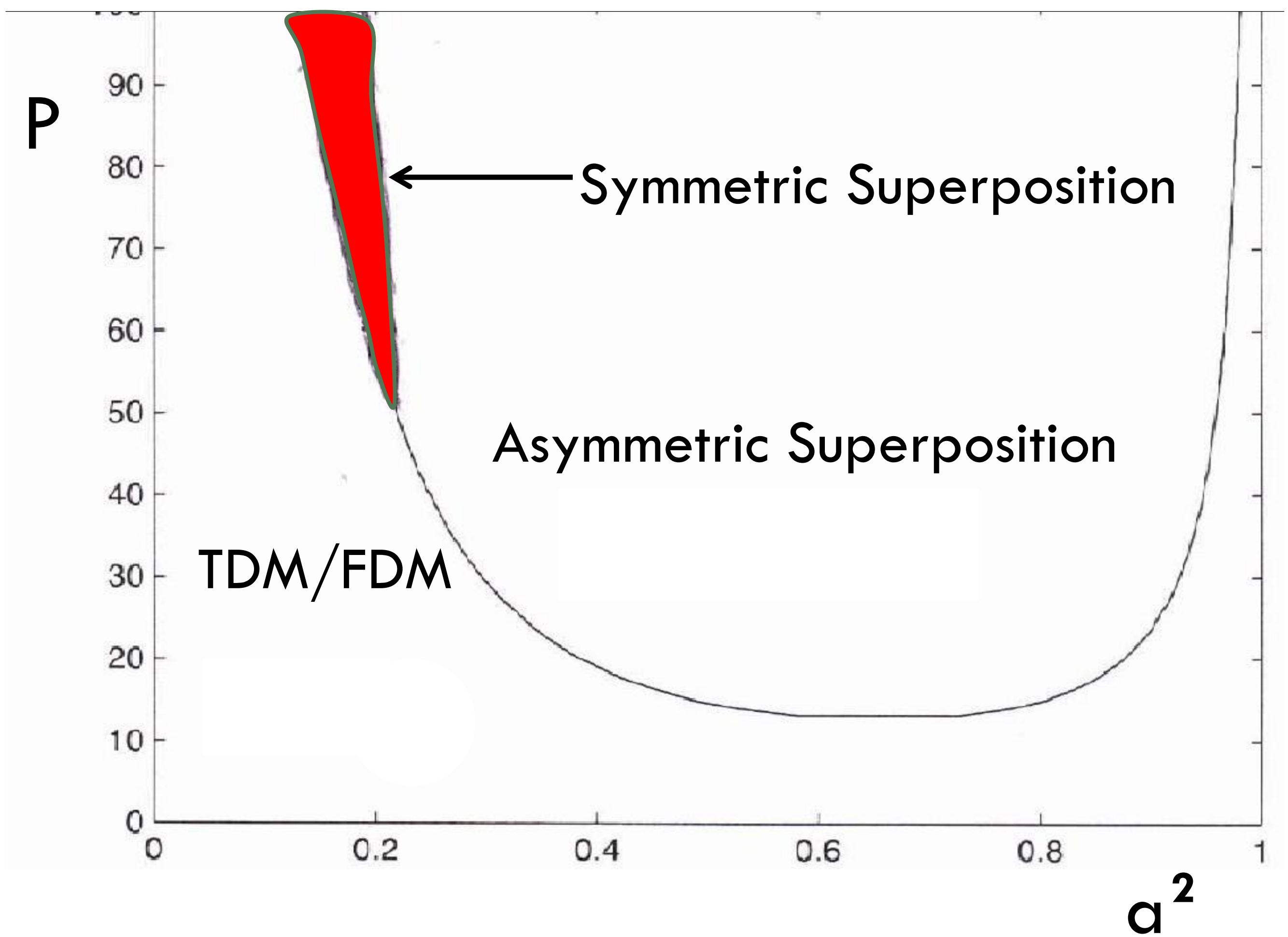


Asymmetric-Superposition vs TDM/FDM

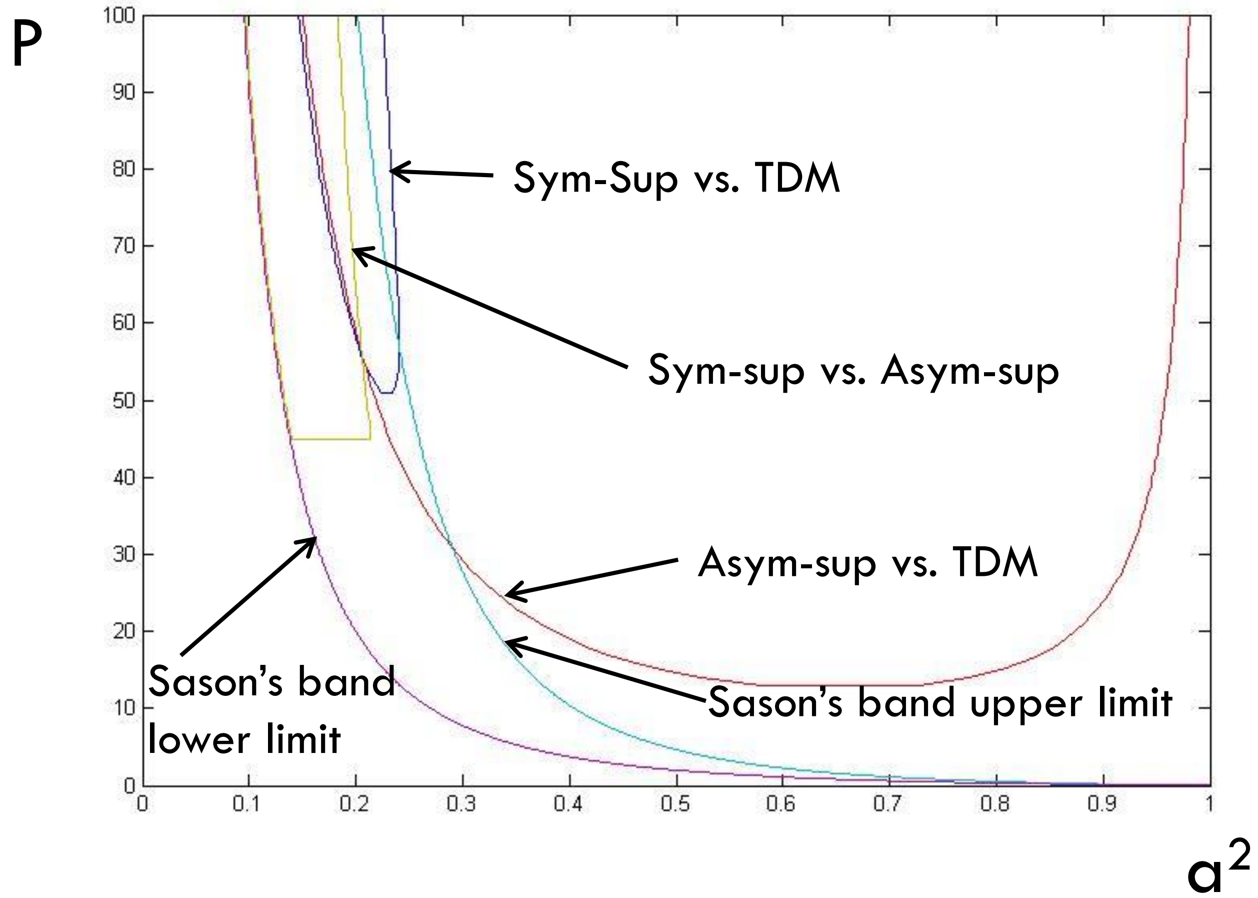


Phase Transitions in Weak Interference

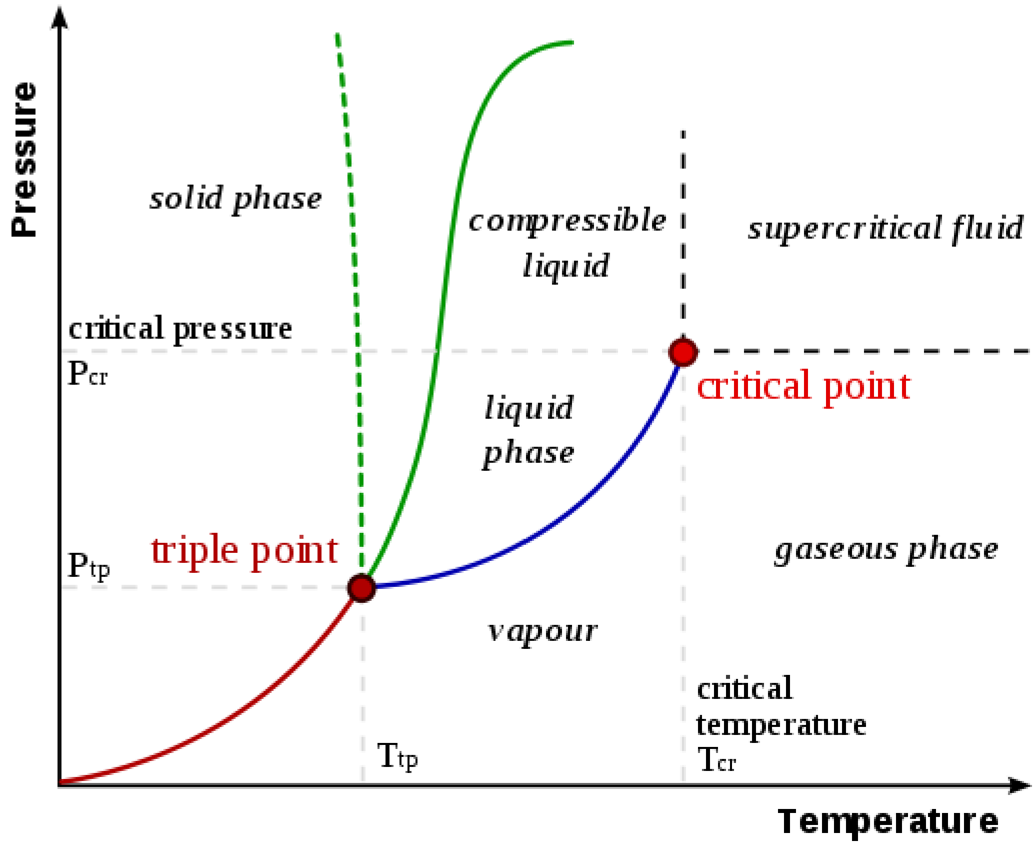
Note: Transitional regions due to convexification along P not included.



Pairwise Phase Transitions



A pleasant resemblance

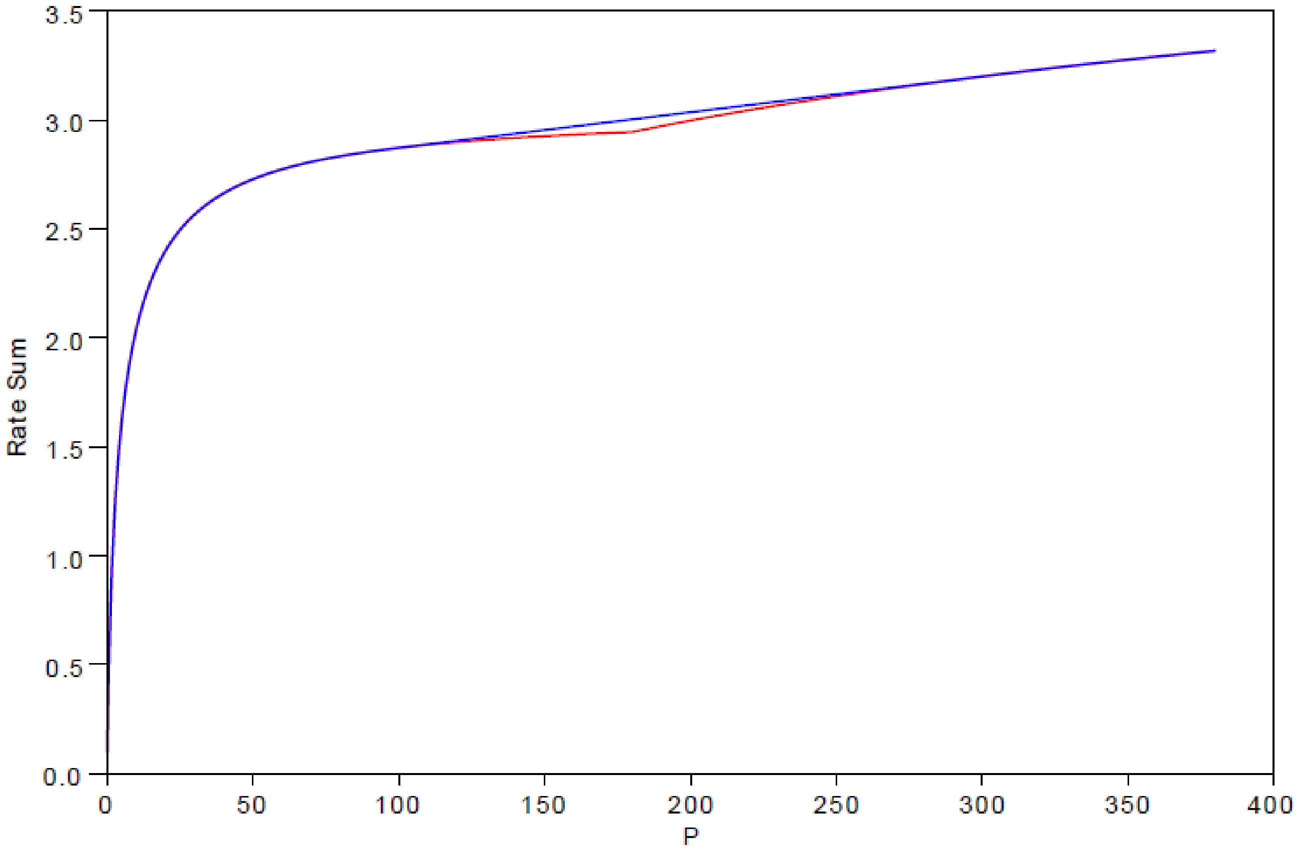


Asymptotically as $P \rightarrow \infty$

$0 < a^2 < 0.087$ -- symmetric superposition is best

$0.087 < a^2 < 1$ – asymmetric superposition is best

As before: Need convexification along P



Final remarks

- Powerful tool: Concave envelopes to transition from one mode to another: time sharing between modes

- Shown a full taxonomy of phase transitions in (a^2, P) parameter space with $0 < a^2 < 1$, $P > 0$:
- 4 pure modes (IAN, TDM, Symmetric Superposition, and Asymmetric Superposition) and
- 4 transitional regions (IAN vs. TDM, TDM vs. Sym-Sup, TDM vs. Asym-Sup, and Sym-Sup vs. Asym-Sup)

Challenges

- Find the full capacity region
- Show Gaussian signalling is best
- Consider channels with parameters (P_1, P_2, a, b)
- Interference Channel is still an open problem

Contatos são bem vindos.

- max@fee.unicamp.br