The physics and geometry of knot homologies

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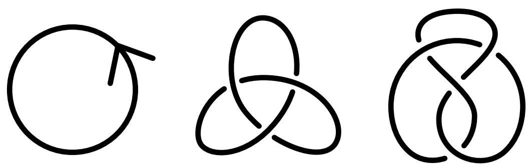
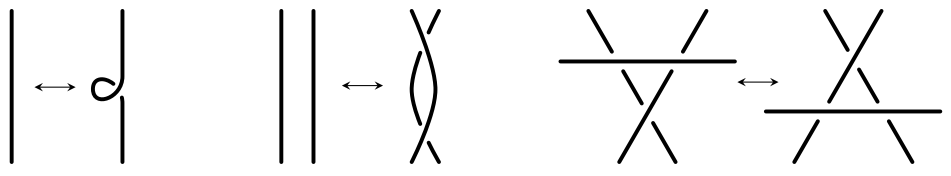
Knots, quantization, knot homologies & super-A-polynomials

Knot invariants & Chern–Simons theory



Pictured are the unknot (0_1), the trefoil knot (3_1), and the figure-8 knot (4_1) with zero, three, and four crossings, respectively.

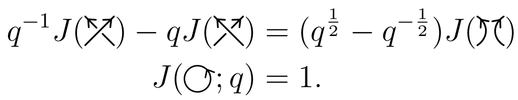
Reidemeister moves



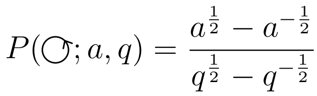
*who assumes full responsibility for all errors that remain in this document.

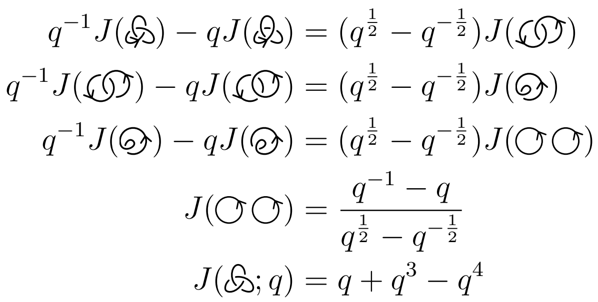
Polynomial knot invariants

- Alexander (1928) $\longrightarrow$ Alexander polynomial $\Delta(q)$
- Jones (1984) $\longrightarrow$ Jones polynomial $J(K;q)$, satisfying the skein relations



- HOMFLY (1985) $\longrightarrow$ HOMFLY polynomial $P(a,q)$, $a = q^2$. Another normalization of the unknot





Puzzles

1. What is the 3-dimensional origin of these knot invariants?
2. Why do knot polynomials have integer coefficients?

For (1), there is Chern–Simons theory (Witten 1989).

$$S_{\mathrm{CS}} = \int_M (A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A),$$

where M is a 3-manifold.

QFT

$$Z_M^G(q) = \int DA \mathrm{e}^{\frac{ki}{4\pi} S_{\mathrm{CS}}},$$

partition function $q = \mathrm{e}^{\frac{2\pi i}{k + c_2(G)}}$.

Wilson loops

$$\left\langle \mathrm{Tr}_R \mathcal{P} \mathrm{e}^{\oint_K A} \right\rangle = \int DA (\mathrm{Tr}_R \mathcal{P} \mathrm{e}^{\oint_K A}) \mathrm{e}^{\frac{ki}{4\pi} S_{\mathrm{CS}}}$$

and

$$J(K;q) = \frac{\left\langle \mathrm{Tr}_{R=\Box} \mathcal{P} \mathrm{e}^{\oint_K A} \right\rangle_{G=\mathrm{SU}(2),\, M=S^3}}{\left\langle \mathrm{Tr}_{R=\Box} \mathcal{P} \mathrm{e}^{\oint_K A} \right\rangle_{G=\mathrm{SU}(2),\, M=S^3}}$$

If $M = S^3$, then

- $G = \mathrm{SU}(2),\ R = \Box$ gives the Jones polynomial
- $G = \mathrm{SU}(2),\ R = \Box$ gives the HOMFLY polynomial P, where

  $$P = P(q^N, q) = P(a, q)$$

  and $a = q^N$.
- $G = \mathrm{SO}$ or Sp give the Kaufmann polynomial

$$J(\text{trefoil}; q) = q + q^3 - q^4$$

Coloured Jones polynomials

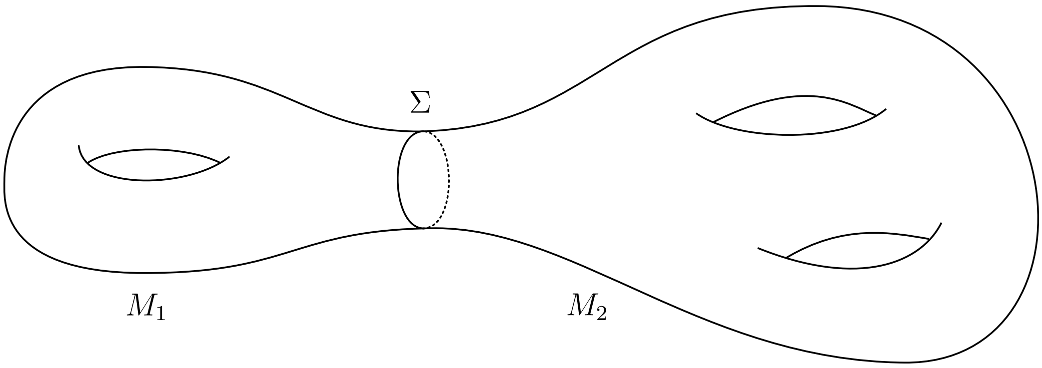
$R = S^3 = \Box\Box \cdots \Box$

$$\begin{aligned} J_{R=\Box\Box}(\text{trefoil}; q) &= q^2 + q^5 - q^7 + q^8 - q^9 - q^{10} + q^{11} \\ J_{R=\Box\Box\Box}(\text{trefoil}; q) &= \text{more and more complicated} \end{aligned}$$

The coloured polynomials are stronger than $R = \Box$, but not strong enough. For example, they don't distinguish mutant knots.

Non-perturbative methods

How to obtain Z from "surgeries".



Wave-functional

$$\psi_{M,\mathcal{O}}(\mathcal{A}) = \langle \mathcal{A} \mid \psi_{M,\mathcal{O}} \rangle = \int_{A|_\Sigma = \mathcal{A}} DA \mathrm{e}^{\frac{ki}{4\pi} S_{\mathrm{CS}}}$$

$|\psi_{M,\mathcal{O}}\rangle \in \mathcal{H}_\Sigma$, where $\mathcal{H}_\Sigma$ is the Hilbert space that arises from canonical quantization of the Chern–Simons $\Sigma \times \mathbb{R}$, i.e.

$$\mathcal{H}_\Sigma = \text{space of conformal blocks in the WZW model with } G \text{ level } k,$$

where G is compact. We have that $\dim \mathcal{H}_\Sigma < \infty$, e.g.

- If $\Sigma = S^2$, then $\dim \mathcal{H}_\Sigma = 1$.
- If $\Sigma = T^2$, then $\mathcal{H}_\Sigma = \{\text{integrable representations of } \widehat{\mathfrak{su}}(N)_k\}$, where by "integrable representations" we mean labelled Young diagrams.

where $\partial M_1 = \partial M_2 = \Sigma = T^2$.

In quantum theory $f \longmapsto \widehat{f}$ acting on $\mathcal{H}_\Sigma$ via $Z = \langle \psi_{M_1,\mathcal{O}_1} | \widehat{f} | \psi_{M_2,\mathcal{O}_2} \rangle$.

Consider $\mathrm{SL}_2 \mathbb{Z}$, generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Then $T, S \longmapsto \widehat{T}, \widehat{S}$.

$$Z(S^2 \times S^1) = \langle 0|1|0 \rangle = \langle 0|0 \rangle = 1.$$

M_1 and M_2 are "solid tori", i.e. $S^1 \times$ disk. Glue using $f = 1$.

$$Z(S^3) = \langle 0|\widehat{S}|0\rangle = S_{00} = \frac{1}{(k+N)^{\frac{N}{2}}} \prod_{j=1}^{N-1} \left(2\sin\left(\frac{\pi_j}{k+N}\right)\right)^{N-j}$$

If $\theta = \operatorname{Tr}_R \mathcal{P}e^{\oint A}$, then $|\psi_\theta\rangle = |R\rangle$. What is the unknot polynomial in S^3?

$$\langle \bigcirc \rangle = \langle 0|\widehat{S}|R\rangle = S_{0R} = \dim_q R = s_R$$

Then $R = \square$ implies

$$\dim_q \square = s_\square(x_i) = \sum_{i=1}^{N} x_i = \frac{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \frac{a^{\frac{1}{2}} - a^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

A-polynomial, volume conjecture & quantization

Knot complement $M = S^3 \backslash K$, where the complement is to be taken of a tubular neighbourhood of the knot. The knot group is $\pi_1(M)$. For example, the knot group of the trefoil knot (3_1) is

$$\langle a, b | aba = bab \rangle$$

Consider a representation $\rho \colon \pi_1(M) \longrightarrow \mathrm{SL}_2\,\mathbb{C}$

Let m denote the meridian and l the longitude, as shown in the figure. Then

$$\rho(m) \cong \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix}, \quad \rho(l) \cong \begin{pmatrix} y & * \\ 0 & y^{-1} \end{pmatrix}$$

so that

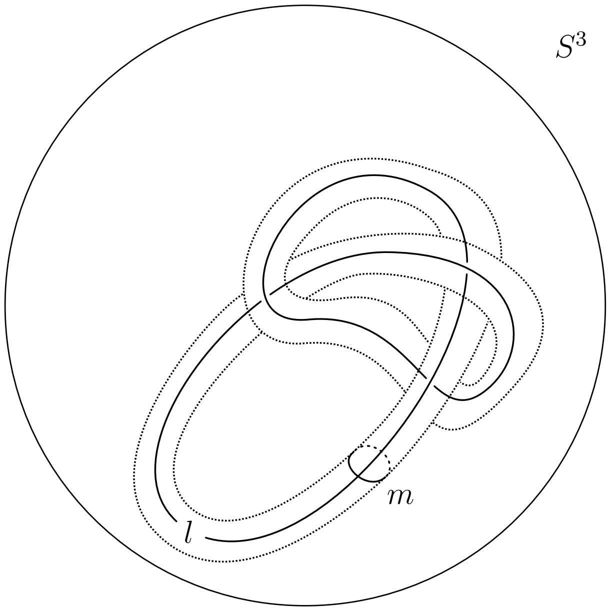
$$\rho \longmapsto (x, y) \in \mathbb{C}^2$$

with (A-)polynomial relation $A(x, y) = 0$.

For example, $A_\bigcirc(x, y) = y - 1$ and $\rho(l) = 1$ implies that $y = 1$.

The properties of the A-polynomial are

- $A(x, y) = (y - 1)(\cdots)$
- $A(x, y) = (\text{monomial})A(\frac{1}{x}, \frac{1}{y})$



- detects mirror knots (that is, knot with opposite orientation) in that $A_{\reflectbox{$K$}}(x,y) = (\text{monomial})A_K(x,y)$, where $\reflectbox{$K$}$ is the mirror knot of K.
- tempered: from a Newton polygon construct face polynomials.

$$A(x,y) = \sum c_{ij} x^i y^j$$
$$A_{3_1}(x,y) = (y-1)(y+x^3)$$

and $f(z) = \sum c_k z^k$, where the roots of f are roots of unity.

Volume conjecture (Kashaev '97)**.** If K is a hyperbolic knot, then

$$\lim_{n \longrightarrow \infty} \frac{1}{2} \ln \left| J_n(K; \mathrm{e}^{\frac{2\pi i}{n}}) \right| = \mathrm{Vol}(S^3 \backslash K),$$

where $J_n = J_{\square\square\cdots\square}$ is the Jones polynomial.

General volume conjecture (Gukov '03)**.**

$$J_n(K; q = \mathrm{e}^{\hbar}) \simeq \mathrm{e}^{\frac{1}{\hbar} s_0(x) + s_1(x) + \hbar s_2(x) + \cdots}$$

as $n \longrightarrow \infty$ and $\hbar \longrightarrow 0$, where $x = \mathrm{e}^{n\hbar} = q^n$ and $s_0(x) = \int_{x_*}^{x} \ln y \frac{\mathrm{d}x'}{x'}$.

Quantum volume conjecture (AJ-conjecture).

$$\widehat{A} J_* = 0.$$

Weyl algebra:

$$\begin{aligned} \widehat{y}\widehat{x} &= q\widehat{x}\widehat{y} \\ \widehat{x} J_n &= x J_n \\ \widehat{y} J_n &= J_{n+1} \\ \exists \widehat{A}(\widehat{x}, \widehat{y}; q) J_* &= 0 \\ a_k J_{n+k} + \cdots + a_1 J_{n+1} + a_0 J_n &= 0, \end{aligned}$$

where $a_i = a_i(\widehat{x}, q)$. Then

$$\widehat{A}(\widehat{x}, \widehat{y}; q) \longrightarrow A(x, y)$$

as $q \longrightarrow 1$ and $\hbar \longrightarrow 0$.

How does this relate to physics & quantization?

$$S_{\text{CS}} = \int A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A$$

M, $\partial M = \Sigma$.

$$\delta S_{\text{CS}} = 0 \Longrightarrow \mathrm{d}A + A \wedge A = 0 \text{ (flat connection)}$$

Classical phase space $\mathcal{M}(G, \Sigma)$, the moduli space of flat connections. We have the embedding

$$\mathcal{M}(G, M) \hookrightarrow \mathcal{M}(G, \Sigma),$$

whose image is a Lagrangian submanifold of $\mathcal{M}(G, \Sigma)$.

For example, let $M = S^3 \backslash K$ and $G = \mathrm{SL}_2 \mathbb{C}$ and consider CS on M. Then flat connections are in 1-to-1 correspondence and labelled by

$$C = \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2 \mathbb{C})/\text{conj.}.$$

Baby example. Each hyperbolic space can be decomposed into "fundamental" tetrahedra.

$$Z(x) = \prod_{i=0}^{\infty}(1-xq^i)^{-1}$$
$$\widehat{y}Z(x) = \prod_{i=1}^{\infty}(1-xq^i)^{-1} = (1-x)Z(x)$$

and $\widehat{A} = \widehat{y} + \widehat{x} - 1$ shows that $A(x,y) = x + y + 1$.

$$Z(x) = \prod_{i=0}^{\infty}(1-xq^i)^{-1} = \mathrm{e}^{-\frac{1}{\hbar}L_{i_2}(x)+\cdots},$$

where $L_{i_k}(x) = \sum_{i=1}^{\infty}\frac{x^i}{i^k}$ and $L_{i_1} = -\ln(1-x)$. But then

$$s_0(x) = -L_{i_2}(x) = \int \ln y \frac{\mathrm{d}x}{x}$$

implies that

$$y = \mathrm{e}^{x\frac{\partial s_0}{\partial x}} = \mathrm{e}^{x\frac{\partial(-L_{i_2}(x))}{\partial x}} = \mathrm{e}^{x\frac{\ln(1-x)}{x}} = 1-x.$$

Example for 3_1.

$$\widehat{A} = \alpha\widehat{y^{-1}} + \beta + \gamma\widehat{y},$$

where

$$\alpha = \frac{x^2(x-q)}{x^2-q}$$
$$\beta = q\left(1 + \frac{1}{x} - x + \frac{q-x}{x^2-q} - \frac{x-1}{x^2q-1}\right)$$
$$\gamma = \frac{q-\frac{1}{x}}{1-qx^2}$$

Also,

$$Z_1 = q + q^3 - q^4$$
$$Z_2 = q^2 + q^5 - q^7 + q^8 - q^9 - q^{10} + q^{11}.$$

$A_{3_1} = (y-1)(y+x^3)$ may be related to $\widehat{A}$ via random matrix theory.

Knot homologies & super-A-polynomial

Super-A-polynomial

The super-A-polynomial $A(x, y; a, t) = 0$ is a new knot invariant that is related to 3-dimensional SUSY Gauge theories and Seiberg–Witten curves via 3d–3d duality.

There is also the quantum super-A-polynomial $\widehat{A}(\widehat{x}, \widehat{y}; a, q, t)$.

Why knot polynomials have integer coefficients

A knot polynomial gives rise to a vector space $\mathcal{H}_{*,*}$. Calculating the Euler characteristic gives

$$J(q) = \sum_{i,j} (-1)^i q^j \dim \mathcal{H}_{i,j},$$

where

$$\mathrm{Kh}(q, t) = \sum_{i,j} t^i q^j \dim \mathcal{H}_{i,j},$$

e.g. $\mathrm{Kh}_{3_1}(q, t) = q + q^3 t^2 + q^4 t^3$ and $t = -1$ gives $J = q + q^3 - q^4$.

$$\Delta(q) \longrightarrow \mathrm{HFK}_{*,*} \longrightarrow \mathrm{HFK}(q, t).$$

The HOMFLY polynomial is given by

$$P(a, q) = \sum_{i,j,k} a^i q^j t^k \dim \mathcal{H}_{i,j,k}$$

giving the superpolynomial $\mathcal{P}(a, q, t)$.

The coloured HOMFLY polynomial is given by

$$P_R(a, q) = \sum_{i,j,k} a^i q^j t^k \dim \mathcal{H}^p_{i,j,k}$$

giving the coloured superpolynomial $\mathcal{P}_R(a, q, t)$.

For 3_1 we have

$$P(a, q) = \frac{a}{q} + aq - a^2$$
$$\mathcal{P}_{R=\square}(a, q, t) = \frac{a}{q} + aqt^2 + a^2 t^3$$

and $a = q^2$ gives the Jones polynomial J.

How do we get $\mathcal{P}_n(a,q,t)$? Refined Chern–Simons theory for $q \longrightarrow (q,t)$ (Aganagic & Shakirov).

$$\langle \bigcirc \rangle_R = \dim_q R = S_R(\cdot, q) \longrightarrow M_R(\cdot, q, t)$$
$$S, T \longrightarrow (ST)^3 = 1,\ S^2 = C$$
$$Z(a,q) \longrightarrow Z(a,q,t)$$

Then

$$\mathcal{H}_{*,*} = \mathcal{H}_{\text{BPS}}.$$

With this

$$\mathcal{P}_n(\text{trefoil}; a,q,t) = \sum_{k=0}^{n-1} a^{n-1} t^{2k} q^{n(k-1)+1} \frac{(q^{n-1}, q^{-1})_k (-atq^{-1}, q)_k}{(q,q)_k},$$

where $(x,q)_k$ is the q-Pochhammer

$$(x,q)_k = \prod_{i=0}^{k-1} (1 - xq^i).$$

Super-A-polynomial

$$\mathcal{P}_n(a,q,t) \longrightarrow \mathrm{e}^{\frac{1}{\hbar} s_0(x;a,t) + \cdots}$$

as $n \longrightarrow \infty$ and $\hbar \longrightarrow 0$, where $x = q^n$ and a, t are fixed.

Claims.

1. For $s_0(x,a,t) = \int \ln y \frac{\mathrm{d}x}{x}$ we have that $A^{\text{super}}(x,y;a,t) = 0$.
2. $\widehat{A}^{\text{super}}(\widehat{x}, \widehat{y}; a, t)\mathcal{P}_* = 0$.
3. $\widehat{A}^{\text{super}}$ gives A^{super} when $q = 1$.

In fact, we have

$$\begin{array}{ccc} A^{\text{super}}(x,y;a,t) & \xrightarrow{t=-1} & A^{\text{super}}(x,y;a) \\ \downarrow {\scriptstyle a=1} & & \downarrow {\scriptstyle a=1} \\ A^{\text{super}}(x,y;t) & \xrightarrow{t=-1} & A(x,y) = 0 \end{array}$$