DIFFERENTIAL FORMS AND THEIR INTEGRALS

(preliminary, incomplete version)

Contents

CHAPTER 0

A review of basic vector calculus

In this chapter we will review some basic facts of vector calculus which will be used extensively along these notes. We will assume the reader familiar with the differential and integral calculus for real valued functions of a real variable, as well as with the basic topology of Euclidean spaces: open and closed sets, continuity, compactness, Cauchy sequences etc. The material in sections 1 and 2 are quite standard in Calculus courses, while the one in section 3 is probably less "popular" at this level.

1. Differentiable functions

We will consider the Euclidean space $\mathbb{R}^n$ with its canonical inner product and associated norm.

For a point $x \in \mathbb{R}^n$ and $r \in \mathbb{R},\ r > 0$, we denote by $B^n(p,r) := \{x \in \mathbb{R}^n : \|x-p\| < r\}$ the *ball of radius r centred at p*. When $p = 0$ we simply write $B^n(r)$ for $B^n(0,r)$.

We will denote by $L(\mathbb{R}^n, \mathbb{R}^m)$ the space of linear maps of $\mathbb{R}^n$ into $\mathbb{R}^m$. There is a natural identification of $L(\mathbb{R}^n, \mathbb{R}^m)$ with $\mathbb{R}^{nm}$, associating to a linear transformation L, the entries (in a fixed order) of the matrix representing L in the canonical bases. This identification induces a scalar product in $L(\mathbb{R}^n, \mathbb{R}^m)$

$$\langle A, B\rangle \;=\; \text{trace } A^t B,$$

where A^t is the transpose of A. Often it is more convenient to consider the *operator norm*, defined by

$$\|L\| = \sup\{\|Lx\| : x \in \mathbb{R}^n,\ \|x\| = 1\}.$$

The two norms are equivalent, as all norms are in a finite dimensional vector space, so for the basic topological concepts like convergence, continuity etc., it does not matter which one we use. In what follows we will consider the operator norm, unless otherwise stated.

Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \longrightarrow \mathbb{R}^m$ a function.

1.1. Definition. f is *differentiable at $x \in U$* if there exist a linear map $\mathrm{d}f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that[1]

$$\lim_{\|h\|\to 0} \frac{\|f(x+h) - f(x) - \mathrm{d}f(x)(h)\|}{\|h\|} = 0.$$

The map $\mathrm{d}f(x)$ is called the *differential of f at x*.

f is *differentiable* in U if it is differentiable at every point of U.

1.2. Remark. For a function $f : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ the *derivative* of f at $x \in U$, $f'(x)$, is defined as

$$f'(x) := \frac{\mathrm{d}f}{\mathrm{d}t}|_{t=x} := \lim_{t\to 0}[f(x+t) - f(x)]t^{-1},$$

[1]Observe that, if $\|h\|$ is sufficiently small, $x + h \in U$.

if the limit exists. The *differential* of f at x is the linear map

$$\mathrm{d}f(x) : \mathbb{R} \longrightarrow \mathbb{R}, \quad \mathrm{d}f(x)h = f'(x)h.$$

The following facts are easy to prove and and we leave the proofs to the reader (Exercise 4.2).

1.3. PROPOSITION.

- *If f, g are differentiable at x and $a \in \mathbb{R}$, then $f+g$ and af are differentiable and* $\mathrm{d}(f+g)(x) = \mathrm{d}f(x) + \mathrm{d}g(x),\ \mathrm{d}[af](x) = a[\mathrm{d}f(x)]$.
- *If f is differentiable at x, f is continuous at x.*
- *If f is differentiable at x, the differential is unique.*
- (The chain rule) *If $f : U \subseteq \mathbb{R}^n \longrightarrow W \subseteq \mathbb{R}^m,\ \ g : W \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^p$ are differentiable at x and $f(x)$ respectively, then $g \circ f$ is differentiable at x and* $\mathrm{d}[g \circ f](x) = \mathrm{d}g(f(x)) \circ \mathrm{d}f(x)$.

1.4. EXAMPLE. Let $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map. Then L is differentiable and $\mathrm{d}L(x) = L,\ \forall\, x \in \mathbb{R}^n$, as follows directly from the definition.

1.5. EXAMPLE. If $B : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^p$ is a bi-linear map, B is differentiable and $\mathrm{d}B(x,y)(z,w) = B(x,w) + B(z,y)$. In particular, if $f, g : U \longrightarrow \mathbb{R}$ are differentiable functions, the map $F : U \longrightarrow \mathbb{R}^2,\ F(x) = (f(x), g(x))$ is differentiable with $\mathrm{d}F(x) = (\mathrm{d}f(x), \mathrm{d}g(x))$. Since the product $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is bilinear, by the chain rule the function fg is differentiable and the product rule $\mathrm{d}(fg)(x) = f(x)\mathrm{d}g(x) + \ g(x)\mathrm{d}f(x)$ holds.

In the theory of real valued functions of one real variable, an elementary but useful result is the Mean Value Theorem.

1.6. THEOREM. [Mean Value Theorem] *If $f : [a, a+h] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable function, then there exists $t_0 \in [0,1]$ such that*

$$f(a+h) - f(a) = f'(a + t_0 h)h.$$

The Theorem extends, with essentially the same proof, to the case of a differentiable function $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$. For functions with values in $\mathbb{R}^m,\ m > 1$, such a Theorem does not hold any longer (see Exercise 4.4) but, at least, we have an inequality. The result will still be called the Mean Value Theorem.

1.7. THEOREM. [Mean Value Theorem] *Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function. Suppose that the segment with endpoints $a, a+h$ is contained in U and f is differentiable at the points of the segment. Then*

$$\|f(a+h) - f(a)\| \leq \|h\| \ \sup\{\|\mathrm{d}f(a+th)\| : t \in [0,1]\}.$$

PROOF. Consider the function $\phi : [0,1] \longrightarrow \mathbb{R}^m,\ \phi(t) = f(a+th)$. ϕ is differentiable, by the chain rule, $\phi(0) = f(a),\ \phi(1) = f(a+h)$ and $\mathrm{d}\phi(t)(1) = \mathrm{d}f(a+th)(h)$. Let $M = \sup\{\|\mathrm{d}\phi(t)\| : t \in [0,1]\}$. It is then sufficient to prove that $\|\phi(1) - \phi(0)\| \leq M$. For this purpose we will show that, given $\epsilon > 0,\ \|\phi(1) - \phi(0)\| \leq M + \epsilon$. Consider the set

$$A = \{t \in [0,1] : \|\phi(s) - \phi(0)\| \leq (M+\epsilon)s,\ \forall\, s \in [0,t]\}.$$

It is easy to see that $A = [0, a]$ for some $a \in (0, 1]$. We wish to prove that $a = 1$. Suppose $a < 1$. Then there exists a positive δ such that $a + \delta < 1$ and for $k \in (0, \delta)$ small enough

$$\phi(a + k) - \phi(a) = \mathrm{d}\phi(a)k + r(k) \quad \text{with } \|r(k)\| \leq \epsilon k$$

(by the definition of differentiability at a). Then $\|\phi(a + k) - \phi(a)\| \leq (M + \epsilon)k$. But $a \in A$, hence $\|\phi(a) - \phi(0)\| \leq (M + \epsilon)a$. Therefore $\|\phi(a + k) - \phi(0)\| \leq (M + \epsilon)(a + k)$. In particular $a + k \in A$, a contradiction. □

1.8. Definition. Let $f : U \longrightarrow \mathbb{R}^m$ be differentiable at $x \in U$ and $X \in \mathbb{R}^n$. The *directional derivative* of f at x in the X direction is defined as

$$\frac{\partial f}{\partial X}(x) := \mathrm{d}f(x)(X).$$

1.9. Remark. For reasons that will be clear later on will use often the notation $X_x(f)$ for $\mathrm{d}f(x)(X)$.

If $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$, $\dfrac{\partial f}{\partial e_i}(x)$ is the i^{th} *partial derivative* at x and will be denoted, as usual, by $\dfrac{\partial f}{\partial x_i}(x)$. If f is differentiable at x and $h = \sum_{i=1}^n \alpha_i e_i$, then

$$\mathrm{d}f(x)h = \sum_{i=1}^n \alpha_i \mathrm{d}f(x)(e_i) = \sum_{i=1}^n \alpha_i \frac{\partial f}{\partial x_i}(x).$$

In particular, if $f(x) = (f_1(x), \ldots, f_m(x))$, where $f_i : U \longrightarrow \mathbb{R}$ are the coordinate functions of f, then the *Jacobian matrix* $[\dfrac{\partial f_j}{\partial x_i}(x)]$ is the matrix that represents $\mathrm{d}f(x)$ in the canonical bases. This is the multidimensional analogue of Remark 1.2.

Let $\gamma : (a, b) \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$ be a differentiable map. We will also say that γ is a *differentiable curve.* For such a function, the tangent vector at $t \in (a, b)$ (or, sometimes, at $\gamma(t)$) is the vector

$$\dot\gamma(t) := \mathrm{d}\gamma(t)(1) = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=t}\gamma(s).$$

It is easy to see that if $\gamma : (-\epsilon, \epsilon) \longrightarrow R^n$ is a differentiable curve with $\dot\gamma(0) = X$,

$$\frac{\partial f}{\partial X}(x) = \mathrm{d}(f \circ \gamma)(0)(1) := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(\gamma(t)).$$

1.10. Remark. The latter fact gives a geometric interpretation of the differential of f: *the image, via* $\mathrm{d}f$*, of the vector tangent to a given curve* γ *is the vector tangent to the image curve,* $f \circ \gamma$.

In particular the right hand side of the formula above does not depend on γ as long the curve passes trough x and its tangent vector, at x, is X. This observation allow us to define the directional derivative, hence partial derivatives, even for a class of not necessarily differentiable functions (in the sense of Definition 1.1). If $f : U \longrightarrow \mathbb{R}^m$ is a function and $X \in \mathbb{R}^n$, we define the directional derivative of f at $x \in U$, in the direction of X as

$$\frac{\partial f}{\partial X}(x) := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(x + tX),$$

if it exists. The partial derivatives may exist even if the function is not differentiable (see Exercise 4.7). However we have

1.11. Proposition. *Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function. If the partial derivatives of f exist and are continuous*[2]*, f is differentiable.*

Proof. We will prove the Proposition for $n = 2$ to avoid notational complications. We want to show that the linear map $L(x, y)(h, k) = \dfrac{\partial f}{\partial x} h + \dfrac{\partial f}{\partial y} k$ is the differential of f at $(x, y) \in \mathbb{R}^2$. Hence we have to show that, given $\epsilon > 0$,

$$\|f(x + h, y + k) - f(x, y) - \frac{\partial f}{\partial x} h - \frac{\partial f}{\partial y} k\| \leq \epsilon \|(h, k)\|,$$

if $\|(h, k)\|$ is sufficiently small. Adding and subtracting $f(x, y + k)$ and using Exercise 4.5, we have that the quantity on the left of the inequality sign is less or equal to

$$\|h\| \sup\{\|\frac{\partial f}{\partial x}(x + th, y + k) - \frac{\partial f}{\partial x}(x, y)\| : t \in [0, 1]\} + \|k\| \sup\{\|\frac{\partial f}{\partial y}(x, y + tk) - \frac{\partial f}{\partial x}(x, y)\| : t \in [0, 1]\}.$$

The conclusion follows from the continuity of the partial derivatives. $\square$

1.12. Remark. Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function. Let $\pi_i : \mathbb{R}^m \longrightarrow \mathbb{R},\ \pi_i(x_1, \ldots, x_m) = x_i$ be the i^{th} projection. Then $f_i(x) = \pi_i \circ f(x)$ are the coordinates of $f(x)$. It is easy to see that f is differentiable at $x \in U$ if and only if the coordinate functions are differentiable at x and, in this case,

$$\mathrm{d}f(x)(X) = (\mathrm{d}f_1(x)(X), \ldots, \mathrm{d}f_m(x)(X)) = \sum [\mathrm{d}f_i(x)X] e_i.$$

Partial derivatives take care of the "opposite" situation. Given a splitting of $\mathbb{R}^n = \mathbb{E}_1 \oplus \mathbb{E}_2$ as a direct sum of complementary subspaces and a point $(x_0, y_0) \in \mathbb{E}_1 \oplus \mathbb{E}_2$, we can consider the inclusion $i_j : \mathbb{E}_j \longrightarrow \mathbb{R}^n,\ i_1(x) = (x, y_0),\ i_2(y) = (x_0, y)$ and the functions $f^{(j)} = f \circ i_j : \mathbb{E}_j \cap U \longrightarrow \mathbb{R}^m$. If f is differentiable at (x_0, y_0), $f^{(1)}$ (resp. $f^{(2)}$) is differentiable at x_0 (resp. y_0) and $\mathrm{d}f(x_0, y_0)(X, Y) = \mathrm{d}f^{(1)}(x_0)(X) + \mathrm{d}f^{(2)}(y_0)(Y)$. So we can define the *partial differentials* relative to the given splitting, $\mathrm{d}_j f = \mathrm{d}f \circ i_j$. The existence of the partial differentials does not implies the existence of the differential of f. However, as in Proposition 1.11, if the partial differentials exist and are continuous, then f is differentiable. Obviously the same arguments work for a decomposition of $\mathbb{R}^n$ into the direct sum of k complementary subspaces. Partial derivatives are, essentially, partial differentials relative to the canonical splitting of $\mathbb{R}^n$ as the direct sum of the coordinate lines.

If $f : U \longrightarrow \mathbb{R}^m$ is a differentiable function, the differential can be seen as a map $\mathrm{d}f : U \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m),\quad x \rightsquigarrow \mathrm{d}f(x)$.

1.13. Definition. We will say that f is *twice differentiable at $x \in U$*, if the function $\mathrm{d}f : U \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is differentiable at x. In this case, the differential of $\mathrm{d}f$ at x will be called the *second differential of f at x* and will be denoted by $\mathrm{d}^2 f(x)$.

Inductively, we define, if it exists, the *k^{th} differential of f at x*, $\mathrm{d}^k f(x)$, as the differential, at x, of $\mathrm{d}^{k-1} f U \longrightarrow L(\mathbb{R}^n, L(\mathbb{R}^n, \ldots)$

We will say that f is of *class C^k* in U if $\mathrm{d}^k f(x)$ exists, $\forall\ x \in U$, and is continuous, as a function of x.

We will say that f is of *class C^∞*, if it is of class C^k, $\forall\ k$. If f is C^∞ we will also say that f is *smooth.*

1.14. Remark. It is easy to produce examples of C^k functions that are not C^{k+1} (see Exercise 4.6). One of the important features of the class of smooth functions is that it is closed under differentiation, i.e. f is smooth if and only if $\mathrm{d}f$ is smooth.

[2] As maps $\dfrac{\partial f}{\partial x_i} : U \longrightarrow \mathbb{R}^m$.

If f is twice differentiable at x, then $\mathrm{d}^2 f(x) \in L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m))$, and so it can be seen as the bilinear map $\mathrm{d}^2 f(x)(X,Y) = \mathrm{d}(\mathrm{d}f)(x)(X)(Y)$. In a similar way, $\mathrm{d}^k f(x)$ can be viewed as a k-multilinear map.

1.15. Theorem. [Schwarz's Theorem] *If* $\mathrm{d}^2 f(x)$ *exists, it is a symmetric bilinear form.*

We will sketch a proof in the case that f is C^2 in Exercise 4.22

1.16. Remark. If f is k times differentiable at x, $\mathrm{d}^k f(x)$ is a k-multilinear *symmetric* map. Moreover we can define higher order partial derivatives. Schwarz Theorem 1.15 and a simple induction imply that if f is of class C^k, the result of successive partial derivatives, up to order k, does not depend on the order of derivations.

1.17. Example. If $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map $\mathrm{d}L(x) = L$, $\forall\, x \in \mathbb{R}^n$. In particular the differential $\mathrm{d}L : \mathbb{R}^n \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is the constant map. Hence $\mathrm{d}^k L = 0$, if $k \geq 2$, and L is C^∞. Similarly, if $B : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^p$ is a bilinear map, $\mathrm{d}B(x,y)(z,w) = B(x,w) + B(z,y)$. In particular $\mathrm{d}B$ is a linear map, hence C^∞, and so is B.

1.18. Example. Let $M(n,\mathbb{R})$ be the space of $n \times n$ matrices with real coefficients. The product map

$$m : M(n,\mathbb{R}) \times M(n,\mathbb{R}) \longrightarrow M(n,\mathbb{R}), \quad m(A,B) = AB,$$

is a bi-linear map, hence smooth. Also the map

$$g : M(n,\mathbb{R}) \times M(n,\mathbb{R}) \longrightarrow L(M(n,\mathbb{R}), M(n,\mathbb{R})), \quad g(A,B)H = AHB$$

is bilinear, hence smooth.

Since the determinant, $\det : M(n,\mathbb{R}) \longrightarrow \mathbb{R}$, is a continuous function, the set of invertible matrixes, $GL(n,\mathbb{R})$, is an open subset of $M(n,\mathbb{R})$. Consider the inversion map

$$\iota : GL(n,\mathbb{R}) \longrightarrow M(n,\mathbb{R}), \quad \iota(A) = A^{-1}.$$

<u>Claim</u> The map ι is smooth.

Proof. Differentiating the identity $(A + tB)(A + tB)^{-1} = \mathbb{1}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(A + tB)^{-1} = -A^{-1}BA^{-1}.$$

So, if ι is differentiable at $A \in GL(n,\mathbb{R})$, $\mathrm{d}\iota(A)(B) = -A^{-1}BA^{-1}$. It is easy to check that, in fact, the formula defines a linear map which is the differential of ι at A. In particular ι is continuous. The differential of ι is then given by the composition

$$GL(n,\mathbb{R}) \xrightarrow{\Delta} GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \xrightarrow{\iota\times\iota} GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \xrightarrow{-g} L(M(n,\mathbb{R}), M(n,\mathbb{R})),$$

where $\Delta : GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R}) \times GL(n,\mathbb{R})$ is the diagonal map, $\Delta(A) = (A,A)$, and g is as above. Hence $\mathrm{d}\iota$ is continuous and ι is of class C^1. At this point a simple induction proves the Claim. $\square$

As we have seen, the differential of a function f at a point x, provides the best linear approximation of $f - f(x)$ in a neighborhood of x. The Taylor formula provides the best polynomial approximation, for functions with more differentiability.

1.19. THEOREM. [Infinitesimal version of Taylor Theorem] *Let $f : U \longrightarrow \mathbb{R}^m$ be a function s times differentiable in an open neighborhood of $a \in U$ and such that $\mathrm{d}^{s+1}f(a)$ exists. Then*

$$f(a+h) = f(a) + \sum_{k=1}^{s+1} \frac{1}{k!}\mathrm{d}^k f(a)(h,\dots,h) + r(h) \quad \textit{with} \quad \lim_{h\to 0} r(h)\|h\|^{-(s+1)} = 0.$$

PROOF. The proof is a simple consequence of the lemma below, which is of interest on its own

1.20. LEMMA. *Let $r : B = B^n(R) \longrightarrow \mathbb{R}^m$ be a function s times differentiable in B and $s+1$ times differentiable in 0. Assume $\mathrm{d}^j r(0) = 0,\ 0 \le j \le s+1$. Then $\lim_{x\to 0} r(x)\|x\|^{-(s+1)} = 0$.*

PROOF. We proceed by induction on s. If $s = 0$ the conclusion follow from the definition of differentiability. Suppose the conclusion true for s. By the mean value Theorem we have

$$\|r(x)\| \le M\|x\|, \qquad M = \sup\{\|\mathrm{d}r(tx)\|| : t \in [0,1]\}.$$

Applying the inductive hypothesis to $\mathrm{d}r$, given $\epsilon > 0$ there exist $\delta > 0$ such that, if $\|y\| < \delta,\ \|\mathrm{d}r(y)\| < \epsilon\|y\|^s$. Hence if $\|x\|\delta,\ M \le \epsilon\|x\|^s$ and $\|r(x)\| \le \epsilon\|x\|^{s+1}$. □

□

In the linear context, i.e. vector spaces and linear maps, we study properties that are invariant for linear isomorphisms, i.e. changes of bases. The analogue in the differential context are properties that are invariant for (local) diffeomorphisms , i.e. change of variables (or coordinates).

1.21. DEFINITION. A map $\phi : U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^m$ between open sets is a C^k *diffeomorphism* if there exists a C^k map $\psi : V \longrightarrow U$, such that $\psi\circ\phi = \mathbb{1}_U,\ \phi\circ\psi = \mathbb{1}_V$. ϕ is a *local diffeomorphism* if $\forall\ x \in U$, there exists an open neighborhood $\tilde{U} \subseteq U$ of x, such that $\phi|_{\tilde{U}}$ is a diffeomorphism onto an open neighborhood $\tilde{V}$ of $\phi(x)$ in V. A local diffeomorphism will also be called a *change of variables* (or *change of coordinates*).

1.22. REMARK. If ϕ is a diffeomorphism, then $\mathrm{d}\phi(x)$ is an isomorphism, by the chain rule. Hence $n = m$.

The following fact will be useful.

1.23. LEMMA. *If $\phi : U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^m$ is a C^k map, $k \ge 1$, between open sets, and ϕ admits a differentiable inverse, then the inverse is of class C^k.*

PROOF. From the chain rule $\mathrm{d}[\phi^{-1}](\phi(x)) \circ \mathrm{d}\phi(x) = \mathbb{1}$. In particular $\mathrm{d}[\phi^{-1}]$ is given by the composition

$$V \xrightarrow{\phi^{-1}} U \xrightarrow{\mathrm{d}\phi} GL(n,\mathbb{R}) \xrightarrow{\iota} GL(n,\mathbb{R}),$$

where ι is the matrix inversion map, that, by Example 1.18, is smooth. Hence $\mathrm{d}[\phi^{-1}]$ is continuous and ϕ^{-1} is of class C^1. In general the argument gives that, if ϕ^{-1} is $C^s,\ s < k$, $\mathrm{d}[\phi^{-1}]$ is also of class C^s so ϕ^{-1} is of class C^{s+1} and this concludes the proof.

□

Let $\mathbb{E}, \mathbb{F}$ be real, finite dimensional vector spaces and $L : \mathbb{E} \longrightarrow \mathbb{F}$ be a linear map. Then, in suitable bases, F has a very simple expression. In fact we can chose a basis $\{e_1,\dots,e_n\}$ of $\mathbb{E}$, such that $\{e_{k+1},\dots,e_n\}$

is a basis of the kernel of F. Then $\{f_1 = F(e_1), \ldots, f_k = F(e_k)\}$ is a basis of the image of F that we can complete with vectors $\{f_{k+1}, \ldots, f_m\}$ to have a basis for $\mathbb{F}$. Then, in terms of coordinates in these bases,

$$F(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

Since a differentiable function is (locally) approximated by a linear one, we can expect something similar to hold, locally, for differentiable maps, up to change of coordinates. In fact this is the case.

1.24. Theorem. [Rank Theorem] *Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function of class C^k. Let $p \in U$ be such that rank $\mathrm{d}f(x) = k$ in an open neighborhood of p. Then there exist open neighborhoods $U', \tilde{U}$ of p and $V', \tilde{V}$ of $f(p)$, and diffeomorphisms $\phi : U' \longrightarrow \tilde{U}$, $\psi : V' \longrightarrow \tilde{V}$ such that:*

$$\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots x_k, 0, \ldots, 0), \qquad for \quad (x_1, \ldots, x_n) \in \tilde{U}.$$

This Theorem will follow from the next three.

1.25. Theorem. [Inverse function Theorem] *Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^k map, $k \geq 1$, such that $\mathrm{d}f(p) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isomorphism. Then the exist an open neighborhood U' of p such that $f|U'$ is a C^k diffeomorphism onto an open neighborhood of $f(p)$.*

Proof. Without loss of generality we may assume $p = 0 = f(p)$. Moreover, by composing f with $\mathrm{d}f(0)^{-1}$, we may assume $\mathrm{d}f(p) = \mathbb{1}$. Consider the function $g(x) = f(x) - x$. Then $g(0) = 0$, $\mathrm{d}g(0) = 0$. Let r be a positive real number such that if $x \in B^n(r)$, $\mathrm{d}f(x)$ is invertible and $\|\mathrm{d}g(x)\| < \frac{1}{2}$.
Claim 1. If $y \in B^n(\frac{r}{2})$ then there exists a unique $x \in B^n(r)$ such that $f(x) = y$.

Proof. Define $x_1 = y$ and, inductively, $x_{n+1} = y - g(x_n)$. By the mean value Theorem we have:

$$\|x_{n+1} - x_n\| = \|g(x_n) - g(x_{n-1})\| \leq \frac{1}{2}\|x_n - x_{n-1}\|$$
$$\|x_{n+1}\| = \|g(x_n) + y\| \leq \|g(x_n)\| + \|y\| < \frac{1}{2}\|x_n\| + \|y\| \leq \frac{1}{2}\|x_n\| + \frac{r}{2} - \epsilon,$$

for some $\epsilon > 0$, independent of n. Hence:

(1) $\|x_{n+1} - x_n\| \leq 2^{-n}\|y\|$ (from the first equation and induction),
(2) $\|x_n\| < r - \epsilon$ (from the second equation and induction).

By condition (1), $\{x_n\}$ is a Cauchy sequence, hence it converges to a point x, and by condition (2), $x \in B^n(r)$. Finally, taking limits, we have $x = y - g(x) = y - f(x) + x$, hence $f(x) = y$.

Let us show that x is unique. Suppose $f(z) = f(x) = y, z \in B^n(r)$. Then $\|x - z\| = \|g(z) - g(x)\| \leq \frac{1}{2}\|z - x\|$ which implies $z = x$. □

So we have a well defined surjective map $f^{-1} : B^n(\frac{r}{2}) \longrightarrow U' = B^n(r) \cap f^{-1}(B^n(\frac{r}{2}))$. Observe that U' is the intersection of two open sets, hence it is open.
Claim 2. $f^{-1} : B^n(\frac{r}{2}) \longrightarrow U'$ is C^k.

Proof. We start by observing that $\|f(x_1) - f(x_2)\| \geq \|x_1 - x_2\| - \|g(x_1) - g(x_2)\| \geq \frac{1}{2}\|x_1 - x_2\|$, hence f^{-1} is continuous. In order to show that f^{-1} is differentiable we observe that, since f is differentiable,

$$f(x) - f(x_1) = \mathrm{d}f(x_1)(x - x_1) + h(x, x_1) \quad \text{with} \quad \lim_{x \to x_1} \frac{h(x, x_1)}{\|x - x_1\|} = 0.$$

Applying $A := \mathrm{d}f(x_1)^{-1}$ to the equality above we have

$$A(y - y_1) + Ah_1(y, y_1) = f^{-1}(y) - f^{-1}(y_1),$$

where $y = f(x), y_1 = f(x_1),\ h_1(y, y_1) = -h(f^{-1}(y), f^{-1}(y_1))$. Then

$$\frac{h_1(y, y_1)}{\|y - y_1\|} = -\frac{h(x, x_1)}{\|x - x_1\|}\frac{\|x - x_1\|}{\|y - y_1\|}.$$

Since $\dfrac{\|x - x_1\|}{\|y - y_1\|} \leq 2,\ \ \lim\limits_{y \to y_1} \dfrac{h_1(y, y_1)}{\|y - y_1\|} = 0$ and $\mathrm{d}[f^{-1}](y_1) = [\mathrm{d}f(x_1)]^{-1}$. Hence f^{-1} is differentiable and the Claim follows from Lemma 1.23. □

□

1.26. Theorem. [Local form of immersions] *Let* $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{n+p}$ *be a* C^k *map,* $k \geq 1$*, such that* $0 \in U, f(0) = 0$*. If* $\mathrm{d}f(0) : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+p} = \mathbb{R}^n \times \mathbb{R}^p$ *is injective, there exists an open neighborhood* U' *of* 0 *and a* C^k *diffeomorphism* ϕ *between neighborhoods of* $0 \in \mathbb{R}^{n+p}$ *such that if* $x \in U'$*,*

$$\phi \circ f(x) = (x, 0).$$

Proof. Up to an isomorphism of $\mathbb{R}^{n+p}$ which sends $\mathrm{d}f(0)e_i$ to e_i, we can assume that $\mathrm{d}f(0)v = (v, 0)$. Consider the function

$$F : U \times \mathbb{R}^p \longrightarrow \mathbb{R}^{n+p}, \qquad F(x, y) = f(x) + y.$$

Observe that $F(x, 0) = f(x)$ and $\mathrm{d}F(0) = \mathbb{1}$. By the Inverse Function Theorem there is a diffeomorphism ϕ between neighborhoods of $0 \in \mathbb{R}^{n+p}$ such that $\phi \circ F = \mathbb{1}$. Then

$$\phi \circ f(x) = \phi \circ F(x, 0) = (x, 0).$$

□

1.27. Theorem. [Local form of subimmersions] *Let* $f : U \subseteq \mathbb{R}^{n+p} \longrightarrow \mathbb{R}^n$ *be a* C^k *map,* $k \geq 1$*, such that* $0 \in U, f(0) = 0$*. If* $\mathrm{d}f(0) : \mathbb{R}^{n+p} \longrightarrow \mathbb{R}^n$ *is surjective, then there exists a* C^k *diffeomorphism* $\psi : U' \longrightarrow V$*, between open neighborhoods of* $0 \in \mathbb{R}^{n+p}$*, such that if* $(x, y) \in U$*,*

$$f \circ \psi(x, y) = x.$$

Proof. Up to an isomorphism of $\mathbb{R}^{n+p} = \mathbb{R}^n \times \mathbb{R}^p$, we can assume that $\ker \mathrm{d}f(0) = \{0\} \times \mathbb{R}^p$ and $\mathrm{d}f(0)(v, 0) = v$. Consider the function

$$F : U \longrightarrow \mathbb{R}^n \times \mathbb{R}^p, \qquad F(x, y) = (f(x, y), y).$$

Then $\mathrm{d}F(0) = \mathbb{1}$ and $f = \pi \circ F$, where $\pi : R^{n+p} \longrightarrow R^n$ is the canonical projection. By the Inverse Function Theorem there exists a local inverse ψ of F. Then

$$f \circ \psi(x, y) = \pi \circ F \circ \psi(x, y) = \pi(x, y) = x.$$

□

At this point we leave to the reader the task of proving Theorem 1.24.

2. Integration

We will recall now the basic fact of Riemann integration theory. We will take a limited approach which will be enough for the purpose of these notes. We will start with the case of functions of one real variable.

Let $[a,b] \subseteq \mathbb{R}$ be a closed interval. A *partition* of the interval is a set $P = \{t_0, \dots, t_k\} \subseteq \mathbb{R}$ such that $a = t_0 < \cdots < t_k = b$. We will set $|P| = \sup\{t_i - t_{i-1}\}$. Given a function $f : [a,b] \longrightarrow \mathbb{R}^m$ and a partition P of $[a,b]$, we define

$$\Sigma(f,P) = \sum_{i=0}^{k-1} (t_{i+1} - t_i) f(t_i).$$

2.1. Definition. A vector $X \in \mathbb{R}^m$ is said to be an *integral* of f on $[a,b]$ if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|\Sigma(f,P) - X\| < \epsilon \quad \text{for all partitions } \ P \text{ with } \ |P| < \delta.$$

If an integral exists we will say that f is *integrable* on $[a,b]$ and we use the notation

$$X = \int_a^b f(t)\mathrm{d}t \quad \text{or, when clear from the contex, simply } \ X = \int_a^b f.$$

It is easy to see that if a function is integrable, the integral is unique.

2.2. Remark. Let $f : [a,b] \longrightarrow \mathbb{R}^m$ be an integrable function. Then we can compute the integral as limit of the sequence $\Sigma(f, P_n)$ where P_n is a sequence of partitions such that $\lim_{n\to\infty} |P_n| = 0$.

The proof of the following Proposition is simple and left to the reader (see Exercise 4.16).

2.3. Proposition. *Let $f, g : [a,b] \longrightarrow \mathbb{R}^m$ be integrable functions, $k \in \mathbb{R}$ and $T : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ be a linear map. Then $f + g$, kf and $T \circ f$ are integrable and*

(1) $\int_a^b f + g = \int_a^b f + \int_a^b g, \qquad \int_a^b kf = k \int_a^b f,$
(2) $\int_a^b T \circ f = T(\int_a^b f)$
(3) $\| \int_a^b f\| \le (b-a)\|f\|_0$.
(4) *The function $F : [a,b] \longrightarrow \mathbb{R}^m$, defined by $F(x) = \displaystyle\int_a^x f(t)\mathrm{d}t$ is well defined and continuous.*

We will denote by $\mathcal{B} := \mathcal{B}([a,b], \mathbb{R}^m)$ the set of bounded functions of $[a,b]$ into $\mathbb{R}^m$. $\mathcal{B}([a,b], \mathbb{R}^m)$ is a real vector space, with the obvious operations, and

$$\|f\|_0 = \sup\{f(t) : t \in [a,b]\}$$

is a norm in $\mathcal{B}$, called the *sup norm* or the norm of *uniform convergence.*

2.4. Definition. A sequence of functions $f_n : [a,b] \longrightarrow \mathbb{R}^m$ in $\mathcal{B}$ is *uniformly convergent to* $f \in \mathcal{B}$ if $\lim_{n\longrightarrow\infty} \|f_n - f\|_0 = 0$.

2.5. Remark. Proposition 2.3 tell us that the set of bounded integrable functions $\mathcal{I}$ is a linear subspace of $\mathcal{B}$, the integral maps $\mathcal{I}$ into $\mathbb{R}^m$ linearly (items (1)) and continuously (item (3)). The next Proposition tell us that $\mathcal{I}$ is closed in $\mathcal{B}$.

2.6. Proposition. *Let $f_n : [a, b] \longrightarrow \mathbb{R}^m$ be a sequence of bounded integrable functions. If the sequence converges uniformly to a function f, the f is integrable and*

$$\int_a^b f = lim_{n\to\infty} \int_a^b f_n.$$

Proof. Set $I_n = \displaystyle\int_a^b f_n$. By Proposition 2.3 (3), $\|I_m - I_k\| \leq (b-a)\|f_m - f_k\|_0$. Hence $\{I_n\}$ is a Cauchy sequence in $\mathbb{R}^m$ and therefore converges to a vector $I \in \mathbb{R}^m$. We claim that I is the integral of f. Fix $\epsilon > 0$. Then there exist n such that $\|f_m - f\|_0 < \epsilon/3(b-a)$, $\|I - I_m\| < \epsilon/3$ if $m > n$. Also there exist $\delta > 0$ such that $\|\Sigma(f_m, P) - I_m\| < \epsilon/3$ if $|P| < \delta$. Observe that $\|\Sigma(f, P) - \Sigma(f_m, P)\| \leq (b-a)\|f - f_m\|_0$. So, if $|P| < \delta$, $\|I - \Sigma(f, P)\| \leq \|I - I_m\| + \|I_m - \Sigma(f_m, P)\| + \|\Sigma(f_m, P) - \Sigma(f, P)\| < \epsilon$ and this prove the claim. □

We will describe classes of integrable functions.

2.7. Definition. A function $f : [a, b] \longrightarrow \mathbb{R}^m$ is a *step function* if there exists a partition P of $[a, b]$, $P = \{t_0, \dots, t_k\}$ and vectors $\{X_0, \dots, X_{k-1}\}$ such that $f(t) = X_i$, $t \in (t_i, t_{i+1})$.

2.8. Lemma. *Let $f : [a, b] \longrightarrow \mathbb{R}^m$ be a step function relative to a partition P. Then f is integrable and*

$$\int_a^b f = \sum (t_{i+1} - t_i)X_i.$$

Proof. We can suppose $f(t_i) = X_i$ (see Exercise 4.19). Observe that $\Sigma(f, P') = \Sigma(f, P)$ if P' is obtained from P adding new points. Therefore $\Sigma(f, P) = \Sigma(f, P \cup Q)$ for all partitions Q and the conclusion follows. □

Since every continuous function is uniform limit of step functions (see Exercise 4.20), combining the last two Proposition we have

2.9. Proposition. *If $f : [a, b] \longrightarrow \mathbb{R}^m$ is continuous, then it is integrable.*

We will recall now the basic relation between differentiation and integration.

2.10. Lemma. *Let $f : [a, b] \longrightarrow \mathbb{R}^m$ be a continuous function and $x \in [a, b]$. Then*

$$\int_a^b f = \int_a^x f + \int_x^b f.$$

Proof. Let P be a partition such that $x \in P$. then $P = P' \cup P''$ where P' is a partition of $[a, x]$ and P'' a partition of $[x, b]$. Since all three integrals exist, by Lemma 4.20, we can compute the integrals as limit of $\Sigma(f, P_n)$ where P_n is a partition as above and $\lim_{n\to\infty} |P_n| = 0$ (see Remark 2.2). Then the conclusion follows easily. □

2.11. Remark. The Lemma still holds for functions that are just integrable. We just have to prove that an integrable function is integrable on any subinterval (see Exercise 4.18).

2.12. Theorem. [Fundamental Theorem of Calculus] *Let $f : [a, b] \longrightarrow \mathbb{R}^m$ be a continuous function. Define*

$$F : (a, b) \longrightarrow \mathbb{R}^m, \qquad F(x) = \int_a^x f.$$

Then F is differentiable and $F'(x) = f(x)$.

PROOF. F is continuous by Proposition 2.3 (item (4)). Fix $x \in (a,b)$ and let $h > 0$ be such that $x+h \in (a,b)$. Then

$$|\frac{F(x+h)-F(x)}{h} - f(x)| = |\frac{1}{h}\int_x^{x+h} f - f(x)| \le \sup\{|f(t)-f(x)| : t \in [x, x+h)\}.$$

Since f is continuous the expression on the right hand side goes to 0, when h goes to 0. The same argument works for $h < 0$ and the Claim follows. $\square$

We will define now the integral of functions of several real variables. We will consider the case of two variables and the reader should not have any difficulty to extend these considerations for n variables.

Let $f : [a,b] \times [c,b] \longrightarrow \mathbb{R}^m$ be a function and let t, s be the first and second coordinate respectively. For $t \in [a,b]$ fixed, we set $f_t(s) = f(t,s)$. Suppose the function f_t integrable, $\forall\ t \in [a,b]$. Then we define the *iterated integral* (if it exists), as

$$\int_a^b \int_c^d f(t,s)\ \mathrm{d}s\ \mathrm{d}t := \int_a^b [\int_c^d f_t(s)\ \mathrm{d}s]\ \mathrm{d}t := \int_a^b \mathrm{d}t \int_c^d f(t,s)\mathrm{d}s.$$

The elementary properties of the iterated integrals follows from the corresponding ones for the integrals of functions of one real variable. For example

$$|\int_a^b \int_c^d f(t,s)\mathrm{d}s\mathrm{d}t| \le (b-a)(d-c)\|f\|_0, \quad \text{where } \|f\|_0 = \sup\{\|f(t,s)\| : (t,s) \in [a.b] \times [c,d]\}.$$

2.13. EXAMPLE. Let $P = \{t_0, \dots, t_k\}$ be a partition of $[a,b]$, $Q = \{s_0, \dots, s_h\}$ a partition of $[c,d]$ and let $X_{ij} \in \mathbb{R}^n$ be fixed vectors. Let $g : [a,b] \times [c,d] \longrightarrow \mathbb{R}^n$ be a function such that $g(t,s) = X_{ij}, t \in (t_i, t_{i+1}),\ s \in (s_j, s_{j+1})$. For $s \in [c,d]$ the function $g_s(t) = g(t,s)$ is a step function and

$$\int_a^b g_s \mathrm{d}t = \sum_0^{k-1}(t_{j+1}-t_j)X_{ij} \quad \text{if } s \in (s_i, s_{i+1}).$$

Therefore $h(s) = \displaystyle\int_a^b g_s \mathrm{d}t$ is also a step function, therefore integrable and

$$\int_c^d \mathrm{d}s \int_a^b g(t,s)\mathrm{d}t = \sum_{ij}(s_{i+1}-s_i)(t_{j+1}-t_j)X_{ij}.$$

Observe, in particular, that the iterated integral does not depend on the order of integration.

2.14. PROPOSITION. *If f is continuous, the iterated integrals exist and*

$$\int_a^b \int_c^d f(t,s)\ \mathrm{d}s\ \mathrm{d}t = \int_c^d \int_a^b f(t,s)\ \mathrm{d}t\ \mathrm{d}s.$$

PROOF. We will start with a general fact

<u>CLAIM</u> Let $U \subseteq \mathbb{R}^n$ and let $f : U \times [a,b] \longrightarrow \mathbb{R}^m$ be a continuous function. Then the function

$$F : U \longrightarrow \mathbb{R}^m, \qquad F(x) = \int_a^b f(x,t)$$

is a continuous function.

PROOF. Fix $x_0 \in U$ and $\epsilon > 0$. The set $V = \{(x,t) \in X \times [a,b] : |f(x,t) - f(x_0,t)| < \epsilon(b-a)^{-1}$ is an open neighborhood of $x_0 \times [a,b]$. Since $[a,b]$ is compact, there exists a neighborhood W of x_0 such that $W \times [a,b] \subseteq V$. In particular, for all $x \in W,\ |f(x,t) - f(x_0,t)| < \epsilon(b-a)^{-1},\ \forall t \in [a,b]$. Hence, if $x \in W$

$$|F(x) - F(x_0)| \leq \int_a^b |f(x,t) - f(x_0,t)| \leq (b-a)\sup\{|f(x,t) - f(x_0,t)|\} < \epsilon.$$

□

The Claim implies, in particular, that a continuous function admits iterated integrals. We will prove now the commutativity relation. More precisely, given $\epsilon > 0$, we will show that, if P, Q are partitions as in Example 2.13, there exists $\delta > 0$ such that, if $|P|, |Q| < \delta$,

$$|\int_a^b \mathrm{d}t \int_c^d f(t,s)\mathrm{d}s - \sum (s_{i+1} - s_i)(t_{j+1} - t_j) f(t_j, s_i)| < \epsilon.$$

The conclusion will follows, since the other integral is, by symmetry, approximated by a sum of the same type. By uniform continuity of f, it follows that there exists $\delta > 0$ such that $|f(t,s) - f(t',s')| < \epsilon/(b-a)(d-c)$ if $|s-s'|, |t-t'|$ are smaller than δ. Consider the function g as in Example 2.13, with $g(t,s) = f(t_j, s_i),\ t \in [t_j, t_{j+1}),\ \ s \in [s_i, s_{i+1})$. Then $\|f - g\| < \epsilon/(b-a)(d-c)$. Therefore

$$|\int_a^b \mathrm{d}t \int_c^d f(t,s)\mathrm{d}s - \sum (s_{i+1} - s_i)(t_{j+1} - t_j) f(t_j, s_i)| = |\int_a^b \mathrm{d}t \int_c^d f(t,s)\mathrm{d}s - \int_a^b \mathrm{d}t \int_c^d g(t,s)\mathrm{d}s| =$$

$$= |\int_a^b \mathrm{d}t \int_c^d [f(t,s) - g(t,s)]\mathrm{d}s \leq (b-a)(d-c)\|f - g\| < \epsilon.$$

□

We will define now the *integral* of a function $f : C = [a,b] \times [c,d] \longrightarrow \mathbb{R}^n$. Let P, Q be partitions of the two intervals and set, in analogy with the 1-dimensional case,

$$\Sigma(f, P, Q) = \sum (t_{i+1} - t_i)(s_{j+1} - s_j) f(t_i, s_j).$$

2.15. DEFINITION. We will say that $X = \lim_{|P|,|Q| \to 0} \Sigma(f, P, Q)$ if, given $\epsilon > 0$ there exist $\delta > 0$ such that $\|X - \Sigma(f, P, Q)\| < \epsilon$ if $|P|, |Q| < \delta$. If such a limit exists we will say that f is integrable and define the *double integral* of f on C as

$$\int_C f(t,s)\mathrm{d}t\mathrm{d}s = X.$$

2.16. LEMMA. *If f is integrable over C and one of the simple integral, let's say $\int_a^p f(t,s)\mathrm{d}t$, exists, $\forall s \in [c,d]$, then the other simple integral exists and the iterated integrals are equal to the double integral.*

PROOF. The claim follows from the general relation between duple limits and iterated limits:

if $\lim\limits_{|P|,|Q| \to 0} \Sigma(f, P, Q)$ exists and $\lim\limits_{|P| \to 0} \Sigma(f, P, Q)$ exists $\forall Q$, $\lim\limits_{|Q| \to 0} [\lim\limits_{|P| \to 0} \Sigma(f, P, Q)] = \lim\limits_{|P|,|Q| \to 0} \Sigma(f, P, Q)$.

□

2.17. THEOREM. [Baby Fubini] *If $f : C \longrightarrow \mathbb{R}^n$ is continuous, then the double integral exists and is equal to the iterated integrals.*

PROOF. This is a corollary of the proof of Theorem 2.14. □

In particular we can define the integral of a continuous function with compact support.

2.18. Definition. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ be a continuous function with compact support and C a rectangle containing the support of F. We define

$$\int_{\mathbb{R}^2} f(t,s)\mathrm{d}t\mathrm{d}s = \int_C f(t,s)\mathrm{d}t\mathrm{d}s.$$

2.19. Remark. It is easy to see that the definition does not depends on the choice of the rectangle C.

Beside Fubini's Theorem, that allows us to reduce the calculus of multiple integrals to the case of simple integrals, the other basic fact on integration that we will need is the formula of *change of variables*.

2.20. Theorem. *Let $U, V \subseteq \mathbb{R}^2$ be open sets and let $F : U \longrightarrow V$ be a diffeomorphism. If $f : V \longrightarrow \mathbb{R}^n$ is an integrable function with compact support,*

$$\int_V f = \int_U |J[F]| f \circ F,$$

where $J[F] := \det[\mathrm{d}F]$ is the Jacobian determinant.

Proof. □

We invite the reader to extend the concepts and results above for the case of integration of function of several variables.

3. Vector fields, distributions and the local Frobenius Theorem

3.1. Definition. Let U be an open set of $\mathbb{R}^n$. A *(tangent) vector field* on U is a smooth map $X : U \longrightarrow \mathbb{R}^n$. We will denote by $\mathcal{H}(U)$ the space of vector fields on U.

3.2. Remark. Let X be a vector field. We want to think of $X(x)$ as a vector *based at* x. This is the reason why we use different names for the same thing[3]. We can make this point more precise as follows.

- The *tangent space of U at $x \in U$* is the vector space

  $$T_xU = \{(x, v) : v \in \mathbb{R}^n\}$$

  with the obvious operations on the second component.
- The *tangent bundle* of U is

$$TU = \cup_{x \in U} T_xU = U \times \mathbb{R}^n.$$

A vector field on U should be defined as a smooth map $\tilde{X} : U \longrightarrow TU$ of the form $\tilde{X}(x) = (x, X(x))$, $X : U \longrightarrow \mathbb{R}^n$. Of course, in our context, we are just complicating notations, but this point of view, that seems silly now, will prove to be useful when these concepts are extended to the case of differentiable manifolds.

We will review now some facts about solutions of differential equations.

[3]B. Russel used to say that "Mathematics is the art of calling different things with the same name and the same thing with different names".

3.3. Definition. Let $X \in \mathcal{H}(U),\ x \in U$. An *integral curve of X with initial condition x* is a smooth map $\gamma_x : (a,b) \subseteq \mathbb{R} \longrightarrow U$ such that:

$$\mathrm{d}\gamma_x(t)(1) := \dot{\gamma}_x(t) = X(\gamma_x(t)), \qquad 0 \in (a,b) \ \text{ and } \ \gamma_x(0) = x.$$

When it is clear from the context, or irrelevant, we will ignore the subscript relative to the initial condition.

The basic result about integral curves is the following

3.4. Theorem. *If $X \in \mathcal{H}(U),\ x \in U$, there exists an integral curve with initial condition $x \in U$. This curve is unique in the sense that two such curves, with the same initial condition, coincide in the intersection of the domains. In particular there is a maximal interval of definition, $(\alpha(x), \beta(x)) \subseteq \mathbb{R}$. Moreover the curve is smooth and depends smoothly on the initial condition.*

3.5. Remark. Smooth dependence on the initial condition means that, for fixed x, there exists a neighborhood U of x and $\epsilon > 0$ such that the map

$$\Gamma : U \times (-\epsilon, \epsilon) \longrightarrow U, \qquad \Gamma(y,t) = \gamma_y(t),$$

is a smooth map.

3.6. Remark. Integral curves exist even if the vector field X is merely continuous. They are unique, in the above sense, if X is *locally Lipschitzian*. If X is of class C^k, the curves are of class C^{k+1}.

3.7. Definition. The vector field is *complete* if its integral curves are defined on all of $\mathbb{R}$.

3.8. Proposition. *If X is complete, the map*

$$\gamma_t : U \longrightarrow U, \qquad \gamma_t(x) = \gamma_x(t),$$

is well defined and smooth. Moreover

(1) $\gamma_0 = \mathbb{1}$,
(2) $\gamma_{t+s} = \gamma_t \circ \gamma_s$.

Proof. The first property is obvious, by definition. As regards the second one, we observe that, for fixed s, the curves $\gamma_{\gamma_x(s)}(t)$ and $\gamma_x(t+s)$ are integral curves of X with the same initial condition. The conclusion follows from the unicity of integral curves. □

In particular γ_t is a diffeomorphism of U with inverse γ_{-t}, and the map $t \in \mathbb{R} \rightsquigarrow \gamma_t$ is a homomorphism of the additive group $\mathbb{R}$ into the group of diffeomorphisms of U.

3.9. Definition. The map Γ (or, sometimes, the maps γ_t) is called the *flow* of X.

3.10. Remark. If X is not complete, the considerations above hold locally. We leave to the reader the task of making this claim precise.

3.11. Definition. A point $x \in U$ is a *singularity* (or a *singular point*) of $X \in \mathcal{H}(U)$, if $X(x) = 0$.

The behavior of X near a singularity could be quite complicated. On the contrary, the behavior near a non singular point is quite simple.

3.12. THEOREM. *Let $X \in \mathcal{H}(U)$, $x \in U$ and $X(x) \neq 0$. Then there exists a neighborhood $\tilde{U}$ of $0 \in \mathbb{R}^n$ and a diffeomorphism ϕ of $\tilde{U}$ onto an open neighborhood of x such that* $\mathrm{d}\phi(y)(e_1) = X(\phi(y))$.

PROOF. We can assume $x = 0$ and $X(0) = e_1$. For $p = (0, x_2, \dots, x_n) \in U$, consider the integral curve of X with initial condition p, $\gamma_p(t)$. Then the map $\phi(p,t) = \gamma_p(t)$ is well defined and smooth if $|t| < \epsilon$, with ϵ sufficiently small and p is in a sufficiently small neighborhood U' of $0 \in \mathbb{R}^{n-1} = \{(x_1, \dots x_n)\} \in \mathbb{R}^n : x_1 = 0\}$. It is clear that $\mathrm{d}\phi(p,t)(e_1) = X(\phi(p,t))$ (see Remark 1.10). Also $\mathrm{d}\phi(0,0) = \mathbb{1}$, hence, by Theorem 1.25, ϕ is a diffeomorphism of a (possible smaller) neighborhood $\tilde{U} \subseteq U' \times (-\epsilon, \epsilon)$ of 0, onto its image. $\square$

We can ask for a natural generalization of Theorem 3.12: given linearly independent vector fields $X_1, \dots, X_k \in \mathcal{H}(U)$, do there exist local coordinates $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ such that $X_i = e_i$? In order to answer this question we will take a slight different approach to vector fields. First a few definitions.

3.13. DEFINITION. An *algebra* (over the reals) is a real vector space $\mathbb{E}$ together with a bilinear map, the *product*, $b : \mathbb{E} \oplus \mathbb{E} \longrightarrow \mathbb{E}$. The algebra is said to be *associative* if $b(x, b(y,z)) = b(b(x,y), z)$ and *commutative* if $b(x,y) = b(y,x) \;\; \forall \; x, y, z \in \mathbb{E}$.

When clear from the context we will write xy for $b(x,y)$.

Examples of such a structure are

- The real or complex numbers with the usual multiplication. They are associative and commutative algebras.
- The spaces $M(n, \mathbb{K})$ of $n \times n$ matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with the usual product of matrices. They are associative but non commutative algebras (if $n > 1$!).
- The space $\mathcal{F}(U)$ of smooth real valued functions defined in $U \subseteq \mathbb{R}^n$.

3.14. DEFINITION. An *algebra homomorphism* $h : \mathbb{E} \longrightarrow \mathbb{E}'$ between the algebras $\mathbb{E}$ and $\mathbb{E}'$ is a linear map such that the image of the product of two elements in $\mathbb{E}$ is the product of the images (in $\mathbb{E}'$).

3.15. DEFINITION. An *ideal* $\mathcal{I}$ of an algebra $\mathbb{E}$ is a vector subspace of $\mathbb{E}$ such that if $x \in \mathcal{I}, y \in \mathbb{E}$, then $b(x,y)$ and $b(y,x)$ are in $\mathcal{I}$

It is not difficult to see that if $\mathcal{I}$ is an ideal of $\mathbb{E}$, the quotient vector space $\mathbb{E}/\mathcal{I}$ has a natural product (and hence a structure of algebra) such that the quotient map is an algebra homomorphism. Moreover, given an algebra homomorphism $h : \mathbb{E} \longrightarrow \mathbb{E}'$, the kernel of h, $\ker h$, is an ideal and, in fact, every ideal $\mathcal{I}$ is the kernel of an algebra homomorphism, the projection $\pi : \mathbb{E} \longrightarrow \mathbb{E}/\mathcal{I}$.

Let $\mathcal{F}(U)$ be the algebra of smooth real valued functions defined in U.

3.16. DEFINITION. A *derivation of* $\mathcal{F}(U)$ (resp. a *derivation at* $x \in U$) is a $\mathbb{R}$-linear map $Y : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ (resp. $Y(x) : \mathcal{F}(U) \longrightarrow \mathbb{R}$), such that:

$$Y(fg) = Y(f)g + fY(g) \quad (\text{resp. } Y(x)(fg) = Y(x)(f)g(p) + f(p)Y(x)(g)) \;\; \forall \; f, g \in \mathcal{F}(U).$$

Both the set of derivations and the set of derivations at x have a natural structure of real vector space. We will denote by $\mathcal{D}er(U)$ and $\mathcal{D}er_x(U)$ these spaces. Observe that $\mathcal{D}er(U)$ is infinite dimensional (if $n > 0$!) while, as we will see soon, $\mathcal{D}er_x(U)$ is n-dimensional.

3.17. Example. Let $X \in \mathbb{R}^n$, $x \in U$. Then the directional derivative of $f \in \mathcal{F}(U)$ at $x \in U$, in the direction X is a derivation of f at x. As we shall soon see, all derivations at x are directional derivatives.

3.18. Example. If $X \in \mathcal{H}(U)$, we define a derivation in $\mathcal{D}er(U)$, still denoted by X, $X(f)(x) := X(x)(f)$, where $X(x)(f)$ is the directional derivative at x, as in the example above. It is easily seen that $X(f) \in \mathcal{F}(U)$ so X is, in fact, a derivation in $\mathcal{D}er(U)$.

Some simple but basic facts are the following:

3.19. Lemma. *Let $f \in \mathcal{F}(U)$ and $X_x \in \mathcal{D}er_x(U)$.*

- *If f vanishes on an open neighborhood V of x, then $X_x(f) = 0$. In particular, if two functions $f, g \in \mathcal{F}(U)$ coincide in a neighborhood of x, then $X_x f = X_x g$.*
- *If f is constant in a neighborhood of x, then $X_x f = 0$.*
- *If f is (locally) a product of functions vanishing at x, then $X_x f = 0$.*

Proof. Let $\phi \in \mathcal{F}(U)$ be a function which vanishes in a neighborhood V_1 of x and is identically 1 outside V (see Exercise 4.13 for the existence of such a function). Then $f = \phi f$ and

$$X_x(f) = (X_x\phi)f(x) + \phi(x)X_x f = 0.$$

The second claim follows from $1 \cdot 1 = 1$ and the definition of a derivation. The third one is also immediate. □

Let $x \in \mathbb{R}^n$. Consider the set

$$\tilde{\mathcal{F}}_x := \{(f, V) : V \text{ is a neighborhood of } \; x, \; f \in \mathcal{F}(V)\}.$$

3.20. Definition. The algebra of *germs of smooth functions at x*, $\mathcal{F}_x$, is the quotient of $\tilde{\mathcal{F}}_x$ by the equivalence relation $(f, U) \sim (g, V) \iff f = g$ in a neighborhood of x (contained in $U \cap V$). The operations are the usual sum and product of functions (which are defined in the intersections of the domains).

3.21. Remark. The advantage of working with germs instead that with functions is that we do not have to worry about the domain of definition of the functions involved. Anyway, when clear from the context we will make no difference between a function and its germ.

We will denote by $\mathcal{D}_x$ the space of derivations of $\mathcal{F}_x$ at x (with the obvious definition). Lemma 3.19 implies, in particular, that an element of $\mathcal{D}er_x(U)$ induces a derivation of $\mathcal{F}_x$. We shall see next that all derivations in $\mathcal{D}_x$ are of this type.

3.22. Theorem. *Given $x \in \mathbb{R}^n$ and a derivation $X_x \in \mathcal{D}_x$, there exist a unique vector $v \in \mathbb{R}^n$ such that $X_x = v(x)$. In particular $\mathcal{D}_x \cong T_x\mathbb{R}^n \cong \mathcal{D}er_x(U)$.*

Proof. Let $f \in \mathcal{F}_x$. In a suitable neighborhood of x consider the Taylor formula

$$f(x_1, \ldots, x_n) = f(x) + \sum_1^n \frac{\partial f}{\partial x_i}(x)(x_i - x_i(x)) + \Phi(x_1 \ldots, x_n),$$

where $\Phi(x_1 \ldots, x_n)$ is a sum of products of functions vanishing at x (see Theorem 1.19 and Exercise 4.26).

Applying X_x to both sides and using Lemma 3.19 we have:

$$X_x(f) = \sum_1^n X_x(x_i)\frac{\partial f}{\partial x_i}(x).$$

Therefore:

$$X = \sum_1^n X(x_i)\frac{\partial}{\partial x_i}(x),$$

and the map that associates to e_i the derivation $\frac{\partial}{\partial x_i}(x)$ extends to an isomorphism of $\mathbb{R}^n$ (or, better T_xU) onto $\mathcal{D}_x$. □

In what follows we will identify T_xU with $\mathcal{D}_x$ and $\mathcal{H}(U)$ with $\mathcal{D}er(U)$.

The composition of two derivations is not a derivation, in general. However the commutator of two derivations is a derivation (Exercise 4.28). This fact suggests the following

3.23. Definition. Let $X, Y \in \mathcal{D}er(U)$. The *Lie product* (or *bracket*) of X and Y is the commutator $[X, Y] := X \circ Y - Y \circ X$.

The following properties are easy to prove and we leave the details to the reader (Exercise 4.29).

3.24. Proposition. *The Lie product* $[\ \cdot, \cdot\ ] : \mathcal{H}(U) \times \mathcal{H}(U) \longrightarrow \mathcal{H}(U)$ *is a* $\mathbb{R}$*-bilinear map. Moreover*

(1) $[X, Y] = -[Y, X]$,

(2) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *(Jacoby identity).*

3.25. Remark. An algebra with a product which satisfies the properties above is called a *Lie algebra.*

3.26. Example. By Theorem 1.15, $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$.

We go back to the original question: given vector fields $X_1, \ldots, X_k \in \mathcal{H}(U)$, linearly independent at each point, there exist local coordinates $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $X_i = \frac{\partial}{\partial x_i}$?

There is a natural necessary condition for a positive answer, the condition being $[X_i, X_j] = 0$ (see Example 3.26). It turns out that the condition is also sufficient, at least locally. We will take a slightly more general approach.

3.27. Definition. Let $U \subseteq \mathbb{R}^n$ be an open set. A *k-dimensional distribution* D on U is a law that associates to a point $x \in U$ a k-dimensional subspace $D_x \subseteq \mathbb{R}^n$. Moreover:

- The distribution D is *smooth* if there exist, locally, k smooth vector fields $X_1, \ldots, X_k$ such that $D_x = span\{X_1(x), \ldots, X_k(x)\}$.
- A smooth distribution D is *involutive* (or *integrable*) if for all vector fields $X, Y \in \mathcal{H}(U)$ such that $X(x),\ Y(x) \in D_x,\ \forall\, x \in U$, then $[X, Y](x) \in D_x$.

3.28. Theorem. [Frobenius Theorem, local version] *Let D be a k-dimensional involutive smooth distribution on $U \subseteq \mathbb{R}^n$. Then there exist (local) coordinates $x_1, \ldots, x_n$ such that $D_x = span\{\frac{\partial}{\partial x_1}(x), \ldots, \frac{\partial}{\partial x_k}(x)\}$.*

3.29. Remark. The word “local” means that the claim of the Theorem holds in a sufficiently small open neighborhood of a fixed point, that we can assume to be $0 \in \mathbb{R}^n$.

PROOF. We will proceed by induction on k. If $k = 1$, the Theorem follows directly from Theorem 3.12. So we assume that the Theorem is true for $(k-1)$-dimensional involutive distributions. Let us suppose that D is a k-dimensional distribution spanned, locally, by smooth vector fields $X_1, \ldots, X_k$. By Theorem 3.12 we can assume that there are coordinates $y_1, \ldots, y_n$ such that $X_1 = \dfrac{\partial}{\partial y_1}$. Consider the set

$$\overline{D} = \{X \in \mathcal{H}(U) : X(x) \in D_x,\ X(y_1) = 0\}.$$

<u>CLAIM 1.</u> $\overline{D}$ is a smooth $(k-1)$-dimensional involutive distribution spanned by the vector fields

$$Y_i = X_i - X_i(y_1)X_1, \quad i = 2, \ldots, k.$$

PROOF. It is easy to see that the vector fields $X_1, Y_2, \ldots, Y_k$ are linearly independent at every point. Moreover $Y_i \in \overline{D}$ and $X_1 \notin \overline{D}$, since $X_1(y_1) = 1$. So $\overline{D}$ is a smooth $(k-1)$-dimensional distribution. Let us show that $\overline{D}$ is involutive. If $Y, Z \in \overline{D}$, $[Y, Z] \in D$ since D is involutive. Moreover $[Y, Z](y_1) = Y(Z(y_1)) - Z(Y(y_1)) = 0$, hence $[Y, Z] \in \overline{D}$. □

Observe that $\overline{D}$ is tangent to the slices $\mathbb{R}^{n-1}_c := \{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_1 = c\}$, since the first coordinate of Y_i is $Y_i(y_1) = 0$. By the inductive hypothesis the are (local) coordinates $z_2, \ldots z_n$, in $\mathbb{R}^{n-1}_0$ such that $\overline{D}_{(0,z)} = span\{\dfrac{\partial}{\partial z_i},\ i = 2, \ldots, k\}$. Consider the coordinates $x_1 = y_1, x_i = z_i, i = 2, \ldots, n$. The proof of the Theorem follow from the following

<u>CLAIM 2.</u> D is spanned by $\dfrac{\partial}{\partial x_i}$, $i = 1, \ldots, k$.

PROOF. We want to show that $Y_1 := X_1 = \dfrac{\partial}{\partial x_1}$, $Y_2, \ldots, Y_k$ are linear combinations of $\dfrac{\partial}{\partial x_i}$, $i = 1, \ldots, k$. For this is sufficient to show that $Y_i(x_j) = 0$ for $i \leq k,\ j > k$ (this is obviously true for $i = 1$).

Since the distribution is involutive, there are real valued smooth functions g_{irs} such that $[Y_i, Y_r] = \sum_{s=1}^k g_{irs} Y_s$. Now

$$Y_1(Y_i)(x_j) = [Y_1, Y_i](x_j) = \sum_1^k g_{irs} Y_s(x_j).$$

Hence the functions $Y_i(x_j)$ are solutions of the system of differential equations

$$\frac{\partial}{\partial x_1} Y_i(x_j) = \sum_1^k g_{irs} Y_s(x_j).$$

This is a *linear homogeneous system of ordinary differential equations*, along the x_1 curves, hence it admits the zero functions as solutions. Now the initial condition, for $x_1 = 0$, is $Y_i(x_j)(0, x_2, \ldots, x_n)$ which vanishes (for $j > k$) since there $x_j = z_j$. Hence, by unicity of the solutions of the initial value problem, the solutions vanish identically. □

□

3.30. REMARK. The Frobenius Theorem is really a result on existence and unicity of solutions of first order partial differential equations. We will sketch the proof of a simple fact that will explain this claim.

Let $a, b : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be smooth functions and consider the problem of finding a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = a, \qquad \frac{\partial f}{\partial y} = b.$$

The Theorem of Schwarz (Theorem 1.15) gives an obvious necessary condition for the existence of such a function, that is $\dfrac{\partial b}{\partial x} = \dfrac{\partial a}{\partial y}$. We will use the Theorem of Frobenius to shows that, at least locally, such condition is also sufficient[4]. Consider the vector fields in $\mathbb{R}^3$

$$X = \frac{\partial}{\partial x} + a\frac{\partial}{\partial z} \qquad Y = \frac{\partial}{\partial y} + b\frac{\partial}{\partial z}.$$

A simple calculation gives $[X, Y] = (\dfrac{\partial b}{\partial x} - \dfrac{\partial a}{\partial y})\dfrac{\partial}{\partial z}$. Hence the distribution spanned by X, Y is involutive if and only if $[X, Y] = 0$. In this case, by the Frobenius Theorem, there is a local diffeomorphism $\Phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $\mathrm{d}\Phi(\dfrac{\partial}{\partial x}) = X$, $\mathrm{d}\Phi(\dfrac{\partial}{\partial y}) = Y$. The "surface" $\Phi(x, y, c)$ has the distribution spanned by X and Y as "tangent space" and, since the normal vector is not horizontal, it projects (locally) onto the plane $e_3^{\perp}$, injectively. Hence it is the graph of a function f that is, as it is easily seen, a solution of our problem.

The differential equation above is the simplest case of a class of differential equation, called *total differential equations*, for which necessary and sufficient conditions for existence and unicity of solutions may be given in terms of the Theorem of Frobenius.

An important fact about Lie product of vector fields is that it "behaves well with respect to smooth maps". First a definition to make the statement precise. Let $F : U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^m$ be a smooth map between open sets.

3.31. Definition. We say that $\tilde{X} \in \mathcal{H}(V)$ is F-related to $X \in \mathcal{H}(U)$ if $\mathrm{d}F(x)(X) = \tilde{X}(F(x))$, $\forall\, x \in U$.

3.32. Proposition. *If $\tilde{X}$, $\tilde{Y} \in \mathcal{H}(V)$ are F-related to $X, Y \in \mathcal{H}(U)$, then $[\tilde{X}, \tilde{Y}]$ is F-related to $[X, Y]$.*

Proof. Let $f \in \mathcal{F}(V)$. We must show that, fixed $x \in U$, $\mathrm{d}F([X, Y](x))(f) = [\tilde{X}, \tilde{Y}](F(x))(f)$.

$$\mathrm{d}F([X, Y](x))(f) = [X, Y](x)(f \circ F) = X(x)(Y(f \circ F)) - Y(x)(X(f \circ F)) = X(x)(\tilde{Y}(f) \circ F) - Y(x)(\tilde{X}(f) \circ F) =$$

$$= \mathrm{d}F(X(x))(\tilde{Y}(f)) - \mathrm{d}F(Y(x))(\tilde{X}(f)) = \tilde{X}(F(x))(\tilde{Y}(f)) - \tilde{Y}(F(x))(\tilde{X}(f)) = [\tilde{X}, \tilde{Y}](x)(f).$$

□

There is an interpretation of the Lie product of vector fields worth mentioning.

3.33. Proposition. *Let $X, Y \in \mathcal{H}(U)$ and let ϕ_t be the (local) flow of X. Then, for $x \in U$,*

$$[X, Y](x) = \lim_{t \to 0} t^{-1}[\mathrm{d}\phi_{-t}(\phi_t(x))Y(\phi_t(x)) - Y(x)].$$

Proof. By Theorem 3.12 we can assume $X = \dfrac{\partial}{\partial x_1}$. Let $Y = \sum y_i \dfrac{\partial}{\partial x_i}$. By linearity we can assume $Y = y_i \dfrac{\partial}{\partial x_i}$. Observe that the flow of X is just translations, i.e. $\phi_t(x_1, \ldots, x_n) = (x_1 + t, \ldots, x_n)$. Then the right hand side is just $\dfrac{\partial Y}{\partial x_1}(x) = \dfrac{\partial y_i}{\partial x_1}(x)\dfrac{\partial}{\partial y_i}(x)$. On the other hand, the left hand side is also $\dfrac{\partial y_i}{\partial x_1}(x)\dfrac{\partial}{\partial y_i}(x)$ (see Exercise 4.30). □

[4]A different proof will be given in Chapter 1.

4. Exercises

4.1. For $L, T \in L(\mathbb{R}^n, \mathbb{R}^n)$ consider the norms $\|L\|_2 = \operatorname{trace} L^t L, \quad \|L\| = \sup\{\|L(x)\| : \|x\| = 1\}$.

(1) Prove that $\|L \circ T\| \leq \|L\|\|T\|$.

(2) Prove that $\|L\|^2 = \lambda$ where λ is the largest eigenvalue of $L^t L$. Conclude that $\|L\| \leq \|L\|_2 \leq n\|L\|$.

4.2. Prove Proposition 1.3.

4.3. Let $f, g : U \longrightarrow \mathbb{R}^n$ be differentiable functions. Define $F : U \longrightarrow \mathbb{R}$, $F(x) = \langle f(x), g(x) \rangle$. Prove that F is differentiable and compute $\mathrm{d}F(x)$ (see Example 1.5).

4.4. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}^2$, $\quad f(t) = (\cos t, \sin t)$. Compute $\mathrm{d}f(t)$ and show that there is no $t_0 \in [0, 2\pi]$ such that $f(2\pi) - f(0) = \mathrm{d}f(t_0)(1)2\pi$ (So the mean value Theorem, in the form 1.6, is not true if the dimension of the target space is greater than 1).

4.5. Let $f : U \longrightarrow \mathbb{R}^m$ be a differentiable function. Use Theorem 1.7 to prove

(1) if $\mathrm{d}f(x) = 0, \ \forall\, x \in U$, then f is locally constant. In particular, if U is connected, f is constant,

(2) if $T \in L(\mathbb{R}^n, \mathbb{R}^m)$. Then

$$\|f(a+h) - f(a) - T(h)\| \leq \|h\| \sup\{\|\mathrm{d}f(a+th) - T\| : t \in [0, 1]\}.$$

4.6. Prove that the function

$$f_k(t) = \begin{cases} t^k & \text{if } \ t > 0 \\ 0 & \text{if } \ t \leq 0 \end{cases}$$

is of class C^{k-1} but is not of class C^k.

4.7. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x, y) = \frac{x^2 y}{x^2 + y^2}, \quad (x, y) \neq (0, 0), \qquad f(0, 0) = 0.$$

Prove that the partial derivatives at $(0, 0)$ exist, but f is not differentiable at $(0, 0)$.

4.8. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function.

(1) Prove that if $f(tx) = tf(x), \quad \forall\, t \in \mathbb{R}$, and f is differentiable at 0, then f is linear.

(2) Prove that if $f(tx) = |t|f(x), \quad \forall\, t \in \mathbb{R}$, and f is differentiable at 0, then f vanishes identically.

(3) Prove that if $f(tx) = t^2 f(x), \quad \forall\, t \in \mathbb{R}$, and f is twice differentiable at 0, then f is bilinear.

(4) Prove that if $f(tx) = t^k f(x), \quad \forall\, t \in \mathbb{R}$ and f is of class C^k, then

$$\mathrm{d}^i f(x)(h_1, \ldots, h_i) = \frac{1}{(k-i)!} \mathrm{d}^k f(0)(x, \ldots, x, h_1, \ldots, h_i).$$

4.9. Let $\|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a norm. Prove that $\|\cdot\|$ is not differentiable at 0, $\|\cdot\|^2$ is differentiable at 0 and twice differentiable at 0 if and only if it is induced by a scalar product.

4.10. Prove that, if $\{x_n\} \subseteq \mathbb{R}^n$ is a Cauchy sequence admitting a subsequence converging to $x \in \mathbb{R}^n$, then the whole sequence converges to x.

4.11. Let $\{x_n\} \subseteq \mathbb{R}^n$ be a sequence. Define convergence for the series $\sum x_n$ and prove the Cauchy convergence criterion for series.

4.12. Let $M(n, \mathbb{R})$ the space of $n \times n$ matrices with real entries. Consider the natural identification with $\mathbb{R}^{n^2}$ and define $\exp : M(n, \mathbb{R}) \longrightarrow M(n, \mathbb{R})$, by:

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

(1) Prove that exp is well defined (i.e. the series converges).
(2) Prove that, if $AB = BA$, then

$$(A + B)^k = \sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i}.$$

Conclude that, if $AB = BA$, $\exp(A + B) = \exp(A) \exp(B)$.
(3) Prove that $\exp(PAP^{-1}) = P \exp(A) P^{-1}$.
(4) Let A be an upper triangular matrix. Compute the diagonal entries of $\exp(A)$.
(5) Show that $\det(\exp(A)) = e^{\mathrm{trace}(A)}$, $\forall A \in M(n, \mathbb{R})$. Conclude that $\exp(A)$ is invertible $\forall A \in M(n, \mathbb{R})$. Hint: put A in upper diagonal form.
(6) Show that exp is differentiable and compute $\mathrm{d}\exp(A)(B)$. Hint: compute $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \exp(A + tB)$.
(7) Show that $\mathrm{d}\exp(0) = \mathbb{1}$. In particular exp maps diffeomorphically a neighborhood of 0 onto a neighborhood of $\exp(0) = \mathbb{1}$. The (local) inverse is the *logarithm*.

4.13. Consider the function

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

(1) Prove that f is smooth.
(2) Let $0 < \delta_1 < \delta_2$. Prove that the function

$$\phi(x) = \frac{f(\|x\|^2 - \delta_1^2)}{f(\|x\|^2 - \delta_1^2) + f(\delta_2^2 - \|x\|^2)}$$

is a well defined smooth function with values in $[0, 1]$, that vanishes for $\|x\| \leq \delta_1$ and is identically 1 for $\|x\| \geq \delta_2$.

4.14. Consider the map $\phi : B^n(1) \longrightarrow \mathbb{R}^n$, $\phi(x) = x(1 - \|x\|^2)^{-\frac{1}{2}}$. Prove that ϕ is a diffeomorphism.

4.15. Use the local form of subimmersions (Theorem 1.27), to prove the following

<u>Theorem</u> [Implicit function Theorem] *Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be an open set. $f : U \longrightarrow \mathbb{R}^m$ a smooth function such that, for $z_0 = (x_0, y_0) \in U$, $f(z_0) = 0$, and $\mathrm{d}_2\, f(z_0) : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is an isomorphism. Then there exists a neighborhood $V \subseteq \mathbb{R}^n$ of x_0 and a unique smooth function $g : V \longrightarrow \mathbb{R}^m$ such that $f(x, g(x)) = 0$ $\forall\, x \in V$. Moreover*

$$\mathrm{d}g((x) = -[\mathrm{d}_2 f(x, g(x))]^{-1} \circ \mathrm{d}_1 f(x, g(x))$$

($\mathrm{d}_i f$ is defined in Remark 1.12).

4.16. Prove Proposition 2.3

4.17. Let $T : [a,b] \subseteq \mathbb{R} \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ be an integrable function and $Y \in \mathbb{R}^m$. Prove that the function $f(t) = T(t)Y$ is integrable and

$$\int_a^b f = [\int_a^b T]Y.$$

4.18. Prove that, if $f : [a,b] \longrightarrow \mathbb{R}^m$ is integrable and $x \in [a,b]$ then $f|_{[a,x]}$ and $f|_{[x,b]}$ are integrable and

$$\int_a^b f = \int_a^x f + \int_x^b f.$$

4.19. Let $f : [a,b] \longrightarrow \mathbb{R}^m$ be such that $f(t) = 0$ for t outside a finite set. Prove that f is integrable and $\int_a^b f = 0$. Conclude that if two functions $f, g : [a,b] \longrightarrow \mathbb{R}^m$ differ only on a finite set, then one is integrable if and only if the other one is integrable and, in this case, the two integral coincide.

4.20. Prove that any continuous function $f : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^m$ is uniform limit of step functions.

4.21. A curve $\gamma : [a,b] \longrightarrow \mathbb{R}^m$ is said to be *rectifiable* if there exists $l = l(\gamma) \in \mathbb{R}$ (called the *length of* γ) such that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $P = \{t_0, \dots, t_k\}$ is a partition with $|P| < \delta$, we have

$$|l - \sum_0^{k-1} \|\gamma(t_{i+1}) - \gamma(t_i)\| \, | < \epsilon.$$

Prove that if γ is of class C^1, γ is rectifiable and $l(f) = \int_a^b \dot\gamma(t)\mathrm{d}t := \int_a^b \mathrm{d}\gamma(t)(1)\mathrm{d}t$.

4.22. Use Fubini's Theorem (Theorem 2.17) to prove Theorem 1.15. Hint: it is nor restrictive to assume $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (why?). If $\dfrac{\partial f}{\partial x \partial y} - \dfrac{\partial f}{\partial y \partial x} > 0$ at $z_0 = (x_0, y_0)$ so it is in a small rectangle $C = [a,b] \times [c,d]$ containing z_0. Show that the integral over C of the difference is zero.

4.23. Let $X : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth vector field. Prove that if $\mathrm{supp}(X) := \overline{\{x \in U : X(x) \neq 0\}}$ is compact, then X is complete.

4.24. Let $X : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth vector field. Prove that, if there is a constant M with $\|X(x)\| \leq M,\ \forall\ x \in \mathbb{R}^n$, then X is complete (hint: show that an integral curve $\gamma : [0,a) \longrightarrow \mathbb{R}^n$ has finite length, if $a < \infty$, so its image has compact closure).

4.25. Give an example of a *non complete* vector field in $\mathbb{R}$.

4.26. Let $f \in \mathcal{F}(U),\ 0 \in U \subseteq \mathbb{R}^n,\ f(0) = 0$. Prove that there exist functions $g_i \in \mathcal{F}(U')$ where $U' \subseteq U$ is an open neighborhood of 0, such that $f(x_i, \dots x_n) = \sum_{i=1}^n x_i g_i(x_i, \dots x_n)$ and $g_i(0) = \dfrac{\partial f}{\partial x_i}(0)$. Hint: write $f(x) = \displaystyle\int_0^1 \frac{\mathrm{d}f(tx)}{\mathrm{d}t}\mathrm{d}t$.

4.27. Consider $\mathcal{F}_0$, the algebra of germs of smooth functions at $0 \in \mathbb{R}^n$ and $\mathcal{I}_0 = \{[f] \in \mathcal{F}_0 : f(0) = 0\}$.

(1) Prove that $\mathcal{I}_0$ is the *unique* maximal (non trivial) ideal of $\mathcal{F}_0$.
(2) Let $\mathcal{I}_0^2$ be the ideal generated by products of two elements in $\mathcal{I}_0$. Prove that $\mathcal{I}_0/\mathcal{I}_0^2$ is a n-dimensional real vector space spanned by the (equivalence classes of) the germs of the coordinate functions. Conclude that $\mathcal{I}_0/\mathcal{I}_0^2$ is canonically isomorphic to $[\mathbb{R}^n]^*$.

4.28. Prove that if $X, Y \in \mathcal{D}er(U)$ then $[X, Y] := X \circ Y - Y \circ X \in \mathcal{D}er(U)$.

4.29. Prove Proposition 3.24.

4.30. Let

$$X = \sum_k a_k(x) \frac{\partial}{\partial x_k}, \qquad Y = \sum_k b_k(x) \frac{\partial}{\partial x_k}$$

be smooth vector fields in $\mathbb{R}^n$.

(1) Compute $[X, Y]$ in the basis $\dfrac{\partial}{\partial x_k}$.

(2) Let $X_1, \ldots, X_p$ be linear independent vectors in $\mathbb{R}^n$. Show that there exist smooth vector fields $\tilde{X}_1, \ldots, \tilde{X}_p$ in $\mathbb{R}^n$ such that, for a fixed $x \in U$, $\tilde{X}_i(x) = X_i$ and $[\tilde{X}_i, \tilde{X}_j] = 0$.

CHAPTER 1

The de Rham cohomology for open sets of $\mathbb{R}^n$

1. Exterior forms

Let $\mathbb{E}$ be a finite dimensional real vector space and $\mathbb{E}^*$ its dual. We will identify, as usual, $\mathbb{E}$ with the double dual $(\mathbb{E}^*)^* := \mathbb{E}^{**}$.

1.1. Definition. A *tensor of type* (p, q) in $\mathbb{E}$ is a multilinear[1] map:

$$t\colon \underbrace{\mathbb{E}^* \times \cdots \times \mathbb{E}^*}_{p \text{ times}} \times \underbrace{\mathbb{E} \times \cdots \times \mathbb{E}}_{q \text{ times}} \longrightarrow \mathbb{R}$$

We will denote by $\mathbb{E}_{(p,q)}$ the space of these tensors. This is a real vector space with the operations of sum of multilinear maps (summing the values) and product by a scalar (multiplying the values by the scalar).

1.2. Examples.

- $\mathbb{E}_{(0,1)} = \mathbb{E}^*$, $\mathbb{E}_{(1,0)} = \mathbb{E}^{**} = \mathbb{E}$.
- A scalar product in $\mathbb{E}$ is an element of $\mathbb{E}_{(0,2)}$.
- It is convenient to define $\mathbb{E}_{(0,0)} := \mathbb{R}$.

We will be interested mainly in tensors of type $(0, q)$. To simplify the notations we will set $\mathbb{E}_q := \mathbb{E}_{(0,q)}$. Beside adding tensors, we can multiply them.

1.3. Definition. Given $\omega \in \mathbb{E}_p$, $\tau \in \mathbb{E}_q$, we define the *tensor product* $\omega \otimes \tau \in \mathbb{E}_{p+q}$ as

$$\omega \otimes \tau(x_1, \dots, x_{p+q}) := \omega(x_1, \dots, x_p)\tau(x_{p+1}, \dots x_{p+q}).$$

1.4. Remark. It is easy to see that the tensor product is associative and distributive (Exercise 7.1) and therefore, suitably extended, defines an associative algebra structure in $\mathbb{E}_* := \oplus \mathbb{E}_p$. With this structure $\mathbb{E}_*$ is called the *tensor algebra*.

1.5. Proposition. *Let* $\{\omega_1, \dots, \omega_n\}$ *be a basis of* $\mathbb{E}_1 = \mathbb{E}^*$. *Then the set* $\{\omega_{i_1} \otimes \cdots \otimes \omega_{i_q} : i_1, \dots, i_q \in \{1, \dots, n\}\}$ *is a basis of* $\mathbb{E}_q$.

Proof. Let $\{e_1, \cdots, e_n\}$ be the dual basis, i.e., $\omega_i(e_j) = \delta_{ij}$. Then:

$$\sum a_{i_1 \cdots i_q} \omega_{i_1} \otimes \cdots \otimes \omega_{i_q}(e_{j_1}, \dots, e_{j_q}) = a_{j_1 \cdots j_q}.$$

[1] i.e. linear in each variable.

It follows, by a standard argument, that the the elements of the set in question are linearly independent. Now, given $\omega \in \mathbb{E}_q$ we define $a_{i_1 \cdots i_q} = \omega(e_{i_1}, \ldots, e_{i_q})$. It is easy to check that $\omega = \sum a_{i_1 \cdots i_q} \omega_{i_1} \otimes \cdots \otimes \omega_{i_q}$, and this concludes the proof.

□

We will be interested in special elements of $\mathbb{E}_q$. Let $\Sigma(p)$ be the group of permutations of $\{1, \ldots, p\} \subseteq \mathbb{N}$. If $\pi \in \Sigma(p)$, we will denote by $|\pi|$ the sign of π, i.e. $|\pi| = 1$ if π is the product of an even number of transpositions and $|\pi| = -1$ otherwise.

1.6. Definition. Let $\omega \in \mathbb{E}_p$. We will say that

- ω is *a symmetric form* if $\quad \omega(x_1, \ldots, x_p) = \omega(x_{\pi(1)}, \ldots, x_{\pi(p)}), \quad \forall\, \pi \in \Sigma(p)$.
- ω is an *exterior form* [2] if $\quad \omega(x_1, \ldots, x_p) = |\pi| \omega(x_{\pi(1)}, \ldots, x_{\pi(p)}), \quad \forall\, \pi \in \Sigma(p)$.

We will denote by $\Sigma^p(\mathbb{E})$ the space of symmetric tensors in $\mathbb{E}_p$ and with $\Lambda^p(\mathbb{E})$ the space of exterior p-forms. These are subspaces of $\mathbb{E}_p$. Clearly $\Lambda^0(\mathbb{E}) = \mathbb{R} = \Sigma^0(\mathbb{E}), \;\; \Lambda^1(\mathbb{E}) = \mathbb{E}_1 = \mathbb{E}^* = \Sigma^1(\mathbb{E})$.

We will be mostly interested in exterior forms and we will describe now the basic example.

1.7. Example. Let $\{e_1, \ldots, e_n\}$ be a fixed basis of $\mathbb{E}$ and $\{\phi_1, \ldots, \phi_n\}$ be the dual basis. Let us fix indexes $1 \leq i_1 < \cdots < i_p \leq n$ and define:

$$\omega_{(i_1, \ldots, i_p)}(x_1, \ldots, x_p) := \det(\phi_{i_j}(x_k)).$$

In other words we consider the matrix whose k^{th} column is given by the coordinates of x_k in the fixed basis, and compute the determinant of the sub matrix obtained considering only the lines $(i_1, \ldots, i_p)$ of the original matrix. The $\omega_{(i_1, \ldots, i_p)}$'s are exterior p-forms since the determinant is multilinear in the columns and, permuting the columns the sign changes according to the parity of the permutation. As we shall see (Proposition 1.20 and Remark 1.18), these forms are a basis of $\Lambda^p(\mathbb{E})$.

1.8. Remark. By Example 1.7 p-forms are, essentially, determinants of $p \times p$ matrices and, therefore, "p-dimensional (oriented) volume elements". So they appear as the *natural integrands of the multiple (oriented) integrals*. This statement will be made precise in the next chapter.

The tensor product of exterior forms is not, in general, an exterior form. But we can "alternate" the tensor product in order to obtain an exterior form. Define the linear operator

$$A : \mathbb{E}_p \longrightarrow \mathbb{E}_p, \qquad A(\tau)(x_1, \ldots, x_p) = \frac{1}{p!} \sum_{\pi \in \Sigma(p)} |\pi| \tau(x_{\pi(1)}, \ldots, x_{\pi(p)}).$$

1.9. Proposition.

(1) *If* $\tau \in \mathbb{E}_p, \quad A(\tau) \in \Lambda^p(\mathbb{E})$.
(2) *If* $\tau \in \Lambda^p(\mathbb{E}), \quad A(\tau) = \tau$.

In particular $A^2 = A$.

[2] The terms alternating tensor or skew symmetric tensor are also used in the literature.

Proof. If $p = 1$ there is nothing to prove, so we assume $p > 1$. For $i, j \in \{1, \dots, p\}$, we will denote by (ij) the element of $\Sigma(p)$ that interchanges i and j and leaves the other integers fixed. If $\pi \in \Sigma(p)$, we set $\pi' = \pi \circ (ij)$. Then $|\pi'| = -|\pi|$ and

$$A(\tau)(x_1, \dots, x_j, \dots, x_i, \dots, x_p) = \frac{1}{p!} \sum_{\pi} |\pi| \tau(x_{\pi(1)}, \dots, x_{\pi(j)}, \dots, x_{\pi(i)}, \dots, x_{\pi(p)}) =$$

$$\frac{1}{p!} \sum_{\pi} |\pi| \tau(x_{\pi'(1)}, \dots, x_{\pi'(i)}, \dots, x_{\pi'(j)}, \dots, x_{\pi'(p)}) =$$

$$\frac{1}{p!} \sum_{\pi'} -|\pi'| \tau(x_{\pi'(1)}, \dots, x_{\pi'(i)}, \dots, x_{\pi'(j)}, \dots, x_{\pi'(p)}) = -A(\tau)(x_1, \dots, x_i, \dots, x_j, \dots, x_p)$$

It is easy to see that the equation above implies that $A(\tau) \in \Lambda^p(\mathbb{E})$ (see Exercise 7.2). Moreover, if $\tau \in \Lambda^p(\mathbb{E})$,

$$A(\tau)(x_1, \dots, x_p) = \frac{1}{p!} \sum_{\pi} |\pi| \tau(x_{\pi(1)}, \dots x_{\pi(p)}) = \frac{1}{p!} \sum_{\pi} |\pi|^2 \tau(x_1, \dots x_p) = \tau(x_1, \dots, x_p)$$

and this proves the second claim.

□

Observe that, in general, $A(\phi \otimes \psi) \neq A(\phi) \otimes A(\psi)$. However we have

1.10. Lemma. *If $\phi_1, \dots, \phi_p \in \mathbb{E}^*$, then:*

$$A(\phi_1 \otimes \dots \otimes \phi_p) = \frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma| \phi_{\sigma(1)} \otimes \dots \otimes \phi_{\sigma(p)}.$$

Proof.

$$A(\phi_1 \otimes \dots \otimes \phi_p)(x_1, \dots, x_p) = \frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma| \phi_1 \otimes \dots \otimes \phi_p(x_{\sigma(1)}, \dots, x_{\sigma(p)}) =$$

$$\frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma| \phi_1(x_{\sigma(1)}) \cdots \phi_p(x_{\sigma(p)}) = \frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma| \phi_{\sigma(1)}(x_1) \cdots \phi_{\sigma(p)}(x_p).$$

□

Using the operator A we can define *product of exterior forms.*

1.11. Definition. The *exterior (or wedge) product* is defined as the map

$$\wedge : \Lambda^p(\mathbb{E}) \times \Lambda^q(\mathbb{E}) \longrightarrow \Lambda^{p+q}(\mathbb{E}), \qquad \wedge(\omega, \tau) := \omega \wedge \tau = \frac{(p+q)!}{p!q!} A(\omega \otimes \tau).$$

(The reason for the coefficient $\frac{(p+q)!}{p!q!}$ will be discuss in Remark 1.19.)

It is easy to prove that the exterior product is distributive (see Exercise 7.3). In particular, suitably extended, it defines an algebra structure on $\Lambda^*(\mathbb{E}) := \oplus \Lambda^p(\mathbb{E})$. $\Lambda^*(\mathbb{E})$ is called the *exterior algebra.*

It is also true that the exterior product is associative, but this fact is a little bit tricky. The proof involves a characterization of the kernel of A. The problem is that A is not an algebra homomorphism, hence we can not conclude, directly, that $\ker A$ is an ideal. We will prove that, in fact, $\ker A$ is an ideal.

Consider the ideal $\mathcal{I} \subseteq \mathbb{E}_*$ generated by $\phi \otimes \phi,\ \phi \in \mathbb{E}^*$. This is the vector subspace of $\mathbb{E}_*$ generated by elements of the form $\tau \otimes \phi \otimes \phi, \psi \otimes \psi \otimes \eta,\ \phi, \psi \in \mathbb{E}^*, \tau, \eta \in \mathbb{E}_*$ or, alternatively, it is the intersection of all ideals containing the elements of the form $\phi \otimes \phi,\ \phi \in \mathbb{E}^*$.

1.12. Theorem. $\ker A = \mathcal{I}$.

Proof. It is easily seen that $\mathcal{I} \subseteq \ker A$. We will prove that $\ker A \subseteq \mathcal{I}$. Consider the quotient algebra $\mathbb{E}_*/\mathcal{I}$. Denote by $\cdot$ the product in this quotient and by $\pi : \mathbb{E}_* \longrightarrow \mathbb{E}_*/\mathcal{I}$ the quotient map, which is an algebra homomorphism. First observe that, if $\phi, \psi \in \mathbb{E}^*$:

$$0 = \pi((\phi + \psi) \otimes (\phi + \psi)) = \pi(\phi \otimes \phi + \phi \otimes \psi + \psi \otimes \phi + \psi \otimes \psi) = \pi(\phi \otimes \psi) + \pi(\psi \otimes \phi),$$

i.e. $\pi(\phi \otimes \psi) = -\pi(\psi \otimes \phi)$. Therefore, for $\phi_1, \ldots, \phi_p \in \mathbb{E}^*$ and $\sigma \in \Sigma(p)$, we have

$$\pi(\phi_{\sigma(1)}, \otimes \ldots, \otimes \phi_{\sigma(p)}) = \pi\left(\phi_{\sigma(1)}\right) \cdots \pi(\phi_{\sigma(p)}) = |\sigma| \pi(\phi_1) \cdots \pi(\phi_p) = |\sigma| \pi(\phi_1 \otimes \cdots \otimes \phi_p).$$

Hence

$$\pi(A(\phi_1 \otimes \cdots \otimes \phi_p)) = \pi(\frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma| \pi(\phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(p)})) = \frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma|^2 \pi(\phi_1 \otimes \cdots \otimes \phi_p) = \pi(\phi_1 \otimes \cdots \otimes \phi_p).$$

So any element in $\ker A$ is in $\mathcal{I} := \ker\ \pi$. □

1.13. Corollary. *Let $\omega \in \mathbb{E}_p,\ \tau \in \mathbb{E}_q$. If $A(\omega) = 0,\quad A(\omega \otimes \tau) = 0 = A(\tau \otimes \omega)$.*

Proof. This follows from the fact that $\ker A$ is an ideal. □

At this point we can prove the announced result

1.14. Proposition. *The wedge product is associative.*

Proof. First we observe that

$$A(A(\omega \otimes \eta) \otimes \theta)) = A(\omega \otimes \eta \otimes \theta) = A(\omega \otimes A(\eta \otimes \theta)).$$

In fact, by 1.9, $A^2 = A$. Hence $A(A(\eta \otimes \theta) - \eta \otimes \theta) = 0$ and, by 1.13, we have that:

$$0 = A(\omega \otimes (A(\eta \otimes \theta) - \eta \otimes \theta)) = A(\omega \otimes A(\eta \otimes \theta) - \omega \otimes \eta \otimes \theta) = A(\omega \otimes A(\eta \otimes \theta)) - A(\omega \otimes \eta \otimes \theta),$$

which proves the second equality. The first one is proved in a similar way.

Therefore, if $\omega \in \Lambda^k(\mathbb{E}), \eta \in \Lambda^l(\mathbb{E}), \theta \in \Lambda^m(\mathbb{E})$, we have:

$$(\omega \wedge \eta) \wedge \theta = \frac{(k+l+m)!}{(k+l)!m!} A((\omega \wedge \eta) \otimes \theta) = \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} A(\omega \otimes \eta \otimes \theta),$$

and the associativity follows from the associativity of the tensor product. □

1.15. Example. Let $\phi_1, \phi_2 \in \mathbb{E}^* = \mathbb{E}_1,\ x_1, x_2 \in \mathbb{E}$. Then:

$$\phi_1 \wedge \phi_2(x_1, x_2) = 2\frac{1}{2}(\phi_1(x_1)\phi_2(x_2) - \phi_1(x_2)\phi_2(x_1)) = \det[\phi_i(x_j)].$$

More generally, an induction on p gives

1.16. Proposition. *Let* $\phi_i \in \mathbb{E}^*, x_j \in \mathbb{E} \ \ i, j = 1, \ldots, p$. *Then:*

$$\phi_1 \wedge \cdots \wedge \phi_p(x_1, \ldots, x_p) = \det[\phi_i(x_j)].$$

In particular if $\sigma \in \Sigma(p)$, $\phi_1 \wedge \cdots \wedge \phi_p = |\sigma| \phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(p)}$.

1.17. Remark. Observe that, by 1.14, the form $\phi_1 \wedge \cdots \wedge \phi_p$ is well defined.

1.18. Remark. In the Example 1.7 the form $\omega_{(i_1,\ldots,i_p)}$ is just $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$.

1.19. Remark. The coefficient $\dfrac{(p+q)!}{p!q!}$ in 1.11 is convenient in order to avoid unpleasant coefficients in 1.16 and also for a geometric reason: let $\mathbb{E}$ be an inner product space, $\{e_1, \ldots, e_n\}$ an orthonormal basis and $\{\phi_1, \ldots, \phi_n\}$ the dual basis (so $\phi_i(e_j) = \langle e_i, e_j \rangle = \delta_{ij}$). Given vectors $x_1, \ldots, x_n \in \mathbb{E}$, $\phi_1 \wedge \cdots \wedge \phi_n(x_1, \ldots, x_n)$ is the "volume" of the parallelepiped of edges the $x_i's$. The coefficient above is such that the "unit cube", i.e. the parallelepiped spanned by the e_i's, has volume 1 (see Definition 1.28).

1.20. Proposition. *Let* $\{\phi_1, \ldots, \phi_n\}$ *be a basis for* $\mathbb{E}^*$. *Then*

$$\{\phi_{i_1} \wedge \cdots \wedge \phi_{i_p} : 1 \le i_1 < \cdots < i_p \le n\}$$

is a basis of $\Lambda^p(\mathbb{E})$. *In particular* $\Lambda^p(\mathbb{E})$ *has dimension* $\binom{n}{p}$ *and* $\Lambda^p(\mathbb{E}) = \{0\}$, *if* $p > n$.

Proof. Let $\{e_1, \ldots, e_n\}$ be the dual basis. First observe that $\phi_1 \wedge \cdots \wedge \phi_n(e_1, \ldots, e_n) = \det[\phi_i(e_j)] = 1$. Also observe that $\phi_i \wedge \phi_j = -\phi_j \wedge \phi_i$ and, in particular, if we interchange two elements in the product $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$ the form changes sign. We will prove now that the forms $\{\phi_{i_1} \wedge \cdots \wedge \phi_{i_p} : i_1 < \cdots < i_p\}$ are linearly independent. Suppose

$$\sum_{i_1 < \cdots < i_p} a_{i_1 \cdots i_p} \phi_{i_1} \wedge \cdots \wedge \phi_{i_p} = 0.$$

We want to show that $a_{i_1 \cdots i_p} = 0$. We will do it for $a_{1 \cdots p}$, the other cases being analogous. Observe that

$$\sum_{i_1 < \cdots < i_p} a_{i_1 \cdots i_p} \phi_{i_1} \wedge \cdots \wedge \phi_{i_p} \wedge \phi_{p+1} \wedge \cdots \wedge \phi_n(e_1, \ldots, e_n) = a_{1 \cdots p} = 0,$$

since the terms with $\{i_1, \ldots, i_p\} \neq \{1, \ldots, p\}$ vanish (they contain two equal indexes), and the conclusion follows. We leave to the reader the task of showing that they span $\Lambda^p(\mathbb{E})$ (Exercise 7.4).

□

1.21. Corollary. *The algebra* $\Lambda^*(\mathbb{E})$ *is a graded commutative algebra*[3], *i.e. if* $\omega \in \Lambda^p(\mathbb{E})$, $\tau \in \Lambda^q(\mathbb{E})$

$$\omega \wedge \tau = (-1)^{pq} \tau \wedge \omega.$$

In particular the square of a form of odd degree is zero.

Proof. As we have seen this is true for products of decomposable elements (i.e. elements of the form $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$). The general case follows from the fact that such forms span the exterior algebra. □

1.22. Remark. There is a restriction, in Proposition 1.20, on the set of indexes with respect to Proposition 1.5 and this is due to the graded commutativity of the exterior algebra.

[3] An algebra $\mathbb{E}$, with product $b : \mathbb{E} \oplus \mathbb{E} \longrightarrow \mathbb{E}$ is a *graded algebra* if there is a sequence of vector subspaces $\mathbb{E}_i$ such that $\mathbb{E} = \oplus \mathbb{E}_i$ and $b(\mathbb{E}_i \oplus \mathbb{E}_j) \subseteq E_{i+j}$. Such an algebra is said to be *graded commutative* if for $\omega \in \mathbb{E}_p$, $\tau \in \mathbb{E}_q$, $b(\omega, \tau) = (-1)^{pq} b(\tau, \omega)$.

Let $L : \mathbb{E} \longrightarrow \mathbb{F}$ be a linear map. Recall that the *transpose* of L is the map

$$L^* : \mathbb{F}^*(= \mathbb{F}_1) \longrightarrow \mathbb{E}^*(= \mathbb{E}_1), \quad L^*(\phi)(x) := \phi(Lx).$$

This map extends to a linear map

$$\mathbb{E}_p(L) : \mathbb{F}_p \longrightarrow \mathbb{E}_p, \quad \mathbb{E}_p(L)(\omega)(x_1, \dots, x_p) = \omega(L(x_1), \dots, L(x_p)).$$

It is simple to see that if $\omega \in \Lambda^p(\mathbb{F})$ then $\mathbb{E}_p(L)(\omega) \in \Lambda^p(\mathbb{E})$. So we get, by restriction, a linear map

$$\Lambda^p(L) := \mathbb{E}_p(L)|_{\Lambda^p(\mathbb{F})} : \Lambda^p(\mathbb{F}) \longrightarrow \Lambda^p(\mathbb{E}),$$

and, by additivity, a linear map $\Lambda^*(L) : \Lambda^*(\mathbb{F}) \longrightarrow \Lambda^*(\mathbb{E})$.

When clear from the context we will write L_p^*, or just L^*, for $\Lambda^p(L)$ and $\Lambda^*(L)$.

1.23. Proposition. $L^*(\omega \wedge \tau) = L^*(\omega) \wedge L^*(\tau)$. *This means that* L *induces a graded algebra homomorphism* $L^* : \Lambda^*(\mathbb{F}) \longrightarrow \Lambda^*(\mathbb{E})$. *Moreover we have the following properties, called the* funtorial properties[4]

(1) $(\mathbb{1}_{\mathbb{E}})^* = \mathbb{1}_{\Lambda^*(\mathbb{E})}$.
(2) *If* $L : \mathbb{E} \longrightarrow \mathbb{F}$ *and* $T : \mathbb{F} \longrightarrow \mathbb{G}$ *are linear maps, then* $(T \circ L)^* = L^* \circ T^*$.

Proof. To prove the first assertion, we just observe that, if $\phi_i \in \mathbb{F}^*, x_j \in \mathbb{E},\ i, j = 1, \dots, p$, we have:

$$L_p^*(\phi_1 \wedge \dots \wedge \phi_p)(x_1, \dots, x_p) = \det[\phi_i(Lx_j)] = \det[L^*(\phi_i)(x_j)] = L^*(\phi_1) \wedge \dots \wedge L^*(\phi_p)(x_1, \dots, x_p).$$

Since $\Lambda^p(\mathbb{E})$ is spanned by elements of the form $\phi_1 \wedge \dots \wedge \phi_p$, by Proposition 1.20, the conclusion follows by linearity. The functorial properties are obvious. $\square$

1.24. Remark. We will meet often, along these notes, "functorial properties". These properties are usually trivial to prove, but important. For example, in the context of Proposition 1.23, they imply that, if L is an isomorphism, then L^* is also an isomorphism (see Exercise 7.16).

Let $\mathbb{E}$ be a finite dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}$.

1.25. Definition. The isomorphisms

$$\flat : \mathbb{E} \longrightarrow \mathbb{E}^*, \quad \flat(x)(y) = \langle x, y \rangle, \quad \sharp : \mathbb{E}^* \longrightarrow \mathbb{E}, \quad \sharp := \flat^{-1},$$

are called the *musical isomorphisms*.

We define an inner product in $\mathbb{E}^*$ by requiring $\flat$ to be an isometry. We can also define an inner product in $\Lambda^p(\mathbb{E})$ extending, by bi-linearity, the formula

$$\langle \phi_1 \wedge \dots \wedge \phi_p, \psi_1 \wedge \dots \wedge \psi_p \rangle = \det(\langle \phi_i, \psi_j \rangle).$$

Observe that, if $\{\omega_i\}$ is an orthonormal basis for $\mathbb{E}^*$, the basis $\{\omega_{i_1} \wedge \dots \wedge \omega_{i_p} : i_1 < \dots < i_p\}$ is orthonormal.

We recall that two bases of a n-dimensional real vector space $\mathbb{E}$ are *equioriented* if the matrix that gives the change of bases has positive determinant. This relation is an equivalence relation and the set of bases of $\mathbb{E}$ is divided into two equivalence classes.

[4] In the language of category theory this means that the law that associate to a finite dimensional real vector space $\mathbb{E}$ the graded algebra $\Lambda^*(\mathbb{E})$ and to a linear maps $L : \mathbb{E} \longrightarrow \mathbb{F}$ the map L^* is a *contravariant* functor from the category of finite dimensional real vector spaces and linear maps, to the category of algebras and their homomorphisms.

1.26. Definition. An orientation on $\mathbb{E}$ is the choice of one of two equivalence classes of equioriented bases. $\mathbb{E}$ is *oriented* if such a choice has been made and the bases in the chosen class will be called *positive*.

1.27. Remark. Naturally an orientation in $\mathbb{E}$ induces an orientation on $\mathbb{E}^*$, by declaring positive the bases that are dual of positive bases of $\mathbb{E}$.

1.28. Definition. Let $\mathbb{E}$ be a n-dimensional *oriented* inner product space and let $\{\omega_1, \ldots, \omega_n\}$ be a positive orthonormal basis of $\mathbb{E}^*$. The *volume form* of $\mathbb{E}$ is the n-form $v = \omega_1 \wedge \cdots \wedge \omega_n$.

1.29. Lemma. *The volume form is well defined, i.e. it does not depend on the choice of the basis.*

Proof. Let $\{\omega_i\}$, $\{\phi_j\}$ be bases of $\mathbb{E}^*$ and $A = (a_{ij})$ such that $\phi_k = \sum a_{kj}\omega_j$. Then

$$\phi_1 \wedge \cdots \wedge \phi_n = \sum_{\sigma \in \Sigma(n)} |\sigma| a_{1\sigma(1)} \cdots a_{n\sigma(n)} \omega_1 \wedge \cdots \wedge \omega_n = \det(A)\omega_1 \wedge \cdots \wedge \omega_n.$$

If the bases are orthonormal and positive, then $A \in SO(n)$. In particular $\det(A) = 1$. □

1.30. Definition. Let $\mathbb{E}$ be a n-dimensional oriented inner product space. The *Hodge (star) operator* is the operator

$$*_p : \Lambda^p(\mathbb{E}) \longrightarrow \Lambda^{(n-p)}(\mathbb{E}), \qquad *_p(\eta)(x_1, \ldots, x_{(n-p)}) := \langle \eta \wedge \flat(x_1) \wedge \cdots \wedge \flat(x_{(n-p)}), v \rangle,$$

where v is the volume form. When clear from the context, we will write simply $*$ instead of $*_p$.

1.31. Remark. Let $\{\omega_i\}$ be a positive orthonormal basis for $\mathbb{E}^*$. Then the Hodge operator may be defined by extending, linearly, the map

$$*(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) = \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-p}},$$

where $\{i_1, \ldots, i_p, j_1, \ldots j_{n-p}\}$ is an even permutation of $\{1, \ldots, n\}$.

The following properties are easily established

1.32. Proposition. $*$ *is a linear isometry and* $*_{n-p} \circ *_p = (-1)^{p(n-p)} \mathbb{1}_{\Lambda^p(\mathbb{E})}$.

2. Differential forms and the de Rham cohomology

2.1. Definition. A *differential p-form* on an open set $U \subseteq \mathbb{R}^n$ is a smooth map $\omega : U \longrightarrow \Lambda^p(\mathbb{R}^n) \cong \mathbb{R}^{\binom{n}{p}}$. When clear from the context we will just say that ω is a differential form or simply a form.

2.2. Remark. According to Remark 3.2 of Chapter 0, we can complicate the definition in order to have one that make sense in the context of smooth manifold. Consider the *bundle of exterior p-forms*

$$\Lambda^p(U) := \cup_{x \in U} \Lambda^p(T_x U)$$

that can be identified with $U \times \Lambda^p(\mathbb{R}^n)$. Then a differential p-form is a smooth map $\tilde{\omega} : U \longrightarrow \Lambda^p(U)$ such that $\tilde{\omega}(x) \in \Lambda^p(T_x U)$, i.e, $\tilde{\omega}(x) = (x, \omega(x))$, $\omega(x) \in \Lambda^p(\mathbb{R}^n)$.

We will denote by $\Omega^p(U)$ the set of differential p-forms on U. $\Omega^p(U)$ has an obvious structure of real vector space. Moreover we can multiply a differential form by a function and this operation is associative and distributive, in the appropriate sense, i.e. $\Omega^p(U)$ is a *module over* $\mathcal{F}(U)$.

A differential form $\omega \in \Omega^p(U)$ induces a $\mathcal{F}(U)$-multilinear map, denoted by the same symbol,

$$\omega : \mathcal{H}(U) \times \cdots \times \mathcal{H}(U) \longrightarrow \mathcal{F}(U), \quad \omega(X_1, \ldots, X_p)(x) = \omega(x)(X_1(x), \ldots, X_p(x)).$$

Conversely, we have

2.3. Theorem. [Tensoriality Criterion] *A* $\mathbb{R}$*-multilinear map*

$$\omega : \mathcal{H}(U) \times \cdots \times \mathcal{H}(U) \longrightarrow \mathcal{F}(U),$$

is induced by a differential form if and only if it is $\mathcal{F}(U)$*-multilinear.*

Proof. Clearly, if ω is induced by a form, it is $\mathcal{F}(U)$-multilinear. Suppose that ω is $\mathcal{F}(U)$-multilinear. Let $x \in U, X_i \in T_xU$. Extend the X_i's to vector fields $\tilde{X}_i \in \mathcal{H}(U)$, $\tilde{X}_i(y) = \sum_j a_{ij}(y)e_j$, and define:

$$\omega(x)(X_1, \ldots, X_p) := \omega(\tilde{X}_1, \ldots, \tilde{X}_p)(x).$$

In order to show that the above equality defines a form it is sufficient to show that it does not depend on the extensions. In fact, by $\mathcal{F}(U)$-multilinearity,

$$\omega(\tilde{X}_1, \ldots, \tilde{X}_p)(x) = \sum_{i_1, \ldots, i_p = 1}^{n} a_{1i_1}(x) \cdots a_{pi_p}(x)\omega(e_{i_1}, \ldots, e_{i_p}).$$

□

2.4. Example. Since $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$, $\Omega^0(U) = \mathcal{F}(U)$.

The basic example of a differential form is the following. Let $f \in \mathcal{F}(U)$. Then the differential of f is the the 1-form

$$(\mathrm{d}f)(x)(X) := X(x)(f), \qquad X \in \mathcal{D}er(U).$$

In particular, we can consider the coordinate functions $x_i : \mathbb{R}^n \longrightarrow \mathbb{R}$. At each point $x \in U$, the differentials at x, $\mathrm{d}x_i(x)$ [5] are a basis of $\Lambda^1(\mathbb{R}^n)$. Therefore $\{\mathrm{d}x_{i_1}(x) \wedge \cdots \wedge \mathrm{d}x_{i_p}(x) : \ 1 \leq i_i < \cdots < i_p \leq n\}$ is a basis of $\Lambda^p(\mathbb{R}^n)$. So we have

2.5. Proposition. *Let* $\omega \in \Omega^p(U)$*. Then* ω *can be written in a unique way as:*

$$\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1, \ldots, i_p} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p},$$

where $\omega_{i_1, \ldots, i_p} \in \mathcal{F}(U)$.

2.6. Example. If $f \in \mathcal{F}(U)$, $\mathrm{d}f = \sum_1^n \dfrac{\partial f}{\partial x_i} \mathrm{d}x_i$.

2.7. Remark. As a real vector space $\Omega^p(U)$ is infinite dimensional (if $n > 0$!), but as a $\mathcal{F}(U)$-module, it is a free module of dimension $\dbinom{n}{p}$.

[5] Since x_i is linear, $\mathrm{d}x_i = x_i$, and $\mathrm{d}x_i$ is the form that associates to a vector its i^{th} coordinate in the canonical basis.

Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open sets and $F : U \longrightarrow V$ a smooth function, $F(x) = (F_1(x), \dots, F_m(x))$. Then $\mathrm{d}F(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map and we have an induced map $F^* : \Lambda^p(\mathbb{R}^m) \longrightarrow \Lambda^p(\mathbb{R}^n)$. This map induces a linear map:

$$F^* : \Omega^p(V) \longrightarrow \Omega^p(U), \qquad F^*(\omega)(X_1, \dots, X_p)(x) := \omega(\mathrm{d}F(x)(X_1), \dots, \mathrm{d}F(x)(X_p)).$$

If $x_1, \dots, x_n$, $y_1, \dots y_m$ are the canonical coordinates in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, we have

$$F^*(\mathrm{d}y_i) = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_j} \mathrm{d}x_j,$$

and therefore, if $\omega = \sum_{i_1, \dots, i_p} \omega_{i_1, \dots, i_p} \mathrm{d}y_{i_1} \wedge \dots \wedge \mathrm{d}y_{i_p}$,

$$F^*(\omega)(x) = \sum_{i_1, \dots, i_p} \omega_{i_1, \dots, i_p}(F(x)) F^*(\mathrm{d}y_{i_1}) \wedge \ldots \wedge F^*(\mathrm{d}y_{i_1}).$$

We have the *functorial properties*:

- $\mathbb{1}_U^* = \mathbb{1}_{\Omega^p(U)}$,
- If $F_1 : U_1 \longrightarrow U_2$ e $F_2 : U_2 \longrightarrow U_3$ are smooth maps, $(F_2 \circ F_1)^* = F_1^* \circ F_2^*$.

In particular, if F is a diffeomorphism, F^* is an isomorphism.

2.8. Example. Let $U \subseteq \mathbb{R}^n$ and $j : U \longrightarrow U \times \mathbb{R}^m, j(x_1 \dots, x_n) = (x_1 \dots, x_n, 0 \dots, 0)$, be the inclusion. If $\omega = f(x_1, \dots, x_{n+m}) \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, i_1 < \dots < i_p$, $j^*\omega = 0$, if $i_p > n$, and $j^*\omega = f(x_1, \dots x_n, 0, \dots, 0) \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}$ is $i_p \le n$.

Differentiating a function can be viewed as a $\mathbb{R}$-linear map:

$$\mathrm{d} : \Omega^0(U) = \mathcal{F}(U) \longrightarrow \Omega^1(U).$$

Now we extend extend now this operation to higher dimensional forms.

2.9. Theorem. *There exists a unique family of $\mathbb{R}$- linear operators $\mathrm{d}^p : \Omega^p(U) \longrightarrow \Omega^{p+1}(U)$, $p = 0, \dots, n$, such that:*

(1) $\mathrm{d}^0 = \mathrm{d}$ *(the usual differential).*
(2) $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$.
(3) *If* $\omega \in \Omega^p(U)$, $\tau \in \Omega^q(U), \mathrm{d}^{p+q}\omega \wedge \tau = \mathrm{d}^p\omega \wedge \tau + (-1)^p \omega \wedge \mathrm{d}^q\tau$.

Moreover, if $F : U \longrightarrow V$ is a smooth map and $\omega \in \Omega^p(V)$, then $\mathrm{d}^p F^*\omega = F^*\mathrm{d}^p\omega$.

When clear from the context we will write simply d *for* d^p.

Proof. Let us suppose that such a family exists. If $\omega = f(x)\, \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}$, we have, by (3),

$$\mathrm{d}\omega = (\mathrm{d}f) \wedge \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p} + f\, \mathrm{d}(\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}).$$

Now, from (1), $\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i$, and, from (2) and (3)

$$\mathrm{d}(\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}) = \sum_j \pm \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}\mathrm{d}x_{i_j} \wedge \dots \wedge \mathrm{d}x_{i_p} = 0.$$

Therefore, if $\omega = \sum_{i_1<\cdots<i_p} \omega_{i_1\ldots i_p} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$,

$$\mathrm{d}\omega = \sum_k \sum_{i_1<\cdots<i_p} \frac{\partial \omega_{i_1\ldots i_p}}{\partial x_k} \mathrm{d}x_k \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}.$$

This shows that if such a family exists, it is unique. Conversely, if we define d^p by the formula above we obtain a family of operators that, as it is easily seen, has the desired properties.

The last claim follows from

$$F^*(\mathrm{d}y_i) = \sum_j \frac{\partial F_i}{\partial x_j} \mathrm{d}x_j = \mathrm{d}(y_i \circ F) = \mathrm{d}(F^*(y_i))$$

and the fact that F^* is an algebra homomorphism. $\square$

2.10. Definition. The operator d is called the *de Rham differential* or the *exterior differential* or simply the *differential*. For reasons that will be clear later, d is also called the *coboundary* operator.

A simple but useful consequence of the properties above is the following

2.11. Corollary. d *is a local operator, i.e. if* $\omega \equiv \tau$ *in an open set* U*, then* $\mathrm{d}\omega = \mathrm{d}\tau$ *in* U.

Proof. The proof is essentially the same as the proof of the first claim in Lemma 3.19 of Chapter 0. $\square$

We can also give an alternative definition of the exterior differential that does not depend on coordinates.

2.12. Proposition. *Let* $\omega \in \Omega^p(U)$, $X_0, \ldots, X_p \in \mathcal{H}(U)$. *Then*

$$\mathrm{d}\omega(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i X_i \cdot \omega(X_0, \ldots, \hat{X}_i, \ldots X_p) + \sum_{i<j} (-i)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p),$$

where $[X_i, X_j]$ *is the Lie product of vector fields defined in Chapter 0.*

Proof. We sketch the proof leaving the details to the reader (Exercise 7.24). First observe that the right hand side of the equality above is $\mathcal{F}(U)$-multilinear, and so, by the tensoriality criterium (Theorem 2.3), it is a differential form. In particular, to compute $\mathrm{d}\omega(X_0, \ldots, X_p)$ at a given point $x_0 \in U$, we can take arbitrarily extensions of the $X_i(x_0)$. So, it will be enough to prove the equality for the case when the X_i's are coordinate vector fields. In this case $[X_i, X_j] = 0$ so the second term on the right hand side vanishes while the first term is just the expression of $\mathrm{d}\omega$ given in Theorem 2.9. $\square$

We have a sequence of vector spaces and $\mathbb{R}$-linear maps:

$$0 \longrightarrow \Omega^0(U) \xrightarrow{\mathrm{d}^0} \Omega^1(U) \xrightarrow{\mathrm{d}^1} \cdots \xrightarrow{\mathrm{d}^{n-1}} \Omega^n(U) \longrightarrow 0$$

which is a *cochain complex*, i.e. $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$, or, equivalently, $\mathrm{Im}\, \mathrm{d}^{p-1} \subseteq \ker\ \mathrm{d}^p$ (see next section for the definition and basic properties of cochain complexes). This sequence is called the *de Rham complex of* U. We define

- $Z^p(U) := \ker\ \mathrm{d}^p$, *the space of p-cocycles* or *closed p-forms.*
- $B^p(U) := \mathrm{Im}\ \mathrm{d}^{p-1}$, *the space p-coboundaries* or *exact p-forms.*
- $H^p(U) := Z^p(U)/B^p(U)$, *the p-dimensional (de Rham) cohomology of* U.

2.13. Remark. Let $\omega,\ \tau$ be closed forms in U. Since $\mathrm{d}(\omega\wedge\tau)=\mathrm{d}\omega\wedge\tau\pm\omega\wedge\mathrm{d}\tau$, $\omega\wedge\tau$ is closed. Moreover if $\tau=\mathrm{d}\beta$, $\omega\wedge\tau=\pm\mathrm{d}(\omega\wedge\beta)$, i.e. $\omega\wedge\tau$ is exact. In particular the wedge product induces a well defined bilinear map $\cup : H^p(U)\oplus H^q(U)\longrightarrow H^{p+q}(U)$, $[\omega]\cup[\tau]=[\omega\wedge\tau]$. This product, suitably extended, defines an algebra structure on $H^*(U):=\oplus H^p(U)$. With this structure $H^*(U)$ is called the *cohomology algebra* of U[6].

Let $U\subseteq\mathbb{R}^n$, $V\subseteq\mathbb{R}^m$ be open sets and $F:U\longrightarrow V$ a smooth function. As we already observed, F induces a map $F^*:\Omega^p(V)\longrightarrow\Omega^p(U)$. Since, by Theorem 2.9, $F^*\circ\mathrm{d}=\mathrm{d}\circ F^*$, F^* maps closed forms to closed forms and exact forms to exact forms. Hence it induces a $\mathbb{R}$-linear map, that we still denote by F^*,

$$F^*:H^p(V)\longrightarrow H^p(U).$$

It is also simple to see that F^* induces an *algebra homomorphism* $F^*:H^*(V)\longrightarrow H^*(U)$ (see Remark 2.13). The functorial properties

- $1\!\!1_U^*=1\!\!1_{H^p(U)}$,
- If $F_1:U_1\longrightarrow U_2$ and $F_2:U_2\longrightarrow U_3$ are smooth maps, then $(F_2\circ F_1)^*=F_1^*\circ F_2^*$

are also easily verified. In particular, if F is a diffeomorphism, F^* is an isomorphism. So the de Rham cohomology is a (differential) topological invariant of U.

3. Algebraic aspects of cohomology

The construction of the de Rham cohomology fits into a general algebraic setting called *homological algebra*. In this section we will discuss some elementary facts that will be used in these notes. For simplicity we will restrict to the case of real vector spaces (not necessarily finite dimensional) although most of the matter could be extended to the case of modules over commutative rings (see Remarks 3.10 and 3.22).

The objects we study are sequences of (real) vector spaces and linear maps of the type

$$\mathcal{E}:=\{(\mathbb{E}^p,\mathrm{d}^p):\mathrm{d}^p:\mathbb{E}^p\longrightarrow\mathbb{E}^{p+1}\}.$$

When we introduce "objects" it is a good strategy to introduce "morphisms" between such objects, i.e. maps that preserves the structure of the objects.

3.1. Definition. A *morphism* $\phi:\mathcal{E}\longrightarrow\mathcal{F}$, between two sequences is a sequence of linear maps $\phi^p:\mathbb{E}^p\longrightarrow\mathbb{F}^p$ such that the diagrams

$$\begin{array}{ccccc}\cdots\longrightarrow & \mathbb{E}^p & \xrightarrow{\mathrm{d}^p} & \mathbb{E}^{p+1} & \longrightarrow\cdots\\ & \downarrow\phi^p & & \downarrow\phi^{p+1} & \\ \cdots\longrightarrow & \mathbb{F}^p & \xrightarrow{\mathrm{d}^p} & \mathbb{F}^{p+1} & \longrightarrow\cdots\end{array}$$

commute, i.e. $\mathrm{d}^p\circ\phi^p=\phi^{p+1}\circ\mathrm{d}^p$ (we are using the same symbols d^p for the linear maps in the two sequences).

The morphism is an *isomorphism* if all ϕ^p are vector spaces isomorphisms.

We have some special sequences.

[6]This product is usually called the *cup product.* The use of this terminology, instead of the more natural wedge product, is due to the fact that the cup product can be defined for different cohomology theories, where the wedge product is not defined.

3.2. Definition. A sequence $\mathcal{E} = \{\mathbb{E}^p, \mathrm{d}^p\}$ is *exact at* $\mathbb{E}^p$ if $\operatorname{Im} \mathrm{d}^{p-1} = \ker \mathrm{d}^p$. The sequence is an *exact sequence* if it is exact at each $\mathbb{E}^p$.

3.3. Examples.

(1) A sequence of the type $\{0\} \longrightarrow \mathbb{E} \xrightarrow{\phi} \mathbb{F}$ is exact at $\mathbb{E}$ if and only if ϕ is injective.
(2) A sequence of the type $\mathbb{E} \xrightarrow{\phi} \mathbb{F} \longrightarrow \{0\}$ is exact at $\mathbb{F}$ if and only if ϕ is surjective.
(3) A sequence of the type $\{0\} \longrightarrow \mathbb{E} \xrightarrow{\phi} \mathbb{F} \longrightarrow \{0\}$ is exact if and only if ϕ is an isomorphism.

3.4. Definition. A sequence of the type:

$$\{0\} \longrightarrow \mathbb{E} \longrightarrow \mathbb{F} \longrightarrow \mathbb{G} \longrightarrow \{0\}$$

is called a short sequence.

3.5. Remark. Short (exact) sequences are important since they are the "building blocks" of long (exact) sequences. Let

$$\cdots \longrightarrow \mathbb{E}^{i-1} \xrightarrow{\phi_{i-1}} \mathbb{E}^i \xrightarrow{\phi_i} \mathbb{E}^{i+1} \longrightarrow \cdots$$

be a sequence. Consider the short sequence

$$\{0\} \longrightarrow \mathbb{E}^{i-1}/\ker \phi_{i-1} \xrightarrow{\tilde{\phi}_{i-1}} \mathbb{E}^i \xrightarrow{\tilde{\phi}_i} \operatorname{Im}(\phi_i) \longrightarrow \{0\}$$

where $\tilde{\phi}_{i-1}$, $\tilde{\phi}_i$ are the induced maps. Since $\operatorname{Im}(\tilde{\phi}_{i-1}) = \operatorname{Im}(\phi_{i-1})$, $\ker(\tilde{\phi}_i) = \ker(\phi_i)$, the long sequence is exact at $\mathbb{E}^i$ if and only if the short sequence is exact.

3.6. Proposition. *A short* exact *sequence*

$$\{0\} \longrightarrow \mathbb{E} \xrightarrow{\phi} \mathbb{F} \xrightarrow{\psi} \mathbb{G} \longrightarrow \{0\}$$

is isomorphic to the sequence

$$\{0\} \longrightarrow \mathbb{E} \xrightarrow{i} \mathbb{E} \oplus \mathbb{G} \xrightarrow{\pi} \mathbb{G} \longrightarrow \{0\},$$

where $i(v) = (v, 0)$ *and* $\pi(v, w) = w$.

Proof. Let $\tilde{\mathbb{G}}$ be a complement[7] of $\operatorname{Im} \phi = \ker \psi$, i.e $\mathbb{F} = \phi(\mathbb{E}) \oplus \tilde{\mathbb{G}}$. The map $\psi \mid_{\tilde{\mathbb{G}}} : \tilde{\mathbb{G}} \longrightarrow \mathbb{G}$ is an isomorphism. Therefore the map $k \colon \mathbb{F} \longrightarrow \mathbb{E} \oplus \mathbb{G}$, $k(v + w) = (\phi^{-1}(v), \psi(w))$ $(v \in \phi(\mathbb{E}), w \in \tilde{\mathbb{G}})$ is the required isomorphism. □

The following result appears often in the applications

3.7. Lemma. [The five Lemma] *Consider the diagram:*

$$\begin{array}{ccccccccc}
\mathbb{E}_1 & \xrightarrow{f_1} & \mathbb{E}_2 & \xrightarrow{f_2} & \mathbb{E}_3 & \xrightarrow{f_3} & \mathbb{E}_4 & \xrightarrow{f_4} & \mathbb{E}_5 \\
\downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & & \downarrow \phi_4 & & \downarrow \phi_5 \\
\mathbb{F}_1 & \xrightarrow{g_1} & \mathbb{F}_2 & \xrightarrow{g_2} & \mathbb{F}_3 & \xrightarrow{g_3} & \mathbb{F}_4 & \xrightarrow{g_4} & \mathbb{F}_5
\end{array}$$

If the squares commute, the lines are exact and the ϕ_i*'s are isomorphisms for* $i = 1, 2, 4, 5$ *then* ϕ_3 *is an isomorphism.*

[7] Recall that a complement of a subspace is obtained by starting from a basis $\{e_\alpha\}$ of the subspace and completing it to a basis of the ambient space with elements $\{f_\beta\}$ and then considering the subspace spanned by the $\{f_\beta\}$.

Proof. Suppose $\phi_3(e_3) = 0$. Then $\phi_4(f_3(e_3)) = g_3(\phi_3(e_3)) = 0$. Therefore $f_3(e_3) = 0$ and, by the exactness of the first line, $e_3 = f_2(e_2)$. Now $g_2(\phi_2(e_2)) = \phi_3(e_3) = 0$. Therefore $\phi_2(e_2) = g_1(\mu_1)$, for some $\mu_1 \in \mathbb{F}_1$, by the exactness of the second line. Since ϕ_1 is surjective, there exists $e_1 \in \mathbb{E}_1$ such that $\phi_1(e_1) = \mu_1$. Finally

$$0 = f_2(f_1(e_1)) = f_2(\phi_2^{-1} g_1 \phi_1(e_1)) = f_2(e_2) = e_3$$

and therefore ϕ_3 is injective. We will show now that ϕ_3 is surjective. Let $\mu_3 \in \mathbb{F}_3$, $\mu_4 = g_3(\mu_3)$ and $e_4 = \phi_4^{-1}(\mu_4)$. Now $\phi_5(f_4(e_4)) = g_4(\mu_4) = 0$ and therefore $f_4(e_4) = 0$, since ϕ_5 is injective. In particular there exists $e_3 \in \mathbb{E}_3$ such that $f_3(e_3) = e_4$. Let $\overline{\mu}_3 = \phi_3(e_3)$ and $\omega = \mu_3 - \overline{\mu}_3$. Now $g_3(\omega) = 0$ and therefore $\omega = g_2(\mu_2)$. Let $e_2 = \phi_2^{-1}(\mu_2)$. We have $\phi_3(f_2(e_2)) = g_2(\phi_2(e_2)) = \omega = \phi(e_3) - \mu_3$ and therefore $\mu_3 = \phi_3(e_3 - f_2(e_2)) \in \operatorname{Im} \phi_3$.

□

3.8. Remark. We observe that in the proof of Theorem 3.7 we use only that ϕ_2, ϕ_4 are isomorphisms, ϕ_1 is surjective and ϕ_5 is injective. However, the lemma is used, generally, as it is stated.

A more general and very important class of sequences is the class of cochain complexes.

3.9. Definition. A sequence $\mathcal{E} = \{\mathbb{E}^p, \mathrm{d}^p\}$ is *semiexact* or a *cochain complex* if $\operatorname{Im} \mathrm{d}^{p-1} \subseteq \ker \mathrm{d}^p$, $\forall p$. Equivalently, it is a cochain complex if $\mathrm{d}^p \circ \mathrm{d}^{p-1} = 0$.

If $\mathcal{E}$ is a cochain complex we define:

- $Z^p(\mathcal{E}) := \ker \mathrm{d}^p$, the *group of p-dimensional cocycles*,
- $B^p(\mathcal{E}) := \operatorname{Im} d^{p-1}$, the *group of p-dimensional coboundaries*,
- $H^p(\mathcal{E}) := Z^p(\mathcal{E})/B^p(\mathcal{E})$, the *$p$-dimensional cohomology group*.

3.10. Remark. Naturally $Z^p(\mathcal{E})$, $B^p(\mathcal{E})$, $H^p(\mathcal{E})$ are vector spaces. The use of the term "group" is due to the fact that they can be defined in the more general context of complexes of Abelian groups, or modules over a commutative ring.

The cohomology gives a measure of how much the complex is not an exact sequence.

3.11. Example. Let $U \subseteq \mathbb{R}^n$ be an open set. The de Rham complex

$$\cdots \longrightarrow \Omega^p(U) \xrightarrow{\mathrm{d}^p} \Omega^{(p+1)}(U) \longrightarrow \cdots$$

is a cochain complex whose cohomology is the de Rham cohomology $H^p(U)$.

Consider now a morphism between two cochain complexes, $\phi : \mathcal{E} \longrightarrow \mathcal{F}$. The commutativity condition implies that cocycles are sent to cocycles and coboundaries to coboudaries. In particular ϕ induces linear maps

$$\phi^{*,p} : H^p(\mathcal{E}) \longrightarrow H^p(\mathcal{F}).$$

When clear from the context we will write simply ϕ^* or ϕ^p.

The following "functorial" properties are easily verified:

- $\mathbb{1}^* = \mathbb{1}$,
- $(\phi \circ \psi)^* = \phi^* \circ \psi^*$.

In particular if ϕ is an isomorphism, ϕ^* is also an isomorphism.

It is convenient to consider also sequences with "decreasing indices", i.e. a sequence of the type

$$\mathcal{E} := \{(\mathbb{E}_p, \partial_p) : \partial_p : \mathbb{E}_p \longrightarrow \mathbb{E}_{p-1}\}.$$

If such a sequence is semiexact, we will call it a *chain complex.* For such a chain complex we define:

- $Z_p(\mathcal{E}) := \ker \partial_p$, the *group of p-dimensional cycles.*
- $B_p(\mathcal{E}) := \operatorname{Im} \partial_{p+1}$, the *group of p-dimensional boundaries.*
- $H_p(\mathcal{E}) := Z_p(\mathcal{E})/B_p(\mathcal{E})$, the *p-dimensional homology group.*

As in the case of cochains, a morphism $\phi : \mathcal{E} \longrightarrow \mathcal{F}$, between two chain complexes, sends cycles to cycles and boundaries to boundaries, so it induces a sequence of maps $\phi_{*,p} : H_p(\mathcal{E}) \longrightarrow H_p(\mathcal{F})$ and the functorial properties are easily verified. When clear from the context we will write simply ϕ_* or $\phi_{*,p}$.

3.12. Remark. Naturally chain and cochain complexes are, essentially, the same objects. For example, changing the index p by $-p$ we pass from a chain complex to a cochain complex. But a more interesting approach is *duality* and we will discuss this now.

Let $\mathcal{E} := \{(\mathbb{E}_p, \partial_p) : \partial_p : \mathbb{E}_p \longrightarrow \mathbb{E}_{p-1}\}$ be a chain complex. We define the *dual complex* $\mathcal{E}^* = \{(\mathbb{E}^p, \mathrm{d}^p)\}$ where $\mathbb{E}^p := (\mathbb{E}_p)^*$ is the dual space of $\mathbb{E}_p$ and $\mathrm{d}^p = (\partial_{p+1})^*$ is the transpose of ∂_{p+1}. It is simple to show that $\mathrm{d}^p \circ \mathrm{d}^{p-1} = 0$ so $\mathcal{E}^*$ is, in fact, a cochain complex. We will denote with H_p (resp. H^p) the homology of $\mathcal{E}$ (resp. the cohomology of $\mathcal{E}^*$). Consider the bi-linear map

$$b : \mathbb{E}^p \times \mathbb{E}_p \longrightarrow \mathbb{R}, \quad b(\phi, c) = \phi(c).$$

Since $(\mathrm{d}\phi)(c) = \phi(\partial c)$, we have that, if $\mathrm{d}\phi = 0,\ \partial c = 0,\ \ b(\phi + \mathrm{d}\tau, c + \partial d) = b(\phi, c)$. Hence b induces a bi-linear map

$$\tilde{b} : H^p \times H_p \longrightarrow \mathbb{R}, \quad \tilde{b}([\phi], [c]) = \phi(c),$$

and therefore a linear map

$$K : H^p \longrightarrow [H_p]^*, \quad K([\phi])([c]) = \phi(c).$$

3.13. Theorem. [Universal coefficient Theorem] *The map K is an isomorphism.*

Proof. We start observing that we have two short exact sequences

$$\{0\} \longrightarrow Z_p \longrightarrow \mathbb{E}_p \xrightarrow{\partial_p} B_{p-1} \longrightarrow \{0\}, \quad \{0\} \longrightarrow B_{p-1} \longrightarrow Z_{p-1} \longrightarrow H_{p-1} \longrightarrow \{0\} \tag{1}$$

where the maps are the obvious ones. By Proposition 3.6, we have the decompositions

$$\mathbb{E}_p \cong Z_p \oplus B_{p-1}, \quad Z_{p-1} \cong B_{p-1} \oplus H_{p-1} \tag{2}$$

Claim 1.: *K is surjective.* Let $[\phi] \in [H_p]^*$. Consider the map $\phi \circ \pi : Z_p \longrightarrow \mathbb{R}$, where $\pi : Z_p \longrightarrow H_p$ is the quotient map. Using the first decomposition in (2), we can extend this map to a map $\tilde{\phi} : \mathbb{E}_p \longrightarrow \mathbb{R}$ with $\tilde{\phi} = 0$ on B_{p-1}. Let $e \in \mathbb{E}_p$. Then $\mathrm{d}\tilde{\phi}(e) = \tilde{\phi}(\partial(e)) = 0$, hence $\tilde{\phi}$ is a cocycle and $K([\tilde{\phi}]) = [\phi]$.

<u>Claim 2.</u>: *K is injective.* Let $\psi \in Z^p$ be such that $\psi(c) = 0 \,\forall\, c \in Z_p$. The map $\phi = \psi \circ \partial^{-1} : B_{p-1} \longrightarrow \mathbb{R}$ is well defined since, by the first sequence in (1), the difference of two elements in $\partial^{-1}(B_{p-1})$ is a cycle. Using the decompositions in (2), we can extend ϕ to a map $\tilde{\phi} : E_{p-1} \longrightarrow \mathbb{R}$. Now, $\forall e \in E_p$, we have:

$$\mathrm{d}\tilde{\phi}(e) = \tilde{\phi}(\partial e) = \psi \circ \partial^{-1}(\partial e) = \psi(e).$$

Hence $[\psi] = [\mathrm{d}\tilde{\phi}] = 0$. □

A useful consequence is the following

3.14. Corollary. *If a sequence is exact, the dual sequence is also exact.*

Proof. An exact sequence is a chain complex with vanishing homology. Therefore the dual sequence is a cochain complex with vanishing cohomology, by Theorem 3.13, hence an exact sequence [8]. □

We will study now when two morphism between cochain (resp. chain) complexes induces the same map in cohomology (resp. homology).

3.15. Definition. An *algebraic homotopy* between two morphisms $\phi, \psi : \mathcal{E} \longrightarrow \mathcal{F}$ of cochain (resp. chain) complexes is a family of maps $K^p : \mathbb{E}^p \longrightarrow \mathbb{F}^{p-1}$ (resp. $K_p : \mathbb{E}_p \longrightarrow \mathbb{F}_{p+1}$), such that:

$$\phi - \psi = \mathrm{d} \circ K + K \circ \mathrm{d} \quad (\text{resp. } \; \phi - \psi = \partial \circ K + K \circ \partial).$$

If there exists such an algebraic homotopy, we will say the the two morphisms are (algebraically) *homotopic*.

From the very definition of induced morphisms we have:

3.16. Proposition. *Two algebraically homotopic maps induce the same morphism in cohomology (resp. in homology).*

Consider now a short exact sequence of cochain complexes:

$$\{0\} \longrightarrow \mathcal{E} \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{G} \longrightarrow \{0\}.$$

In particular ϕ_i is injective and ψ_i is surjective. In general, at the cohomology level, ϕ^* is not injective and ψ^* is not surjective. In any case, we still have a good relation between the cohomology groups of the three complexes.

3.17. Theorem. [Algebraic Mayer-Vietoris Theorem] *In the situation above there exists a family of linear maps $\Delta^{*,p} : H^p(\mathcal{G}) \longrightarrow H^{p+1}(\mathcal{E})$ such that the sequence:*

$$\cdots \longrightarrow H^p(\mathcal{E}) \xrightarrow{\phi^*} H^p(\mathcal{F}) \xrightarrow{\psi^*} H^p(\mathcal{G}) \xrightarrow{\Delta^{*,p}} H^{p+1}(\mathcal{E}) \longrightarrow \cdots$$

is a (long) exact sequence. When clear from the context we will write simply Δ^p or Δ^*.

[8] We could also give a direct proof, and the reader is invited to do so (Exercise 7.18).

Proof. We have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathbb{E}^p \arrow[r, "\mathrm{d}^p"] \arrow[d, "\phi_p"] & \mathbb{E}^{p+1} \arrow[r, "\mathrm{d}^{p+1}"] \arrow[d, "\phi_{p+1}"] & \mathbb{E}^{p+2} \arrow[r] \arrow[d, "\phi_{p+2}"] & \cdots \\
\cdots \arrow[r] & \mathbb{F}^p \arrow[r, "\mathrm{d}^p"] \arrow[d, "\psi_p"] & \mathbb{F}^{p+1} \arrow[r, "\mathrm{d}^{p+1}"] \arrow[d, "\psi_{p+1}"] & \mathbb{F}^{p+2} \arrow[r] \arrow[d, "\psi_{p+2}"] & \cdots \\
\cdots \arrow[r] & \mathbb{G}^p \arrow[r, "\mathrm{d}^p"] \arrow[d] & \mathbb{G}^{p+1} \arrow[r, "\mathrm{d}^{p+1}"] \arrow[d] & \mathbb{G}^{p+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the columns are exact and the rows are the cochain complexes under consideration. The idea is to construct a map from $\mathbb{G}^p$ to $\mathbb{E}^{p+1}$. A natural choice would be $(\phi_{p+1})^{-1} \circ \mathrm{d}^p \circ \psi_p^{-1}$. The point is that this map is not well defined. Let us see how we can overcome this problem. Consider a cocycle $c \in \mathbb{G}^p$. Since ψ_p is surjective, there exists $b \in \mathbb{F}^p$ such that $c = \psi_p(b)$. The element $\mathrm{d}^p(b) \in \mathbb{F}^{p+1}$ is in $\ker \psi_{p+1}$ since the diagrams commute and c is a cocycle. Since $\ker \psi_{p+1} = \operatorname{Im} \phi_{p+1}$ we have $\mathrm{d}^p(b) = \phi_{p+1}(a)$ for some $a \in \mathbb{E}^{p+1}$ and this a is unique since ϕ_{p+1} is injective. Observe that $\mathrm{d}^{p+1}(a) = 0$, since $\phi_{p+2}(d^{p+1}(a)) = \mathrm{d}^{p+1}(\phi_{p+1}(a)) = \mathrm{d}^{p+1} \circ \mathrm{d}^p(b) = 0$ and ϕ_{i+2} is injective. Therefore a is a cocycle. We define: $\Delta^*\colon H^p(\mathcal{G}) \longrightarrow H^{p+1}(\mathcal{E})$, $\Delta^*([c]) = [a]$. We have to show that $[a]$ is well defined. The first choice we made was $b \in \mathbb{F}^p$. If b' is an other choice, i.e. $\psi_p(b') = \psi_p(b)$, then $b - b' \in \ker \psi_p = \operatorname{Im} \phi_p$. Therefore $b' - b = \phi_p(a')$, for some $a' \in \mathbb{E}^p$, and $b' = b + \phi_p(a')$. So, changing b by $b + \phi_p(a')$, we change a by $a + \mathrm{d}^p(a')$ and this does not change $[a]$. Next we shall show that $[a]$ does not depend on the choice of $c \in [c]$. Consider $c + \mathrm{d}^p(c')$. Since $c' = \psi_{p-1}(\tilde{b})$, for some $\tilde{b} \in \mathbb{F}^{p-1}$, we have $c + \mathrm{d}^{p-1}(c') = c + \mathrm{d}^{p-1}(\psi_{p-1}(\tilde{b})) = c + \psi_p(\mathrm{d}^{p-1}(\tilde{b})) = \psi_p(b + \mathrm{d}^{p-1}(\tilde{b}))$. Therefore b is replaced by $b + \mathrm{d}^{p-1}(\tilde{b})$ and this does not change $\mathrm{d}^p(b)$ and, therefore, $[a]$.

It is easy to see that Δ^* is linear. We leave to the reader the task of proving the exactness of the sequence (Exercise 7.22). □

3.18. Remark. The map Δ^* is well defined in cohomology but *not* at the cocycles level.

3.19. Definition. The sequence in Theorem 3.17 is called the (algebraic) *Mayer-Vietoris sequence*. The maps Δ^* are called the *Mayer-Vietoris coboundaries*[9].

3.20. Remark. Naturally we have a similar sequence in homology, associated to a short exact sequence of chain complexes. The similar maps $\Delta_{*,p}$ or simply Δ_*, are called the *Mayer-Vietoris boundaries*. We leave the details to the reader.

An important aspect of the Mayer-Vietoris (co)boundaries is that they are "natural" in the sense of the following Proposition, whose proof we leave to the reader (Exercise 7.22).

[9]The name "coboundaries" cames from the fact that, in the case of the de Rham cohomology, they are, essentially, the coboundaries operators d^p (see Remark 4.14).

3.21. Proposition. *A morphism between short exact sequences of (co)chain complexes induces a morphism between the associated Mayer-Vietoris exact sequences, i.e. the Mayer-Vietoris (co)boundaries commute with the induced maps.*

3.22. Remark. As suggested in Remark 3.10, instead of chain and cochain complexes of vector spaces we could consider chain and cochain complexes of Abelian groups (or modules over a commutative ring). Almost all we have done in this section extends to the case of complexes of Abelian groups. The "almost" refers to two exceptions:

• Proposition 3.6 does not hold in this more general setting. For example the sequence of abelian groups

$$\{0\} \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow \{0\}, \quad \cdot 2(a) := 2a,$$

is a short exact sequence, but it is *not* isomorphic to the sequence

$$\{0\} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow \{0\}.$$

A short exact sequence of Abelian groups that verify Proposition 3.6 is called a *split short exact sequence.* A sufficient condition for splitting is given by the following simple fact

3.23. Proposition. *A short exact sequence of Abelian groups*

$$\{0\} \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \{0\}$$

splits if and only if there is a map $r : C \longrightarrow B$ such that $\psi \circ r = \mathbb{1}_C$. This always happens if C is free [10].

• We can consider "duality" in the context Abelian groups. If G is such a group, $G^* := Hom(G, \mathbb{Z})$ is the group of homomorphisms from G to $\mathbb{Z}$. Therefore we can define the dual of a chain complex of Abelian groups. However Theorem 3.13 does not hold in this context. In fact, one of the points in the proof was that the sequence of vector spaces

$$\{0\} \longrightarrow B_{p-1} \longrightarrow \mathbb{Z}_{p-1} \longrightarrow H_{p-1} \longrightarrow \{0\}$$

splits. As observed above, this is not the case, in general, for short exact sequences of Abelian groups. However, if H_{p-1} is a free Abelian group, then the sequence splits, by Proposition 3.23, and the Theorem is true. In general, there is still a relation between the homology of a chain complex of Abelian groups and the cohomology of the dual complex, still known as *the Universal Coefficient Theorem.*

4. Basic properties of the de Rham cohomology

The natural problem that the de Rham cohomology treats is the problem of (indefinite) integration, i.e. the problem of solving the equation $d\omega = \beta$, for a given $\beta \in \Omega^{p+1}(U)$. A necessary condition for the existence of a solution ω is $d\beta = 0$. In general the problem has two aspects:

• *The local problem*: given $x \in U$, $\beta \in \Omega^{p+1}(U)$ does there exist a neighborhood $V \subseteq U$ of x and a solution $\omega \in \Omega^p(V)$ of the equation $d\omega = \beta|V$? In this case, as we shall see, the condition $d\beta = 0$ is also sufficient.

[10] A *free Abelian group* G is an Abelian group that admits a basis, i.e. a subset $\mathcal{B} \subseteq G$ such that for any Abelian group H and map $\phi : \mathcal{B} \longrightarrow H$, there exists a homomorphism $\tilde{\phi} : G \longrightarrow H$, extending ϕ.

• *The global problem*: given $\beta \in \Omega^{p+1}(U)$, does there exist a solution $\omega \in \Omega^p(U)$ of the equation $\mathrm{d}\omega = \beta$? In this case, the condition $\mathrm{d}\beta = 0$ is no longer sufficient, in general, and the answer will depend on the particular β and/or the *topology* of U.

We will start computing the de Rham cohomology in some simple cases.

4.1. Example. For $U = \mathbb{R}^0$ we have the obvious fact

$$H^p(\mathbb{R}^0) \simeq \begin{cases} \mathbb{R} & \text{if } p = 0 \\ \{0\} & \text{if } p > 0 \end{cases}$$

4.2. Example. Let $U = \coprod_\alpha U_\alpha$ be the union of disjoint open sets U_α. Then $\Omega^p(U) = \prod_\alpha \Omega^p(U_\alpha)$ (direct product) and the differential preserves the decomposition, i.e. if $\omega = \{\omega_\alpha\}$, $\mathrm{d}\omega = \{\mathrm{d}\omega_\alpha\}$. It follows that

$$H^p(U) \cong \prod_\alpha H^p(U_\alpha).$$

4.3. Example. Let us analyze the 0-dimensional cohomology. In this case, the only exact 0-form is the zero form so $H^0(U)$ is the space of closed 0-forms, i.e. functions in $\mathcal{F}(U)$ with zero differential. Such a function is locally constant, in particular it is constant on the connected components of U. It follows that $H^0(U)$ is the direct product of copies of $\mathbb{R}$, as many as the connected components of U.

Let us take a further look at the 0-dimensional cohomology. Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open *connected* sets, and $F : U \longrightarrow V$ a smooth map. As we observe in 4.3, the zero dimensional cohomology of U is the space of constant functions, and the same for V. Given a 0-form $f \in \Omega^0(V) = \mathcal{F}(V)$, $F^*(f) = f \circ F$ and therefore $F^* : H^0(V) \longrightarrow H^0(U)$ is an isomorphism. Modulo the identification of the zero dimensional cohomology groups with $\mathbb{R}$, we have $F^* = \mathbb{1} : \mathbb{R} \longrightarrow \mathbb{R}$.

We want to look now at the induced maps in higher dimensional cohomology groups. The question is the following: when do two smooth maps $F_i : U \longrightarrow V$, $i = 0, 1$ induce the same morphism in cohomology?

We will give a sufficient condition in terms of *homotopy*.

4.4. Definition. Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open sets and $F_i : U \longrightarrow V$, $i = 0, 1$ be smooth functions.

- A *homotopy* between the two functions is a smooth map[11]

$$H : U \times [0, 1] \subseteq \mathbb{R}^{n+1} \longrightarrow V,$$

  such that $H(x, i) = F_i(x), i = 0, 1$.
- We will say that the two functions are *homotopic* if there exist a homotopy between them. In this case we write $F_0 \sim F_1$.
- We will say that U and V are *homotopy equivalent* if there exist functions $F : U \longrightarrow V$, $G : V \longrightarrow U$, such that $G \circ F \sim \mathbb{1}_U$, $F \circ G \sim \mathbb{1}_V$. F (resp. G) is called a homotopy inverse of G (resp. of F)[12].
- We will say that U is *contractible* if U is homotopy equivalent to $\mathbb{R}^0$.

[11]A map $f : V \subseteq \mathbb{R}^N \longrightarrow \mathbb{R}^M$, defined in a *non necessarily open subset* $V \subseteq \mathbb{R}^N$ is smooth, if for all $p \in V$, f extends to a smooth map defined in an open neighborhood of p.

[12]Observe that a homotopy inverse *is not*, in general, unique.

4.5. Example. A subset $U \subseteq \mathbb{R}^n$ is star shaped if there exists $p \in U$ such that, for all $q \in U$, the segment joining p and q is contained in U. For example convex sets are star shaped. Star shaped subsets are contractible since the map $H(q,t) := tp + (1-t)q$ is a homotopy between $\mathbb{1}_U$ and the constant map $F(q) = p$. It follows that $\mathbb{1}$ and F are homotopy inverses.

4.6. Remark. Given a homotopy $H : U \times [0,1] \longrightarrow V$, there is a smooth function $\overline{H} : U \times \mathbb{R} \longrightarrow V$, such that $\overline{H}(x,i) = F_i(x), i = 0,1$. In fact, if $\lambda : \mathbb{R} \longrightarrow [0,1]$ is a smooth function such that $\lambda(t) = 0$ if $t \leq 0$, $\lambda(t) = 1$ if $t \geq 1$, just take $\overline{H}(x,t) = H(x,\lambda(t))$.

A homotopy between two functions may be viewed as a curve in the space of smooth maps joining the two functions. Also it may be viewed as a "smooth deformation" of one function to the other.

4.7. Theorem. [Homotopy invariance for cohomology] *If $F_i : U \longrightarrow V, i = 0,1$ are two homotopic smooth functions, then $F_0^* = F_1^* : H^p(V) \longrightarrow H^p(U)$, for all p.*

Proof. By Remark 4.6 we can suppose that there is a homotopy $H : U \times \mathbb{R} \longrightarrow V$. Let $j_i : U \longrightarrow U \times \mathbb{R}$, $i = 0,1$, $j_i(x) = (x,i)$, be the canonical inclusions. We claim that it is sufficient to prove that $j_0^* = j_1^*$. In fact, if so, we have:

$$F_0^* = (H \circ j_0)^* = j_0^* \circ H^* = j_1^* \circ H^* = (H \circ j_1)^* = F_1^*.$$

To prove that $j_0^* = j_1^*$ we will construct an algebraic homotopy between j_0^* and j_1^* (at the cochain level, see Definition 3.15 and Proposition 3.16), i.e. an $\mathbb{R}$-linear map $\tilde{H} : \Omega^p(U \times \mathbb{R}) \longrightarrow \Omega^{p-1}(U)$ such that

$$\tilde{H}\mathrm{d}\omega + \mathrm{d}\tilde{H}\omega = j_1^*\omega - j_0^*\omega.$$

Let us construct such a map. If $\omega \in \Omega^p(U \times \mathbb{R})$, $\omega = \mathrm{d}t \wedge \alpha + \beta$, with

$$\alpha = \sum_{i_1 < \ldots < i_{p-1}} \alpha_{i_1,\ldots,i_{p-1}}(x,t)\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}}, \quad \beta = \sum_{j_1 < \cdots < j_p} \beta_{j_1,\ldots,j_p}(x,t)\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p}.$$

We define

$$\tilde{H}(\omega) = \sum_{i_1 < \ldots < i_{p-1}} \left(\int_0^1 \alpha_{i_1,\ldots,i_{p-1}}(x,t)\mathrm{d}t \right) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}}.$$

Then

$$\mathrm{d}\omega = -\mathrm{d}t \wedge \mathrm{d}\alpha + \mathrm{d}\beta = -\mathrm{d}t \wedge \sum_{j,i_1 < \cdots < i_p} \frac{\partial \alpha_{i_1 \ldots i_{p-1}}}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}} +$$

$$+\mathrm{d}t \wedge \sum_{j_1 < \cdots < j_p} \frac{\partial \beta_{j_1,\ldots,j_p}}{\partial t} \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p} + \gamma$$

where γ does not contain terms with $\mathrm{d}t$. Therefore

$$\tilde{H}\mathrm{d}\omega = \sum_{j_1 < \cdots < j_p} \left(\int_0^1 \frac{\partial \beta_{j_1,\ldots,j_p}}{\partial t} \mathrm{d}t \right) \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p} \quad -$$

$$\sum_{j,i_1 < \cdots < i_p} \left(\int_0^1 \frac{\partial \alpha_{i_1 \ldots i_{p-1}}}{\partial x_j} \quad \mathrm{d}t \quad \right) \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}},$$

$$\mathrm{d}\tilde{H}\omega = \sum_{j,i_1 < \cdots < i_p} \left(\int_0^1 \frac{\partial \alpha_{i_1 \ldots i_{p-1}}}{\partial x_j} \mathrm{d}t \right) \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}},$$

and (see Example 2.8)

$$\tilde{H}\mathrm{d}\omega + \mathrm{d}\tilde{H}\omega = \sum_{j_1<\cdots<j_p} \left(\int_0^1 \frac{\partial \beta_{j_1,\ldots,j_p}}{\partial t}\, \mathrm{d}t \right) \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p} =$$

$$= \sum_{j_1<\cdots<j_p} [\beta_{j_1,\ldots,j_p}(x,1) - \beta_{j_1,\ldots,j_p}(x,0)]\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p} = j_1^*\omega - j_0^*\omega.$$

□

From 4.7, and the funtorial properties, we have

4.8. Corollary. *If* $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ *are homotopically equivalent open sets, then they have isomorphic cohomology.*

In particular we have the so called *Poincaré Lemma*

4.9. Corollary. [Poincaré Lemma] *If* U *is a contractible open set in* $\mathbb{R}^n$, $H^p(U) = \{0\}$ *if* $p \geq 1$.

4.10. Remark. Theorem 4.7 allows to define the map induced in cohomology by a *continuous map*. In fact, as we shall see in the Appendix, a continuous map $F : U \longrightarrow V$ is homotopic, via a continuous homotopy $H : U \times [0,1] \longrightarrow V$, to a smooth map $\tilde{F} : U \longrightarrow V$ and if there is a continuous homotopy between two smooth maps, there is a smooth one. So $F^* := \tilde{F}^*$ is well defined and invariant by continuous homotopies.

A basic method to compute the cohomology of an open set $U \subseteq \mathbb{R}^n$ is to write U as union of two, possibly simpler open sets U_1, U_2, and look for relations between the cohomology of U, U_i and $V := U_1 \cap U_2$.

4.11. Lemma. *Consider the sequence :*

$$\{0\} \longrightarrow \Omega^p(U) \overset{(j_1^*, j_2^*)}{\longrightarrow} \Omega^p(U_1) \oplus \Omega^p(U_2) \overset{(k_1^* - k_2^*)}{\longrightarrow} \Omega^p(V) \longrightarrow \{0\},$$

where $j_i : U_i \longrightarrow U$ *and* $k_i : V \longrightarrow U_i$ *are the inclusions. Then the sequence is a short exact sequence of cochain complexes.*

Proof. Observe that $j_i^*\omega = \omega|_{U_i}$ and, if $(\omega_1, \omega_2) \in \Omega^p(U_1) \oplus \Omega^p(U_2)$, $(k_1^* - k_2^*)(\omega_1, \omega_2) = \omega_1|_V - \omega_2|_V$ (see Example 2.8). So the exactness of the sequence is obvious, except for the surjectivity of $(k_1^* - k_2^*)$. To prove that $(k_1^* - k_2^*)$ is surjective we consider a partition of unity dominated by the covering $\{U_1, U_2\}$, i.e. smooth functions $\phi_i : U \longrightarrow [0,1]$, $i = 1,2$ such that:

$$\phi_1(x) + \phi_2(x) = 1 \quad \forall x \in U, \qquad \mathrm{supp}(\phi_i) := \overline{\{x \in U : \phi_i(x) > 0\}} \subseteq U_i$$

(see Theorem 6.2 for a proof of the existence of partitions of unity).

Given $\omega \in \Omega^p(V)$, we define:

$$\omega_i(x) = \begin{cases} \phi_j(x)\omega(x) & \text{if } x \in V \\ 0 & \text{if } x \in U_i \setminus V \end{cases}$$

where $i \neq j$. Then ω_i is well defined in U_i since ϕ_j vanishes outside $\overline{U_j}, j \neq i$. Moreover,

$$(k_1^* - k_2^*)(\omega_1, -\omega_2) = \omega_1|_V + \omega_2|_V = \phi_2\omega + \phi_1\omega = \omega.$$

Therefore $(k_1^* - k_2^*)$ is surjective. $\square$

At this point Theorem 3.17 gives:

4.12. Theorem. [Mayer Vietoris sequence for de Rham cohomology] *There exists a sequence of linear maps* $\Delta^* : H^p(V) \longrightarrow H^{p+1}(U)$, *such that the sequence below is exact:*

$$\cdots \longrightarrow H^p(U) \xrightarrow{(j_1^*, j_2^*)} H^p(U_1) \oplus H^p(U_2) \xrightarrow{(k_1^* - k_2^*)} H^p(V) \xrightarrow{\Delta^*} H^{p+1}(U) \longrightarrow \cdots$$

4.13. Definition. The sequence above is called the *Mayer-Vietoris sequence for the de Rham cohomology* and the maps Δ^* are called the *Mayer-Vietories coboundaries.*

4.14. Remark. The Mayer-Vietoris coboundaries can be decribed explicitly. If $[\omega] \in H^p(V)$, $\Delta^*[\omega]$ is the class of the form

$$\tau(x) = \begin{cases} -\mathrm{d}(\phi_2\omega)(x) & \text{if } x \in U_1 \\ \mathrm{d}(\phi_1\omega)(x) & \text{if } x \in U_2 \end{cases}$$

Since d commutes with induced maps, so does Δ^*. We invite the reader to check the details.

4.15. Example. Let us apply the Mayer-Vietoris sequence to compute the cohomology of $\Sigma_n := \mathbb{R}^n \backslash \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x_i| \le 1\}$.

Consider the open sets:

$$U_1 = \{(x_1, \dots, x_n) \in \Sigma_n : x_n > -1/2\}, \quad U_2 = \{(x_1, \dots, x_n) \in \Sigma_n : x_n < 1/2\}.$$

The following facts are easy to prove:

- $\Sigma_n = U_1 \cup U_2$.
- The U_i's are contractible. In fact the projection $(x_1, \dots, x_n) \rightsquigarrow (x_1, \dots, x_{n-1}, 2)$ is a homotopy equivalence between U_1 and the hyperplane $x_n = 2$. The latter is contractible since it is convex. Similarly for U_2.
- $U_1 \cap U_2$ is homotopy equivalent to $\Sigma_{(n-1)}$ (the projection of $U_1 \cap U_2$ into the hyperplane $x_n = 0$ is a homotopy equivalence).

If $n = 1$, Σ_1 is the disjoint union of two contractible sets, hence by Corollary 4.8 and Example 4.3

$$H^p(\Sigma_1) \cong \begin{cases} \mathbb{R} \oplus \mathbb{R} & \text{if } p = 0 \\ \{0\} & \text{if } p > 0 \end{cases}$$

For the case $n \ge 2$ we will prove that

$$H^p(\Sigma_n) = \begin{cases} \mathbb{R} & \text{if } p = 0, n-1 \\ \{0\} & \text{if } p \ne 0, n-1 \end{cases}$$

We proceed by induction. Let $n = 2$. Since Σ_2 and the U_i's are connected, $H^0(\Sigma_2) \cong H^0(U_i) \cong \mathbb{R}$. Consider the Mayer-Vietoris sequence:

$$\begin{aligned} &\{0\} \longrightarrow H^0(\Sigma_2) \longrightarrow H^0(U_1) \oplus H^0(U_2) \longrightarrow H^0(\Sigma_1) \longrightarrow H^1(\Sigma_2) \longrightarrow H^1(U_1) \oplus H^1(U_2) \longrightarrow \\ &\longrightarrow \cdots \longrightarrow H^{p-1}(\Sigma_1) \longrightarrow H^p(\Sigma_2) \longrightarrow H^p(U_1) \oplus H^p(U_2) \longrightarrow \cdots . \end{aligned}$$

The first row reduces to:

$$\{0\} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow H^1(\Sigma_2) \longrightarrow \{0\}.$$

The first arrow is injective hence the kernel of the second one, as well as its image, are 1-dimensional. Hence the the third one is surjective with 1-dimensional kernel and $H^1(\Sigma_2) \cong \mathbb{R}$.

From the second row we get $H^p(\Sigma_2) = \{0\}$ if $p > 1$. Hence the formula is true for $n = 2$.

Suppose now $n \geq 3$ and that the formula is true for $n-1$. Consider again the Mayer-Vietoris sequence:

$$H^{p-1}(\Sigma_n) \longrightarrow H^{p-1}(U_1) \oplus H^{p-1}(U_2) \longrightarrow H^{p-1}(\Sigma_{n-1}) \longrightarrow H^p(\Sigma_n) \longrightarrow H^p(U_1) \oplus H^p(U_2) \longrightarrow$$

If $p > 1$ we have $H^p(\Sigma_n) \cong H^{p-1}(\Sigma_{n-1})$, and, for $p = 1$ we get

$$\{0\} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow H^1(\Sigma_n) \longrightarrow \{0\}.$$

Hence $H^1(\Sigma_n) = \{0\}$ and the formula holds true for n.

4.16. Remark. Observe that the inclusion $\Sigma_n \longrightarrow \mathbb{R}^n \setminus \{0\}$ is a homotopy equivalence (Exercise 7.27).

5. An application: the Jordan-Alexander duality Theorem

It is convenient, as we shall see, in order to avoid special arguments for the 0-dimensional case and to have more clean statements, to introduce *reduced cohomology*. Define

$$\Omega^{-1}(U) := \mathbb{R} \qquad \mathrm{d}^{(-1)} : \Omega^{-1}(U) \longrightarrow \Omega^0(U), \qquad \mathrm{d}^{(-1)}(a) := a \in \Omega^0(U).$$

Then the sequence

$$\{0\} \longrightarrow \Omega^{-1}(U) \xrightarrow{d^{(-1)}} \Omega^0(U) \xrightarrow{d} \Omega^1(U) \longrightarrow \cdots$$

is a cochain compex called the *augmented de Rham complex*.

5.1. Definition. The *reduced de Rham cohomology* of U, $\tilde{H}^p(U)$, is the cohomology of the augmented de Rham complex.

5.2. Remark. It is clear that $\tilde{H}^{-1}(U) = \{0\}$, $H^0(U) \cong \tilde{H}^0(U) \oplus \mathbb{R}$ and $\tilde{H}^p(U) = H^p(U)$, if $p > 0$. In particular $\tilde{H}^p(U) = \{0\}$, $\forall \;\; p \geq 0$, if U is contractible.

The basic properties, such as homotopy invariance and the Mayer-Vietoris exact sequence, continue to be true for the reduced cohomology and we will leave the proofs to the reader (see Exercise 7.23).

We will discuss now a nice application of the Mayer-Vietoris argument, the so called *Jordan-Alexander duality principle*, that has, as a simple consequence, the celebrated Jordan closed curve Theorem. We will follow closely [**4**].

Let $F_i, \;\; i = 1, 2$ be closed subsets of $\mathbb{R}^n$. Suppose that there exists a homeomorphism $\phi : F_1 \longrightarrow F_2$. It is natural to ask if there exists some relation between the complementary sets $\mathbb{R}^n \setminus F_i$. The illusion that they are homeomorphic or, at least, homotopy equivalent is soon frustrated. For example consider $F_1 = \{x \in \mathbb{R}^2 : \|x\| = 1\} \cup \{x \in \mathbb{R}^2 : \|x\| = 2\}$ and $F_2 = \{x \in \mathbb{R}^2 : \|x\| = 1\} \cup \{x \in \mathbb{R}^2 : \|x - (3, 0)\| = 1\}$. The complement of F_1 is homotopy equivalent to the disjoint union of a point and two circles, while the complement of F_2 is homotopy equivalent to the disjoint union of two points and the wedge [13] of two circles. It is easily seen that these spaces *are not* homotopy equivalent.

[13] Recall that the wedge of two topological spaces is the space obtained from the disjoint union identifying a fixed point in the first space with one in the second one.

5.3. Remark. (For the reader familiar with the concept of foundamental group,) The fact that the complements of two homeomorphic closed set are not homotopy equivalent is important in several contexts, for example in Knot Theory. Recall that a knot in $\mathbb{R}^3$ is a function $\gamma : S^1 \longrightarrow \mathbb{R}^3$ which is a homeomorphism onto its image. Two knots are equivalent if there exists an isotopy, i.e. a homotopy through homeomorphisms, which takes one into the other. One of the most important invariants for equivalence classes of knots is the fundamental group of the complement of the image. Now, the images of two knots are homeomorphic and if the complements were homotopy equivalent, they would have isomorphic fundamental group and so the invariant would be trivial.

There is, however, an interesting relation between the complements of homeomorphic closed sets.

5.4. Theorem. [Jordan Alexander duality Theorem] *Let $F_1, F_2 \subseteq \mathbb{R}^n$, be homeomorphic closed sets. Then*

$$\tilde{H}^k(\mathbb{R}^n \setminus F_1) \cong \tilde{H}^k(\mathbb{R}^n \setminus F_2).$$

Proof. We will consider $\mathbb{R}^n$ as the subspace of vectors in $\mathbb{R}^{n+k}$ with the last k coordinates zero. The proof of the Theorem will be an easy consequence of the following two Lemmas.

5.5. Lemma. *Let $F \subsetneq \mathbb{R}^n$ be a closed subset. Then $\tilde{H}^{i+1}(\mathbb{R}^{n+1} \setminus F) \cong \tilde{H}^i(\mathbb{R}^n \setminus F), \quad i \geq -1$.*

Proof. Consider the subsets of $\mathbb{R}^{n+1}$:

- $Z_+ := \mathbb{R}^{n+1} \setminus (F \times \{t \in \mathbb{R} : t \leq 0\})$.
- $Z_- := \mathbb{R}^{n+1} \setminus (F \times \{t \in \mathbb{R} : t \geq 0\})$.
- $Z := Z_+ \cup Z_- = \mathbb{R}^{n+1} \setminus F$.
- $Z_+ \cap Z_- \sim \mathbb{R}^n \setminus F$.

The orthogonal projection of Z_+ onto the hyperplane $x_{n+1} = 1$ is a homotopy equivalence. Hence the reduced cohomology of Z_+ vanishes in all dimensions. The same is true for Z_- and the Lemma follows from the Mayer-Vietoris sequence for the reduced cohomology:

$$\tilde{H}^i(Z_+) \oplus \tilde{H}^i(Z_-) = \{0\} \longrightarrow \tilde{H}^i(Z_+ \cap Z_-) \longrightarrow \tilde{H}^{i+1}(Z) \longrightarrow \tilde{H}^{i+1}(Z_+) \oplus \tilde{H}^{i+1}(Z_-) = \{0\}.$$

□

5.6. Corollary. *If $F \subseteq \mathbb{R}^n$ is a closed set, then $\tilde{H}^{i+k}(\mathbb{R}^{n+k} \setminus F) \cong \tilde{H}^i(\mathbb{R}^n \setminus F), \quad \forall\, i \geq -k$.*

5.7. Lemma. *Let $F_i \subseteq \mathbb{R}^n, i = 1, 2$ be closed subsets and $\phi : F_1 \longrightarrow F_2$ a homeomorphism. Then $\mathbb{R}^{2n} \setminus F_1 \times \{0\}$ is homeomorphic to $\mathbb{R}^{2n} \setminus \{0\} \times F_2$.*

Proof. Let $\psi = \phi^{-1}$. The homeomorphisms ϕ, ψ extend, by Tietze's Theorem[14], to continuous maps $\Phi, \Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Define:

- $L : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n},\ L(x, y) = (x, y - \Phi(x))$.
- $R : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n},\ R(x, y) = (x - \Psi(y), y)$.

[14] Tietze's Theorem states that a continuous real valued function defined in a closed subset of $\mathbb{R}^n$ extends to a continuous function defined in the all of $\mathbb{R}^n$ (this fact is true, more generally, for normal topological spaces).

The maps L, R are homeomorphisms. In fact $L^{-1}(x,y) = (x, y + \Phi(x)), R^{-1}(x,y) = (x + \Psi(y), y)$. Consider $\Gamma := \{(x,y) \in \mathbb{R}^{2n} : x \in F_1,\ y = \phi(x)\} = \{(x,y) \in \mathbb{R}^{2n} : y \in F_2,\ x = \psi(y)\}$. We have $L(F_1 \times \{0\}) = \Gamma = R(\{0\} \times F_2)$ and therefore a homeomorphism:

$$\mathbb{R}^{2n} \setminus (F_1 \times \{0\}) \xrightarrow{L} \mathbb{R}^{2n} \setminus \Gamma \xrightarrow{R^{-1}} \mathbb{R}^{2n} \setminus (\{0\} \times F_2).$$

□

The proof of the Theorem is, at this point, immediate:

$$\tilde{H}^i(\mathbb{R}^n \setminus F_1) \cong \tilde{H}^{i+n}(\mathbb{R}^{2n} \setminus F_1) \cong \tilde{H}^{i+n}(\mathbb{R}^{2n} \setminus F_2) \cong \tilde{H}^i(\mathbb{R}^n \setminus F_2).$$

□

As an immediate consequence of the Jordan-Alexander duality we have have the celebrated Jordan curve Theorem.

5.8. Theorem. [Jordan curve Theorem] *Let $\gamma : S^1 \longrightarrow \mathbb{R}^2$ be a homeomorphism onto its image*[15]*. Then $\mathbb{R}^2 \setminus \gamma(S^1)$ has exactly two connected components.*

Proof. Consider the unit circle $S^1 \subseteq \mathbb{R}^2$. It is clear that the complement of S^1 in $\mathbb{R}^2$ has exactly two connected components and therefore $\tilde{H}^0(\mathbb{R}^2 \setminus S^1) \cong \mathbb{R}$. By the duality Theorem 5.4, $\tilde{H}^0(\mathbb{R}^2 \setminus \gamma(S^1)) \cong \mathbb{R}$ and therefore the complement of $\gamma(S^1)$ in $\mathbb{R}^2$ has also exactly two connected components. □

5.9. Remark. It is clear that the argument in the proof of Theorem 5.8 may be extended to the case of a closed hypersurface $M^n \subseteq \mathbb{R}^{n+1}$ (see Chapter 3 for definitions) any time we have a "model", i.e. a closed hypersurface homeomorphic to M^n and information on the complement of the model. For example this happens in the case of closed oriented surfaces in $\mathbb{R}^3$ or in the case of closed hypersurfaces of $\mathbb{R}^{n+1}$, homeomorphic to a sphere. A different approach will be discussed in Chapter 4 (Theorem **??**).

6. Appendix: partitions of unity and smooth approximations

In order not to interrupt the flow of the arguments, we left, in the previous sections, a couple of "gaps", namely the proof of existence of partitions of unity (in the proof of Theorem 4.12) and the approximation of continuous maps by smooth ones (see Remark 4.10). In this Appendix we will fill these gaps.

Partitions of unity is a basic tool that allows to glue together locally defined objects (such as functions, forms etc.) in order to obtain a globally defined object. We start with the basic definition.

6.1. Definition. Let $U \subset \mathbb{R}^n$ be an open set and let $\{V_\alpha\}$ be an open covering of U. A *partition of unity dominated by the covering* $\{V_\alpha\}$ is a family of smooth functions $\lambda_i : \mathbb{R}^n \longrightarrow [0,1]$ such that:

(1) For all i there exist α such that $\mathrm{supp}(\lambda_i) := \overline{\{x \in \mathbb{R}^n : \lambda_i(x) \neq 0\}} \subseteq V_\alpha$.
(2) For all $x \in U$ there exists a neighborhood U_x of x, such that $U_x \cap \mathrm{supp}(\lambda_i) = \emptyset$ for all but finitely many of the λ_i's.
(3) For $x \in U$, $\sum_i \lambda_i(x) = 1$ (observe that, by (2), the sum is finite).

Our aim is to prove the following result:

[15] Such a map is usually called a *Jordan curve*.

6.2. THEOREM. *Let $U \subset \mathbb{R}^n$ be an open set and let $\{V_\alpha\}$ be an open covering of U. Then there exist a partition of unity dominated by $\{V_\alpha\}$.*

PROOF. We will use the following notations:

$$B(p,r) = \{x \in \mathbb{R}^n : \|x-p\| < r\}, \quad D(p,r) = \{x \in \mathbb{R}^n : \|x-p\| \le r\} = \overline{B(p,r)}\ .$$

We recall (Exercise 4.13 of Chapter 0) that given $\delta_1, \delta_2 \in \mathbb{R},\ \ 0 < \delta_1 < \delta_2$,and $p \in \mathbb{R}^n$, there exists a smooth function $\phi : \mathbb{R}^n \longrightarrow [0,1]$ such that $\phi(x) = 0$ in $B(p,\delta_1)$ and $\phi(x) = 1$ in $\mathbb{R}^n \setminus B(p,\delta_2)$.

<u>CLAIM 1.</u> Let $K \subseteq \mathbb{R}^n$ be a compact set and $V \subseteq \mathbb{R}^n$ an open set with $K \subseteq V$. Then there exist a smooth function $\psi : \mathbb{R}^n \longrightarrow [0,1]$ such that $\psi(x) = 1$, if $x \in K$ and $\psi(x) = 0$ if $x \notin V$.

PROOF. For any $p \in K$ consider $\delta(p)$ such that $D(p, 2\delta(p)) \subseteq V$. Since K is compact, there is a finite number of points, $p_1, \dots, p_r \in K$, such that $K \subseteq \bigcup D(p_i, \delta(p_i))$. For each i we have a function $\phi_i : \mathbb{R}^n \longrightarrow [0,1]$ such that $\phi_i(x) = 0,\ \ x \in D(p_i, \delta(p_i))$ and $\phi(y) = 1,\ \ y \notin D(p_i, 2\delta(p_i))$. Then the function

$$\psi(x) = 1 - \phi_1(x) \cdots \phi_r(x)$$

has the required properties. □

<u>CLAIM 2.</u> There exist a continuous proper function[16] $\phi : U \longrightarrow [0,\infty)$.

PROOF. Since $\mathbb{R}^n$ is homeomorphic to the open ball $B(0,1)$ (Exercise 4.14, Chapter 0) and the composition of a proper continuous function with a homeomorphism is still proper, we can assume that $U \subseteq B(0,1)$. For $x \in U$, define $d(x)$ to be the distance of x to the boundary of U. Then $d : U \longrightarrow \mathbb{R}$ is a *positive* continuous function. Consider $\phi : U \longrightarrow [0,\infty),\ \ \phi(x) = d(x)^{-1}$. Then ϕ is continuous and for all $n \in \mathbb{N}, \phi^{-1}[0,n]$ is a closed bounded set in U, hence compact. So ϕ is proper. □

We will now prove the Theorem. Consider a proper function $\phi : U \longrightarrow [0,\infty)$ and set

$$A_n = \phi^{-1}[n, n+1], \qquad W_n = \phi^{-1}(n - \frac{1}{2}, n + \frac{3}{2}).$$

Then A_n is compact and therefore may be covered with a finite number of balls $B_{k,n}$ such that each disk $D_{k.n} := \overline{B_{k,n}}$ is contained in some $V_\alpha \cap W_n$. For each such disk we have a smooth function $\phi_{k,n} : U \longrightarrow [0,1]$ vanishing outside $V_\alpha \cap W_n$ and identically 1 in $D_{k,n}$. It is clear from the construction that the A_n's cover U and so, for all $x \in U$, there is at least one of the $\phi_{n.k}$'s not vanishing at x. Also $W_n \cap W_{n+2} = \emptyset$ so the supports of the $\phi_{n,k}$ are a locally finite covering and $\sum_{k,n} \phi_{k,n}(x) < \infty,\ \forall x \in U$. So the family of functions

$$\lambda_{n,k} = \frac{\phi_{n,k}}{\sum_{i,j} \phi_{i,j}}$$

is a well defined partition of unity dominated by the covering V_α. □

6.3. REMARK. Observe that the partition of unity we constructed is a *countable* set of smooth functions.

We shall prove now that a continuous function may be approximate by a smooth function, homotopic to it. The proof is a good example of how to use partition of unity.

[16] A function is *proper* if the inverse image of a compact set is compact.

6.4. THEOREM. *Let $U \subseteq \mathbb{R}^n$ be an open set and let $F : U \longrightarrow W \subseteq \mathbb{R}^m$ be a continuous function which is smooth on a closed subset $N \subseteq U$. Then, given a real valued positive continuous function $\delta : U \longrightarrow \mathbb{R}$ there exists a smooth function $G : U \longrightarrow W$ such that $\|F(x) - G(x)\| < \delta(x),\ \forall x \in U$ and $F(x) = G(x)$ if $x \in N$. Moreover $G \sim F$.*

PROOF. We recall that F smooth on N means that for all $x \in N$ there exists a neighborhood V_x of x and a smooth extension h_x of $F|_{[V_x \cap N]}$. For $x \in U$ we consider a neighborhood V_x of x and a function $h_x : V_x \longrightarrow \mathbb{R}$ with the following conditions:

(1) $F(V_x)$ is contained in a subset of an open ball contained in W.
(2) If $x \in N$, h_x is a smooth extension of $F|_{[V_x \cap N]}$ and $\|h_x(y) - F(x)\| < \dfrac{\delta(x)}{2}$.
(3) If $x \notin N$, $V_x \cap N = \emptyset$ and $h_x(y) = F(x),\ \forall y \in V_x$.
(4) $\forall y \in V_x,\ \ \|F(y) - F(x)\| < \dfrac{\delta(x)}{2} < \delta(y)$.

Consider a smooth partition of unity, λ_i, dominated by the covering V_x. Then $\forall\, i$ there exists $x = x(i)$ with $\mathrm{supp}(\lambda_i) \subseteq V_{x(i)}$. For every i fix such a $x(i)$ and set

$$G(z) = \sum_i \lambda_i(z) h_{x(i)}(z).$$

Then G is a smooth function since in a neighborhood of a point G is a finite sum of smooth functions.

Let $z \in N$ and $\lambda_{i_1}, \dots \lambda_{i_k}$ be the functions of the partition which do non vanish at z. Then $h_{x(i_j)}$ is an extension of F, hence equal, in z, to $F(z)$. Hence $G(z) = F(z)$ and G is an extension of $F|_N$.

Let $y \in U \setminus N$. If $\lambda_i(y) \neq 0$, $y \in \mathrm{supp}(\lambda_i) \subseteq V_{x(i)}$. Hence $\|F(y) - h_{x(i)}\| < \delta(x(i))/2$. Hence

$$\|F(y) - G(y)\| = \|\sum_i \lambda_i(y)F(y) - \sum_i \lambda_i h_{x(i)}(y)\| \leq \sum_i \lambda_i(y)\|F(y) - h_{x(i)}(y)\| < \frac{\delta(x(i))}{2} < \delta(y).$$

Finally $H(x,t) = tF(x) + (1-t)G(x)$ is the required homotopy (observe that, by (1), $H(x,t) \in W$).

□

6.5. COROLLARY. *If two smooth maps $F, G : U \longrightarrow W$ are homotopic via a continuous homotopy, then they are homotopic via a smooth one.*

7. Exercises

7.1. Prove that the tensor product of tensors is associative and distributive.

7.2. Prove that $\omega \in \mathbb{E}_p$ is an exterior form if and only if

$$\omega(x_1, \dots, x_i, \dots, x_j, \dots x_p) = -\omega(x_1, \dots, x_j, \dots, x_i, \dots x_p).$$

7.3. Prove that the exterior product is distributive with respect to the sum.

7.4. Complete the proof of Proposition 1.20.

7.5. Prove that $\phi_1, \dots, \phi_p \in \mathbb{E}^*$ are linearly independent if and only if $\phi_1 \wedge \dots \wedge \phi_p \neq 0$.

7.6. Prove that two sets of linearly independent elements of $\mathbb{E}^*$, $\{\phi_1, \dots, \phi_p\}$ and $\{\psi_1, \dots, \psi_p\}$ span the same subspace of $\mathbb{E}^*$, if and only if $\phi_1 \wedge \dots \wedge \phi_p = d\ \psi_1 \wedge \dots \wedge \psi_p,\ d \in \mathbb{R}$. In this case, d is the determinant of the matrix that gives the change of basis for the subspace.

7.7. Let $\omega \in \Lambda^*(\mathbb{E})$, $\omega = \sum_0^n \omega_i$, $\omega_i \in \Lambda^i(\mathbb{E})$. Prove that ω is invertible in $\Lambda^*(\mathbb{E})$[17] if and only if $\omega_0 \neq 0$.

7.8. Let $\mathbb{E}$ be a n-dimensional vector space. Let $\pi : \mathbb{E}^* \times \cdots \times \mathbb{E}^* \longrightarrow \Lambda^p(\mathbb{E})$ be the p-linear extension of $(\phi_1, \ldots, \phi_p) \longrightarrow \phi_1 \wedge \cdots \wedge \phi_p$. Prove that the following universal property of the exterior algebra holds:

- (UP$\wedge$) If $\mathbb{K}$ is a vector space and $b : \mathbb{E}^* \times \cdots \times \mathbb{E}^* \longrightarrow \mathbb{K}$ is an alternated p-linear map, then there exists a unique linear map $l : \Lambda^p(\mathbb{E}) \longrightarrow \mathbb{K}$ such that $l \circ \pi = b$.

7.9. Prove that the universal property (UP$\wedge$) characterizes $\Lambda^p(\mathbb{E})$ i.e., given a vector space $\mathbb{L}$ and a p-linear map $\tilde{\pi} : \mathbb{E}^* \times \cdots \times \mathbb{E}^* \longrightarrow \mathbb{L}$ such that $(\tilde{\pi}, \mathbb{L})$ verifies UP$\wedge$, then $\mathbb{L} \cong \Lambda^p(\mathbb{E})$.

7.10. Prove that $\Lambda^p(\mathbb{E}^*) \cong [\Lambda^p(\mathbb{E})]^*$.

7.11. Let $\mathbb{E}$ be a n-dimensional vector space and let $\{\phi_1, \ldots, \phi_n\}$ be a basis for $\mathbb{E}^*$. Define

$$\Phi : \mathbb{E} \longrightarrow \Lambda^{n-1}(\mathbb{E}), \qquad \Phi(x_1)(x_2, \ldots, x_n) = \det[\phi_i(x_j)].$$

(1) Prove that Φ is an isomorphism.
(2) Let $\{e_1, \ldots, e_n\} \subseteq \mathbb{E}$ be the dual basis. Prove that $\Phi(e_1) = \phi_2 \wedge \cdots \wedge \phi_n$.

7.12. Let $v \in \Lambda^n(\mathbb{E}) \setminus \{0\}$. Define a map:

$$b_v : \Lambda^p(\mathbb{E}) \times \Lambda^{(n-p)}(\mathbb{E}) \longrightarrow \mathbb{R}, \qquad b_v(\omega, \tau)v := \omega \wedge \tau.$$

(1) Prove that b_v is non degenerate and hence defines an isomorphism $\tilde{b}_v : \Lambda^p(\mathbb{E}) \longrightarrow [\Lambda^{(n-p)}(\mathbb{E})]^*$.
(2) For $p = 1$ relate the map $\tilde{b}_v$ with the map Φ of exercise 7.11.

7.13. A form $\omega \in \Lambda^p(\mathbb{E})$ is *decomposable* if $\omega = \phi_1 \wedge \cdots \wedge \phi_p$, $\phi_i \in \mathbb{E}^*$. By Proposition 1.20, any p-form is a sum of decomposable forms.

(1) Show that, if $\dim(\mathbb{E}) = n$, any $(n-1)$-form is decomposable (hint: see exercise 7.11).
(2) Show that, if $\dim(\mathbb{E}) = 4$ and $\{\phi_1, \ldots, \phi_4\}$ is a basis of $\mathbb{E}^*$, then $\phi_1 \wedge \phi_2 + \phi_3 \wedge \phi_4$ is *not* decomposable.

7.14. Let $\phi_1, \ldots, \phi_r \in \mathbb{E}^*$ be linearly independent. Let $\psi_1, \ldots, \psi_r \in \mathbb{E}^*$ be such that $\sum_i \phi_i \wedge \psi_i = 0$. Prove that $\psi_i = \sum_j a_{ij} \phi_j$ with $a_{ij} = a_{ji}$.

7.15. Let $\mathbb{E}$ be a n-dimensional vector space. A vector space $G(\mathbb{E})$, with an associative product denoted by $\wedge$, is called a *Grassman algebra for* $\mathbb{E}$ if

(1) $G(\mathbb{E})$ contains a subspace isomorphic to $\mathbb{R} \oplus \mathbb{E}$ and is generated, as an algebra, by this subspace,
(2) $1 \wedge x = x, x \wedge x = 0, \ \forall x \in \mathbb{E}$,
(3) $\dim(G(\mathbb{E})) = 2^n$.

Prove that $G(\mathbb{E})$ is isomorphic, as an algebra, to $\Lambda^*(\mathbb{E}^*)$.

7.16. Prove, using the functorial properties, that if $L : \mathbb{E} \longrightarrow \mathbb{F}$ is an isomorphism, $L^* : \Lambda^*(\mathbb{F}) \longrightarrow \Lambda^*(\mathbb{E})$ is an isomorphism (see Remark 1.24).

[17] i.e. there exists $\omega^{-1} \in \Lambda^*(\mathbb{E})$ such that $\omega \wedge \omega^{-1} = 1$.

7.17. Let $\phi \in \mathbb{E}^* \setminus \{0\}$ and $\omega \in \Lambda^p(\mathbb{E})$. Show that, if $\phi \wedge \omega = 0$, then there exists $\tau \in \Lambda^{p-1}$ such that $\omega = \phi \wedge \tau$. Conclude that the sequence:

$$\cdots \longrightarrow \Lambda^{p-1}(\mathbb{E}) \xrightarrow{\phi\wedge} \Lambda^p(\mathbb{E}) \xrightarrow{\phi\wedge} \Lambda^{p+1}(\mathbb{E}) \longrightarrow \cdots$$

is exact. Hint: choose a basis containing ϕ.

7.18. Prove directly, i.e. without using Theorem 3.13, Proposition 3.14.

7.19. Let $\mathbb{L}$ be a finite dimensional real Lie algebra, i.e. a finite dimensional real vector space with a bi-linear map $[\ ,\ ] : \mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L}$, $(X, Y) \longrightarrow [X, Y]$ such that, $\forall\ X, Y, Z \in \mathbb{L}$ we have:

(1) $[X, Y] = -[Y, X]$,

(2) $[[X, Y]Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity).

Define a map $\mathrm{d}^p : \Lambda^p(\mathbb{L}) \longrightarrow \Lambda^{p+1}(\mathbb{L})$,

$$\mathrm{d}^p(\omega)(X_1, \ldots, X_{p+1}) = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1 \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}).$$

Show, at least for $p = 1$, that $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$. In particular the sequence $\{\Lambda^p(\mathbb{L}), \mathrm{d}^p\}$ is a cochain complex and its cohomology is called the *cohomology of the Lie algebra* $\mathbb{L}$.

7.20. Let $\mathcal{E} = \{\{0\} \longrightarrow \mathbb{E}_n \longrightarrow \cdots \longrightarrow \mathbb{E}_0 \longrightarrow \{0\}\}$ be a chain complex. Assume that the $\mathbb{E}_i$'s are finite dimensional and let H_i be the homology groups of the complex. Prove that

$$\chi(\mathcal{E}) := \sum_0^n (-1)^i \dim(\mathbb{E}_i) = \sum_0^n (-1)^i \dim(H_i).$$

$\chi(\mathcal{E})$ is called the *Euler characteristic of the complex.*

7.21. Let $\mathcal{E}, \mathcal{F}$ be chain complexes as in Exercise 7.20, and let $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ be a morphism. Prove that:

$$\lambda(\phi) := \sum (-1)^i \mathrm{trace}(\phi_i) = \sum (-1)^i \mathrm{trace}(\phi_{*,i}).$$

$\lambda(\phi)$ is called the *Leftchetz number of* ϕ (this number is of great importance in fixed point theory).

7.22. Show that the (algebraic) Mayer-Vietoris sequence (Theorem 3.17) is exact and the (co)boundaries are natural (Proposition 3.21).

7.23. Show that the Mayer-Vietoris sequence for the reduced cohomology (see Definition 5.2) is exact.

7.24. Give details of the proof of Proposition 2.12.

7.25. Use Example 4.15 and Remark 4.16 to prove the *Theorem of invariance of dimension*:

THEOREM: If $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a homeomorphism, then $n = m$.

7.26. Redo the computations in Example 4.15, using reduced cohomology.

7.27. Prove the Claim in Remark 4.16

7.28. Consider the points $p_i = (0, \ldots, 0, i) \in \mathbb{R}^n,\ i = 1, \ldots, 25$. Compute $\tilde{H}^k(\mathbb{R}^n \setminus \{p_1, \ldots, p_{25}\}$.

7.29. Let $U \subseteq \mathbb{R}^n$ be an open set and let $\omega \in \Omega^1(U)$ be a *closed* 1-form such that $\omega(x) \neq 0, \quad \forall x \in U$. Consider a smooth function $f : U \longrightarrow \mathbb{R}$ and the 1-form $\tau = f\omega$. Prove that τ is closed if and only if there exists a function $g : U \longrightarrow \mathbb{R}$ such that $\mathrm{d}f = g\omega$.

7.30. Let $U = \{(x, y) \in \mathbb{R}^2 : x > 0\}$.

(1) Consider $\tau = -yx^{-2}\mathrm{d}x + x^{-1}\mathrm{d}y \in \Omega^1(U)$. Prove that τ is exact and find a function $f : U \longrightarrow \mathbb{R}$ such that $\tau = \mathrm{d}f$.

(2) Consider $\omega = -y(x^2+y^2)^{-1}\mathrm{d}x+x(x^2+y^2)^{-1}\mathrm{d}y \in \Omega^1(U)$. Prove that ω is exact and find a function $g : U \longrightarrow \mathbb{R}$ such that $\omega = \mathrm{d}g$.

Remark. The form ω is defined in $W = \mathbb{R}^2 \setminus (0, 0)$. It is closed but *not* exact in W (see next Chapter).

7.31. Let $U \subseteq \mathbb{R}^n$ be an open set and $v = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ be the volume form. We will identify vectors fields and 1-forms via the "musical isomorphisms" $\flat : \mathcal{H}(U) \longrightarrow \Omega^1(U)$ and its inverse $\sharp : \Omega^1(U) \longrightarrow \mathcal{H}(U)$. Also $*$ will denote the Hodge operator. We define the classical differential operators of calculus:

- The *gradient* $\nabla : \mathcal{F}(U) \longrightarrow \mathcal{H}(U),\ \nabla f := \sharp\mathrm{d}f = \sum \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}$.
- The *divergence* div $: \mathcal{H}(U) \longrightarrow \mathcal{F}(U),\ \ \operatorname{div}\left(\sum X_i \frac{\partial}{\partial x_i}\right) = \sum \frac{\partial X_i}{\partial x_i}$.
- The (geometers) *Laplacian* $\Delta : \mathcal{F}(U) \longrightarrow \mathcal{F}(U),\ \ \Delta f = -\operatorname{div} \nabla f$.
- The *rotational* rot $: \Omega^1(U) \longrightarrow \Omega^{n-2}(U)\ \ \operatorname{rot} \omega = *\mathrm{d}\omega$.

.

Prove that:

(1) $\Delta f = -\mathrm{d} * (\mathrm{d}f) = -\sum_1^n \frac{\partial^2 f}{\partial x_i^2}$.

(2) $\Delta(fg) = g\Delta f + f\Delta g - 2\langle\nabla f, \nabla g\rangle$.

(3) ω is closed if and only if $\operatorname{rot} \omega = 0$.

(4) $\operatorname{rot} \nabla f = 0$.

(5) If $n = 3$ compute rot $\sum X_i \frac{\partial}{\partial x_i}$ and show that $\operatorname{div} \operatorname{rot} \omega = 0$.

7.32. Let $U \subseteq \mathbb{R}^n$ be an open set. Show that $H^n(U) = \{0\}$ if and only if $\forall\ f \in \mathcal{F}(U)$ there exists a vector field $X \in \mathcal{H}(U)$ such that $\operatorname{div} X = f$.

Remark: It can be shown that the Laplacian $\Delta : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ is surjective (this is a non trivial fact). In particular the equation $\operatorname{div} X = f$ has a solution $\forall\ f \in \mathcal{F}(U)$. Hence $H^n(U) = \{0\}$.

7.33. Identify $\mathbb{R}^2$ with the complex line $\mathbb{C}$, $\ (x, y) \longrightarrow x + iy, i = \sqrt{-1}$. If $U \subseteq \mathbb{R}^2$ is an open set and $f : U \longrightarrow \mathbb{C}$, we will write $f(z) := f(x, y) = u(x, y) + iv(x, y), u, v \in \mathcal{F}(U)$. f is said to be *holomorphic* if it is C^1 and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{(Cauchy-Riemann equations)}.$$

It can be shown that a holomorphic function is smooth, and, even more than that, complex analytic, i.e. it is locally the sum of its (complex) Taylor series.

(1) Show that the Cauchy-Riemann equations just say that the differential $\mathrm{d}f(z) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is $\mathbb{C}$-linear (i.e. commutes with multiplication by $i = \sqrt{-1}$).

(2) Define *complex* 1-forms:

$$\mathrm{d}z := \mathrm{d}x + i\mathrm{d}y, \quad f\mathrm{d}z := (u+iv)\mathrm{d}z := (u\mathrm{d}x - v\mathrm{d}y) + i(u\mathrm{d}y + v\mathrm{d}x).$$

and the complex derivative $f'(z)$ by the identity $f'(z)\mathrm{d}z = \mathrm{d}f$. Prove that f is holomorphic if and only if the real and imaginary parts of $f\mathrm{d}z$ are closed. In this case $f'(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$.

(3) Prove that if $f = u + iv$ is holomorphic, then $u, v : U \longrightarrow \mathbb{R}$ are harmonic functions (i.e. $\Delta u = \Delta v = 0$).

(4) Show that, if U is star shaped, given a harmonic function $u : U \longrightarrow \mathbb{R}$, there exists a harmonic function $v : U \longrightarrow \mathbb{R}$ such that $f(x,y) = u(x,y) + iv(x,y)$ is holomorphic. The function v is unique, up to an additive constant (if U is connected), and is called the *harmonic conjugate* of u.

7.34. Let $\mathbb{E}$ be a real m-dimensional inner product vector space and $J : \mathbb{E} \longrightarrow \mathbb{E}$ an isometry such that $J^2 = -\mathbb{1}$.

(1) Prove that the dimension of $\mathbb{E}$ is even, say $m = 2n$, and J induces a structure of complex vector space on $\mathbb{E}$.

(2) Prove that there exist orthonormal vectors $\{e_1, \dots, e_n\} \subseteq \mathbb{E}$ such that the set $\{e_i, J(e_i), i = 1, \dots n\}$ is an orthonormal basis for $\mathbb{E}$.

(3) Prove that $\omega(x,y) := \langle x, J(y)\rangle$ is an exterior form.

(4) Let $\phi_i = \flat e_i, \psi_i = \flat J(e_i)$. Prove that $\omega = -\sum \phi_i \wedge \psi_i$.

(5) Prove that $\omega^n = (-1)^n n! * 1$

7.35. Let $U \subseteq \mathbb{R}^n$ be an open. Let $\{X_1, \dots, X_n\} \subset \mathcal{H}(U)$ be vector fields linearly independent at every point of U and let $\{\omega_1, \dots, \omega_n\} \subseteq \Omega^1(U)$ be the dual basis. Consider the k-dimensional distribution D spanned by $\{X_1, \dots, X_k\}$ (see Definition 3.27 of Chapter 0).

Let $\omega \in \Omega^*(U), \quad \omega = \sum \omega_l, \quad \omega_l \in \Omega^l(U)$. We will say that ω *annihilates* D, if $\omega_i(V_1, \dots, V_i) = 0$ whenever the vectors are in D. Define the annihilator of D,

$$\mathcal{I}(D) = \{\omega \in \Omega^*(U) : \omega \ \text{ annihilates } D\}.$$

(1) Prove that $\mathcal{I}(D)$ is an ideal of $\Omega^*(U)$.

(2) Prove that $\mathcal{I}(D)$ is generated (as an algebra) by $\{\omega_{k+1} \dots, \omega_n\}$.

(3) Prove that an ideal of $\Omega^*(U)$ is generated by $n-k$ linearly independent 1-forms if and only is it is the annihilator of a (unique) k-dimensional distribution.

(4) Prove that the distribution D is involutive if and only if for $\omega \in \mathcal{I}(D)$, $\mathrm{d}\omega \in \mathcal{I}(D)$. In this case the ideal $\mathcal{I}(D)$ is called a *differential ideal.*

CHAPTER 2

Integration and the singular homology of open sets of $\mathbb{R}^n$

In Remark 1.8 of Chapter 1, we observed that p-forms are "p-dimensional (oriented) volume elements" and hence the natural integrands for the (oriented) multiple integrals. In this Chapter we will make this statement precise, we will introduce the singular homology of open sets in $\mathbb{R}^n$ and see how integration gives a duality between singular homology and the de Rham cohomology.

1. Integration on singular chains and Stokes Theorem

1.1. Definition. Let $U \subseteq \mathbb{R}^n$ be an open set and $\omega = f(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \in \Omega^n(U)$. Let $D \subseteq U$ be the closure of an open bounded set. We define

$$\int_D \omega = \int_D f(x_1, \ldots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where the integral on the right hand side is the usual Riemann integral.

1.2. Remark. The integral defined above is "oriented" in the sense that if $\omega_\sigma = f(x)\mathrm{d}x_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}x_{\sigma(n)}$, $\sigma \in \Sigma(n)$, then

$$\int_D \omega = |\sigma| \int_D \omega_\sigma.$$

In particular the integral depends on the ordering of the coordinates, i.e., it depends on the choice of an orientation in R^n, while the usual Riemann integral of a function does not depend on such a choice (see also Exercise 7.3).

In order to define the integral of a p-form, we first define the "domain of integration".

1.3. Definition.

- A *p-simplex in* $\mathbb{R}^n$ is the *convex hull*[1] of $(p+1)$ points $\{v_0, \ldots, v_p\} \subset \mathbb{R}^n$ in general position[2]. The points v_i are called the *vertices* of the simplex. Such a simplex will be denoted by $[v_0, \ldots, v_p]$. Any subset of $q+1$ (distinct) vertices determines a q-simplex called a *face* of the original one.
- Let $\{e_1, \ldots, e_p\}$ be the canonical basis of $\mathbb{R}^p$ and $e_0 = 0$. The *standard p-simplex*, $\Delta^p \subset \mathbb{R}^p$ is the simplex with vertices $\{e_0, e_1, \ldots, e_p\}$.
- A *differentiable* (or *smooth*) singular p-simplex in U is a smooth map $\sigma : \Delta^p \longrightarrow U$ (i.e. σ extends to a smooth map of an open neighborhood of Δ^p) . If it is clear from the context we shall omit the term differentiable.

[1]We recall that the convex hull of a subset of $\mathbb{R}^n$ is the smallest convex set that contains the given set. More precisely, it is the intersection of all convex sets that contain the given set.

[2]The points $\{v_0, \ldots, v_p\}$ are in general position if they are not contained in any affine subspace of dimension less than p. This is equivalent to the fact that the vectors $\{v_i - v_0 : i = 1, \ldots p\}$ are linearly independent.

1.4. Remark. Given a p-simplex $[v_0, \dots, v_p]$, a point v in the simplex can be written in a unique way in the form $v = \sum_{i=0}^p \lambda_i v_i$ with $\lambda_i \in [0,1] \subset \mathbb{R}$ and $\sum_{i=0}^p \lambda_i = 1$. The numbers λ_i are the *barycentric coordinates of* v. The *barycenter* of of the simplex is the point $b = (p+1)^{-1} \sum_0^p v_i$ (see Exercise 7.1).

1.5. Example. An important example of a singular simplex is the following: Let $\{v_0, \dots, v_p\}$ be points of $\mathbb{R}^n$, not necessarily in general position. Define $L(v_0, \dots, v_p)$ to be the singular simplex of $\mathbb{R}^n$ that maps the point of Δ^p with barycentric coordinates $\{\lambda_0, \dots, \lambda_p\}$ to the point $\sum_{i=0}^p \lambda_i v_i \in \mathbb{R}^n$. This simplex will be called the *linear simplex with vertices* $\{v_0, \dots, v_p\}$.

1.6. Definition. Let $\omega \in \Omega^p(U)$ be a differential p-form and $\sigma : \Delta^p \to U$ a singular p-simplex. We define the *integral of* ω *over* σ as

$$\int_\sigma \omega := \int_{\Delta^p} \sigma^* \omega,$$

where the integral on the right hand side is in the sense of Definition 1.1.

1.7. Example. If $f \in \mathcal{F}(U)$ is a smooth function, i.e. a 0-form, and let p be a point in U, i.e. a 0-simplex. Then the integral of the form on the simplex is just $f(p)$.

1.8. Example. If $\omega = \sum \omega_i \mathrm{d}x_i \in \Omega^1(U)$ is a 1-form and $\sigma : \Delta^1 \longrightarrow U$ a smooth 1-simplex, then

$$\sigma^*\omega = \tilde{\omega}(t)\mathrm{d}t, \quad \text{with} \quad \tilde{\omega}(t) = \sigma^*\omega(t)(1) = \omega(\sigma(t))(\mathrm{d}\sigma(t)(1)) = \omega(\sigma(t))(\dot{\sigma}(t)) = \sum_{i=1}^n \omega_i(\sigma(t))\dot{\sigma}_i(t),$$

where $\sigma_i(t) = \langle \sigma(t), e_i \rangle$ is the i^{th} coordinate of σ. Hence

$$\int_\sigma \omega = \int_0^1 \left[\sum_{i=1}^n \omega_i(\sigma(t))\dot{\sigma}_i(t) \right] \mathrm{d}t.$$

The fundamental result in the elementary integration theory is Stokes Theorem. It relates the integral of a p-form on a domain to the integral of a primitive on the boundary. For $p = 1$ Stokes Theorem is just the *fundamental Theorem of calculus*

$$\int_a^b \mathrm{d}f = \int_{\partial[a,b]} f = f(b) - f(a) \quad \text{(see Example 1.7)}.$$

We will define now the ingredients necessary to state this Theorem in higher dimensions. We start by introducing more general domains of integration for a p-form.

1.9. Definition. A *singular p-chain* is a (formal, finite) linear combination of singular p-simplices in U, with real coefficients. The set $C_p(U)$ of all such p-chains is a real vector space, with the obvious operations.

If $\omega \in \Omega^p(U), c \in C_p(U),\ c = \sum a_i \sigma_i$, we define the integral of ω on c by:

$$I(c, \omega) := \int_c \omega := \sum a_i \int_{\sigma_i} \omega.$$

Next we have to define the boundary of a p chain. Intuitively, the boundary of a singular simplex will be the restriction of the simplex to the boundary of the standard p-simplex Δ^p (which is a chain and not a simplex). More precisely

1.10. DEFINITION. The *boundary operator* $\partial_p : C_p(U) \longrightarrow C_{p-1}(U)$ is defined as the linear extension of

$$\partial_p \ \sigma := \sum_0^p (-1)^i \sigma \circ F_i,$$

where σ is a singular p-simplex and $F_i : \Delta^{p-1} \longrightarrow \Delta^p$ is the linear simplex $F_i = L(e_0, \ldots, \hat{e_i}, \ldots, e_p)$.

1.11. REMARK. The signs in the definition above guarantee that the $(p-1)$ faces of Δ^p are taken with the *induced orientations*.

1.12. EXAMPLE. For a linear simplex, we have the formula

$$\partial_p L(v_0, \ldots, v_p) = \sum_{i=0}^p (-1)^i L(v_0, \ldots, \hat{v_i}, \ldots, v_p).$$

In our context we have the following version of the classical Stokes Theorem:

1.13. THEOREM. [Stokes Theorem] *If* $c \in C_{p+1}(U),\ \omega \in \Omega^p(U)$*, then*

$$I(\partial c, \omega) := \int_{\partial c} \omega = \int_c \mathrm{d}\omega := I(c, \mathrm{d}\omega).$$

PROOF. By linearity, it is sufficient to prove the Theorem when c is a singular simplex $\sigma : \Delta^{p+1} \longrightarrow U$. In this case

$$\int_\sigma \mathrm{d}\omega = \int_{\Delta^{p+1}} \sigma^* \mathrm{d}\omega = \int_{\Delta^{p+1}} \mathrm{d}\sigma^*\omega$$

(see Theorem 2.9 of Chapter 1 for the last equality). Also

$$\int_{\partial\sigma} \omega = \int_{\partial\Delta^{p+1}} \sigma^*\omega,$$

where $\partial\Delta^{p+1}$ is the linear chain $\sum_{i=0}^{p+1}(-1)^i L(e_0, \ldots, \hat{e_i}, \ldots e_{p+1}) \in C_p(\Delta^{p+1})$.

Now $\eta := \sigma^*\omega = \sum_i f_i(x_1, \ldots, x_{p+1}) \mathrm{d}x_1 \wedge \cdots \hat{\mathrm{d}x_i} \cdots \wedge \mathrm{d}x_{p+1}$. Again by linearity, it is sufficient to prove the Theorem for each monomial. Since we can permute coordinates, up to sign, it is not restrictive to assume

$$\eta = f(x_1, \ldots, x_{p+1}) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p.$$

Then:

$$\mathrm{d}\eta = (-1)^p \frac{\partial f}{\partial x_{p+1}} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p+1}.$$

Hence, by Fubini's Theorem

$$\int_{\Delta^{p+1}} \mathrm{d}\eta = (-1)^p \int_{\Delta^{p+1}} \frac{\partial f}{\partial x_{p+1}} \mathrm{d}x_1 \cdots \mathrm{d}x_{p+1} = (-1)^p \int_{\Delta^p} \left[\int_0^{1-\sum_i^p x_i} \frac{\partial f}{\partial x_{p+1}} \mathrm{d}x_{p+1} \right] \mathrm{d}x_1 \cdots \mathrm{d}x_p =$$

$$= (-1)^p \int_{\Delta^p} \left[f(x_1, \ldots, x_p, 1 - \sum_{i=1}^p x_i) - f(x_1, \ldots, x_p, 0) \right] \mathrm{d}x_1 \cdots \mathrm{d}x_p,$$

where Δ^p is the standard simplex $\{e_0, \ldots e_p\} \subseteq \mathbb{R}^p \subseteq \mathbb{R}^{p+1}$.

Now $\partial\Delta^{p+1} = L(e_1, \ldots e_{p+1}) + (-1)^{p+1} L(e_0, \ldots, e_p) + \gamma$ where γ is a chain of linear simplices that are faces of Δ^{p+1} containing both e_0 and e_{p+1}. Since on each of such faces at least one of the first p coordinates vanishes, $\eta = 0$ on γ. Hence:

$$\int_{\partial\Delta^{p+1}} \eta = \int_{L(e_1, \ldots e_{p+1})} \eta + (-1)^{p+1} \int_{L(e_0, \ldots e_p)} \eta =$$

$$= (-1)^p \int_{\Delta^p} f(x_1, \ldots, x_p, 1 - \sum_{i=1}^{p} x_i) \mathrm{d}x_1 \cdots \mathrm{d}x_p + (-1)^{p+1} \int_{\Delta^p} f(x_1, \ldots, x_p, 0) \mathrm{d}x_1 \cdots \mathrm{d}x_p = \int_{\Delta^{p+1}} \mathrm{d}\eta.$$

□

2. Singular homology

We will now look a little deeper into the boundary operator.

2.1. Lemma. $\partial_{(p-1)} \circ \partial_p = 0$.

Proof. Let σ be a singular simplex. From Example 1.12 we have

$$\partial_p(\sigma) = \sum_{i=0}^{p} (-1)^i \sigma \circ L(e_0, \ldots, \hat{e_i}, \ldots, e_p).$$

Therefore:

$$\partial_{(p-1)}\partial_p(\sigma) = \sum_{i=0}^{p} (-1)^i \sum_{j<i} (-1)^j \sigma \circ L(e_0, \ldots, \hat{e_j}, \ldots, \hat{e_i}, \ldots, e_p) +$$

$$+ \sum_{i=0}^{p} (-1)^i \sum_{j>i} (-1)^{(j-1)} \sigma \circ L(e_0, \ldots, \hat{e_i}, \ldots, \hat{e_j}, \ldots, e_p).$$

Note that the term $\sigma \circ L(e_0, \ldots, \hat{e_i}, \ldots, \hat{e_j}, \ldots, e_p)$, i, j fixed, appears twice in the above sum with opposite signs, and therefore $\partial_{(p-1)}\partial_p(\sigma) = 0$.

□

In particular the sequence:

$$\cdots \longrightarrow C_{(p+1)}(U) \overset{\partial_{(p+1)}}{\longrightarrow} C_p(U) \overset{\partial_p}{\longrightarrow} C_{(p-1)}(U) \overset{\partial_{(p-1)}}{\longrightarrow} \cdots,$$

is a chain complex and we define:

- $Z_p(U) := \ker \partial_p$ the group of *p-dimensional cycles.*
- $B_p(U) := \operatorname{Im} \partial_{(p+1)}$ the group of *p-dimensional boundaries.*
- $H_p(U) := Z_p(U)/B_p(U)$ the p^{th} *dimensional (singular smooth) homology group.*

From Stokes Theorem 1.13 we get

2.2. Theorem. *If $a \in Z_p(U)$, $I(a, \mathrm{d}\omega) = 0$. If $\sigma \in Z^p(U)$, $I(\partial b, \sigma) = 0$. Therefore $I : C_p(U) \times \Omega^p(U) \to \mathbb{R}$ induces a bilinear map*

$$\tilde{I} : H_p(U) \times H^p(U) \longrightarrow \mathbb{R}, \qquad \tilde{I}([c], [\omega]) := I(c, \omega).$$

2.3. Remark. The classical Theorem of de Rham states that the linear map induced by $\tilde{I}$,

$$dR_U : H^p(U) \longrightarrow [H_p(U)]^*, \quad dR_U([\omega])([c]) = \int_c \omega,$$

is an isomorphism, called de *de Rham isomorphism.* We will prove this Theorem in the next section.

Let $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a smooth map. Then F induces a linear map $F_* : C_p(U) \longrightarrow C_p(V)$, obtained by extending by linearity the map which sends a singular simplex $\sigma : \Delta^p \longrightarrow U$ to the singular simplex $F \circ \sigma : \Delta^p \to V$. It is easy to check that F_* commutes with the boundary operator and hence it is a morphism between chain complexes. Therefore it induces a morphism in homology, that we will denote with the same symbol,

$$F_* : H_p(U) \longrightarrow H_p(V).$$

The following *functorial properties* are easily established[3]

- $(1\!\!1_U)_* = 1\!\!1_{H_p(U)}$,
- $(G \circ F)_* = G_* \circ F_*$.

An important feature of the de Rham map is that it is *natural*, in the following sense:

2.4. PROPOSITION. *Let $F : U \longrightarrow V$ be a smooth map. Then*

$$[F_*]^*(dR_V(\omega)) = dR_U(F^*\omega).$$

PROOF. Let $\sigma \in C_p(U), \omega \in \Omega^p(V)$. Then

$$\int_{F\circ\sigma} \omega = \int_\sigma F^*\omega$$

(essentially by definition), and the conclusion follows. □

Now we will look at some examples that are the analogues, for homology, of Examples 4.1 4.2, and 4.3 of Chapter 1.

2.5. EXAMPLE. Let $U = \mathbb{R}^0$. Then there is a unique singular p-simplex, the constant one. Its boundary is the alternated sum of $(p+1)$ elements, all equal to the (unique) $(p-1)$-simplex. Therefore the boundary operator is null if p is odd and it is the identity if p is even. The complex of singular chains is given by:

$$\longrightarrow C_{(2p+1)}(U) = \mathbb{R} \xrightarrow{0} C_{2p}(U) = \mathbb{R} \xrightarrow{1\!\!1} C_{(2p-1)}(U) = \mathbb{R} \xrightarrow{0} \cdots \xrightarrow{0} C_0(U) = \mathbb{R} \longrightarrow \{0\}.$$

Therefore:

$$H_p(\mathbb{R}^0) \simeq \begin{cases} \mathbb{R} & \text{if } p = 0 \\ \{0\} & \text{if } p > 0 \end{cases}$$

2.6. REMARK. It might appear more natural and, in fact, some times more convenient, to define chains and homology using *singular cubes*, i.e., smooth maps of the unit cube $[0,1]^p \subseteq \mathbb{R}^p$ into U. Since a p-cube has always an even number of $(p-1)$-faces, this construction gives, for $U = \mathbb{R}^0$, a chain complex with p-dimensional chain group $\mathbb{R}$ and null boundary operators. The homology is isomorphic $\mathbb{R}$ in all dimensions, something unpleased if we wont the cohomology to be the dual of homology. However if we take the quotient of the complex of singular cubes by a suitable subcomplex, we obtain a new complex whose homology is the same as the homology of the complex of singular simplices.

[3]This means that the homology is a *covariant* functor from the category of open sets of $\mathbb{R}^n$ and smooth maps into the category of (graded) vector spaces and linear maps.

2.7. Example. Let $U = \coprod_\alpha U_\alpha$ be the disjoint union of the open sets U_α. Since Δ^p is connected, the image of a singular simplex is contained in some U_α. Therefore $C_p(U) = \bigoplus_\alpha C_p(U_\alpha)$ (direct sum) and the boundary maps preserve the decomposition, i.e. if $c = \{c_\alpha\},\ \partial c = \{\partial c_\alpha\}$. It follows that

$$H_p(U) \cong \bigoplus_\alpha H_p(U_\alpha).$$

2.8. Remark. We observe explicitly that we are dealing with *finite* linear combinations of simplices, hence we have a direct sum instead of a direct product, as in the case of cohomology. Furthermore, this is in agreement with the de Rham Theorem 2.3, since the dual of the direct sum of vector spaces is the direct product of the duals.

2.9. Example. Let us analyze the 0-dimensional homology. Let us suppose first that U is connected. A singular 0-simplex is a constant map, i.e. a point in U. Such a simplex is a cycle, by definition. On the other hand, given two points in U they can be joined by a smooth curve, i.e. a 1-simplex[4]. The boundary of such simplex is the difference of the two points, so the two points are in the same homology class. It follows that $H_0(U) \cong \mathbb{R}$. Also, as in the case of cohomology, if $U \subseteq \mathbb{R}^n,\quad V \subseteq \mathbb{R}^m$ are connected open sets and $F : U \longrightarrow V$ is a smooth map, the induced map $F_* : H_0(U) \longrightarrow H_0(V)$ is an isomorphism.

If U is not connected, let us say with connected components U_α, it follows from Example 2.7 that

$$H_0(U) \cong \bigoplus_\alpha \mathbb{R}.$$

Next we will prove the *homotopy invariance for homology*:

2.10. Theorem. *Let $F, G : U \to V$ be homotopic smooth maps. Then $F_* = G_*$.*

Proof. Let $H : U \times [0,1] \to V$ be a homotopy between F and G. We will construct an *algebraic homotopy* between the induced maps, i.e. a map $\tilde{H}_p : C_p(U) \longrightarrow C_{(p+1)}(V)$ such that

$$\partial \circ \tilde{H}(\sigma) = G_*(\sigma) - F_*(\sigma) - \tilde{H} \circ \partial\sigma. \tag{3}$$

The Theorem then follows since if $c \in Z_p(U), G_*(c) - F_*(c) \in B_p(V)$, i.e. $[G_*(c)] = [F_*(c)]$ in $H_p(V)$.

Consider the product $\Delta^p \times [0,1] \subset \mathbb{R}^{p+1}$. If σ is a singular p-simplex of U, we consider the map $H \circ (\sigma \times \mathbb{1}) : \Delta^p \times [0,1] \longrightarrow V$. The problem is that $\Delta^p \times [0,1]$ is not a simplex[5]. The strategy will be to subdivide $\Delta^p \times [0,1]$ into simplices and to take a suitable alternated sums of the restrictions of $H \circ (\sigma \times \mathbb{1})$ to such simplices.

Consider $v_i = (e_i, 0), w_i = (e_i, 1)$, and the linear $(p+1)$-simplices $L(v_0, \ldots, v_i, w_i, \ldots w_p)$. If $\sigma : \Delta^p \longrightarrow U$ is a singular p-simplex, we define

$$\tilde{H}(\sigma) = \sum_{i=0}^{p} (-1)^i H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \ldots, v_i, w_i, \ldots, w_p) \in C_{p+1}(V).$$

Extending the formula by linearity we get a morphism $\tilde{H} : C_p(U) \longrightarrow C_{p+1}(V)$.

[4] A connected open set $U \subseteq \mathbb{R}^n$ is path connected.

[5] This is a case in which would be more convenient to work with singular cubes instead that simplices since the product of two cubes is a cube (Remark 2.6).

We observe that, geometrically, the left hand side of equation (3) is the restriction of $\sigma \times \mathbb{1}$ to the boundary of the prism $\Delta^p \times [0,1]$ while the right hand side is, with appropriate signs, the restriction of $\sigma \times \mathbb{1}$ to the bases of the prism, $\Delta \times \{0,1\}$, essentially $G_*(\sigma) - F_*(\sigma)$, plus the restriction of $\sigma \times \mathbb{1}$ to the "lateral faces" $\partial\Delta^p \times [0,1]$. So "morally" $\tilde{H}$ is an algebraic homotopy. Formally, using 1.12 and the functorial properties, we get:

$$\partial\tilde{H}(\sigma) = \sum_i \sum_{j\leq i} (-1)^i(-1)^j H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \ldots, \hat{v_j}, \ldots, v_i, w_i, \ldots w_p) +$$
$$+ \sum_i \sum_{j\geq i} (-1)^i(-1)^{j+1} H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \ldots, v_i, w_i, \ldots, \hat{w_j}, \ldots, w_p).$$

For $i = j$ the terms on the right hand side cancel except for

$$H \circ (\sigma \times \mathbb{1}) \circ L(\hat{v_0}, w_0, \ldots, w_p) = G \circ \sigma \quad \text{and} \quad - H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \ldots, v_p, \hat{w_p}) = -F \circ \sigma.$$

The rest of the sum is the opposite of

$$\sum_i \sum_{j<i} (-1)^{i-1}(-1)^j H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \ldots, \hat{v_j}, \ldots, v_i, w_i, \ldots w_p) +$$
$$+ \sum_i \sum_{j>i} (-1)^i(-1)^j H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \ldots, v_i, w_i, \ldots, \hat{w_j}, \ldots, w_p) = \tilde{H}\partial(\sigma).$$

Hence $\tilde{H}$ is then an algebraic homotopy. □

From Theorem 2.10 and the funtorial properties we have

2.11. Corollary. *If $F : U \longrightarrow V$ is a homotopy equivalence, then $F_* : H_p(U) \to H_p(V)$ is an isomorphism. In particular, a contractible space has the same homology as $\mathbb{R}^0$.*

2.12. Remark. As in the case of cohomology, the homotopy invariance allows us to define the map induced in homology by a continuous map (see Remark 4.10 in Chapter 1).

We also have a Mayer-Vietoris exact sequence for homology. Let $U_i \subseteq \mathbb{R}^n, i = 1, 2$ be open sets and define $U = U_1 \cup U_2, V = U_1 \cap U_2$. Consider the sequence of chain complexes

$$\{0\} \longrightarrow C_p(V) \overset{((j_1)_*,(j_2)_*)}{\longrightarrow} C_p(U_1) \oplus C_p(U_2) \overset{((k_1)_*-(k_2)_*)}{\longrightarrow} C_p(U) \longrightarrow \{0\},$$

where $j_i : V \to U_i, \quad k_i : U_i \to U$ are the inclusions and the boundary maps are the obvious ones.

We would like to proceed like in the case of cohomology. The problem we have here is that the sequence above is not exact. More precisely, $((k_1)_* - (k_2)_*)$ is not surjective, since a chain in U might not be the sum of chains in U_i. To overcome this problem, we consider the chain complex $C_p(U_1 + U_2) \subseteq C_p(U)$ spanned by the singular simplices of U_1 and U_2. Substituting $C_p(U)$ with this complex, we have a short exact sequence of chain complexes. The point that makes this idea work is the following result

2.13. Theorem. [Small simplices Theorem] *The inclusion $C_p(U_1 + U_2) \longrightarrow C_p(U)$ induces an isomorphism in homology.*

The proof requires some new constructions and we will give it in the Appendix in order not to interrupt the flow of our discussion.

Using Theorem 2.13 and Theorem 3.17 of Chapter 1, we deduce, as for cohomology

2.14. THEOREM. *There are linear maps* $\Delta_{*,p} : H_p(U) \longrightarrow H_{(p-1)}(V)$ *such that the sequence*

$$\cdots \longrightarrow H_p(V) \xrightarrow{((j_1)_*,(j_2)_*)} H_p(U_1) \oplus H_p(U_2) \xrightarrow{((k_1)_*-(k_2)_*)} H_p(U) \xrightarrow{\Delta_{*,p}} H_{(p-1)}(V) \longrightarrow \cdots$$

is a (long) exact sequence. Again, we will often write Δ_* *or* Δ_p *for* $\Delta_{*,p}$.

2.15. DEFINITION. The exact sequence above is called the *Mayer-Vietoris sequence for singular homology* and the maps Δ_*, the *Mayer-Vietoris boundary operators*.

2.16. REMARK. As in the case of cohomology (Remark 4.14), we can give an explicit description of the maps $\Delta_* : H_p(C_*(U_1 + U_2)) \longrightarrow H_{p-1}(U_1 \cap U_2)$. Let $c_1 + c_2$ be a p-cycle in $C_p(U_1 + U_2)$. Then $\partial(c_1 + c_2) = 0$, hence $\partial c_1 = -\partial c_2 \in C_{p-1}(U_1 \cap U_2)$. Moreover $\partial\partial c_1 = 0$ hence ∂c_1 defines a homology class $[\partial c_1] \in H_{p-1}(U_1 \cap U_2)$. Then $\Delta([c_1 + c_2]) = [\partial c_1]$. We leave to the interested reader the task of proving this claim. It is also clear, from this interpretation, that Δ_* commutes with morphisms induced by smooth maps.

3. The de Rham Theorem for open sets of $\mathbb{R}^n$

Let $U \subseteq \mathbb{R}^n$ be an open set. As we have seen, integration induces a linear map:

$$dR : H^p(U) \longrightarrow [H_p(U)]^*, \quad dR([\omega])([c]) = \int_c \omega.$$

We have already announced that this map is an isomorphism and the aim of this section is to prove this fact. We will start with a Lemma, known as the *Mayer Vietoris argument*, useful in several situations.

3.1. LEMMA. [Mayer Vietoris argument][6] *Let* $U \subseteq \mathbb{R}^n$ *be an open set and* $\mathcal{P}$ *a statement about the open subsets* $V \subseteq U$. *Suppose that:*

(1) $\mathcal{P}$ *is true for open convex sets,*
(2) *If* $\mathcal{P}$ *is true for disjoint sets, then it is true for their union,*
(3) *If* $\mathcal{P}$ *is true for two sets and for their intersection, then it is true for their union.*

Then $\mathcal{P}$ *is true for* U.

PROOF. First we observe that $\mathcal{P}$ is true for the union of n convex sets. In fact, for $n = 2$ this follows from (3) observing that the intersection of two convex sets is convex. Suppose that $\mathcal{P}$ is true for the union of $(n-1)$ convex sets. Let $V_1, \ldots, V_n$ be convex sets and $V = V_1 \cup \ldots \cup V_{(n-1)}$. Then $\mathcal{P}$ is true for V_n and, by the inductive hypothesis, for V. But it is also true for $V \cap V_n$ since

$$V \cap V_n = (V_1 \cap V_n) \cup \ldots \cup (V_{(n-1)} \cap V_n)$$

is the union of $(n-1)$ convex sets. From (3), $\mathcal{P}$ is true for the union of all the V_i's.

Let $\phi : U \longrightarrow [0, \infty)$ be a proper function (see Claim 2. in the proof of Theorem 6.2, Chapter 1). Define:

$$A_n = \phi^{-1}([n, n+1]).$$

[6] The lemma is also called the *onion lemma* and the reason for this will be clear from the proof (see [**2**]).

Since ϕ is proper, A_n is compact and we can cover it with a finite number of open convex sets, $U_{k,n}$, contained in $\phi^{-1}\left((n-\frac{1}{2}, n+\frac{3}{2})\right)$. Let $U_n = \cup_k U_{k,n}$. Now $\mathcal{P}$ is true for U_n, since it is a finite union of convex sets. Let us consider $U_{even} = \cup_n U_{2n}$ and $U_{odd} = \cup_n U_{2n+1}$. Then, by (2), $\mathcal{P}$ is true for U_{even} and U_{odd} since each one is a disjoint union of sets for which $\mathcal{P}$ is true. Finally $U_{even} \cap U_{odd} = \cup_{n,k,h} U_{k,2n} \cap U_{h,2n+1}$ and therefore it is a disjoint union of sets that are finite unions of convex sets. Therefore, by (3), $\mathcal{P}$ is true for $U = U_{even} \cup U_{odd}$. □

We can now prove the de Rham Theorem.

3.2. Theorem. *The map $dR_U : H^p(U) \longrightarrow [H_p(U)]^*$ is an isomorphism.*

Proof. We are going to use Lemma 3.1. Let $V \subseteq U$ be an open set and consider the statement

$$\mathcal{P}(V) := dR_V : H^p(V) \longrightarrow [H_p(V)]^* \text{ is an isomorphism.}$$

Clearly the statement is true for convex sets. In fact they are contractible and we have only to check the statement in dimension 0, which is trivial. Also, if it is true for a family of disjoint open sets, it is also true for their union (recall that the dual of the direct sum is the direct product).

Let us suppose that $\mathcal{P}$ is true for the open sets V, W and for $V \cap W$. Consider the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H^p(V\cap W) \arrow[r] \arrow[d, "dR_{V\cap W}"] & H^{p+1}(V\cup W) \arrow[r] \arrow[d, "dR_{V\cup W}"] & H^{p+1}(V)\oplus H^{p+1}(W) \arrow[r] \arrow[d, "dR_V \oplus dR_W"] & \cdots \\
\cdots \arrow[r] & [H_p(V\cap W))]^* \arrow[r] & [H_{p+1}(V\cup W)]^* \arrow[r] & [H_{p+1}(W)]^* \oplus [H_{p+1}(W)]^* \arrow[r] & \cdots
\end{tikzcd}$$

where the upper row is the Mayer-Vietoris sequence for cohomology and the lower row is the *dual* of the Mayer-Vietoris sequence in homology. The latter is exact by Proposition 3.14 of Chapter 1. Since integration commutes with induced maps (Proposition 2.4), the diagram above are commutative. Since $dR_{V\cap W}$ and $dR_V \oplus dR_W$ are isomorphisms by hypothesis, it follows from the five Lemma (Lemma 3.7 of Chapter 1) that $dR_{V\cup W}$ is an isomorphism. So $\mathcal{P}$ verifies the hypothesis of Lemma 3.1 and hence $dR = dR_U$ is an isomorphism. □

3.3. Remark. Starting with the singular complex $\mathcal{C}(U) = \{C_p(U), \partial_p\}$, we can consider the *dual complex* $\mathcal{C}^*(U) = \{C_p(U)^*, \partial_p^*\}$ (see Remark 3.12 of Chapter 1). The cohomology of $\mathcal{C}^*(U)$ is called the *singular cohomology* of U and it is isomorphic, by Theorem 3.13 of Chapter 1, to the dual of the singular homology of U. So the de Rham Theorem states that the singular cohomology and the de Rham cohomology are isomorphic. The de Rham cohomology $H^*(U) = \oplus_{p\geq 0} H^p(U)$ has a natural product, induced by the exterior product of forms, which is distributive, associative and graded commutative, (see Remark 2.13 of Chapter 1). In the singular cohomology it is possible to introduce, by geometric arguments, a product, also called the *cup product*, which is distributive, associative and graded commutative. The de Rham Theorem actually says that dR, extended by linearity, is an isomorphism of *algebras*.

3.4. Remark. Singular homology is usually defined by starting with *continuous simplices* i.e., continuous maps $\sigma : \Delta^p \longrightarrow U$ [7]. The singular (continuous) chain complex $\mathcal{C}^0(U) = \{C_p^0(U), \partial_p\}$ is defined in the obvious

[7]Here U can be any topological space.

way, i.e. the space $C_p^0(U)$ is the vector space with basis the singular continuous simplices and the boundary operator is defined just as in the smooth case. The basic properties, such as homotopy invariance and the Mayer-Vietoris exact sequence, are also proved just as in the smooth case. The inclusion $\mathcal{C}(U) \longrightarrow \mathcal{C}^0(U)$ is a morphism of chain complexes, so it induces a map between the homology groups. Using the same arguments as in the proof of the de Rham Theorem, it is easy to prove that the inclusion induces an isomorphism in homology.

4. Tensor product of vector spaces and the Künneth's Theorem

A natural problem is the following: given open sets $U_1 \subseteq \mathbb{R}^n$ and $U_2 \subseteq \mathbb{R}^m$ find a relation between the cohomology groups of U_1, U_2 and $U_1 \times U_2 \subseteq \mathbb{R}^n \times \mathbb{R}^m$. In order to describe this relation we need some preliminary algebraic facts. To start with we need a slightly different approach to tensors.

4.1. Definition. Let $\mathbb{E}, \mathbb{F}$ be two real vector spaces (not necessarily finite dimensional). Consider the vector space freely generated by $\{(x, y) : x \in \mathbb{E}, y \in \mathbb{F}\}$ and the subspace generated by the elements

- $(x_1 + x_2, y) - (x_1, y) - (x_2, y), \quad (x, y_1 + y_2) - (x, y_1) - (x, y_2), \quad x_i \in \mathbb{E},\ y_i \in \mathbb{F}.$
- $r(x, y) - (rx, y), \quad r(x, y) - (x, ry), \quad x \in \mathbb{E},\ y \in \mathbb{F},\ r \in \mathbb{R}.$

The quotient space is called the *tensor product* of $\mathbb{E}$ and $\mathbb{F}$ and will be denoted by $\mathbb{E} \otimes \mathbb{F}$. The class of (x, y) in $\mathbb{E} \otimes \mathbb{F}$ will be denoted by $x \otimes y$.

In other words we may think of $\mathbb{E} \otimes \mathbb{F}$ as the vector space of finite (formal) linear combinations of elements of the type $x \otimes y$ with the "calculus rules"

- $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2, \quad \forall x, x_1, x_2 \in \mathbb{E}, y, y_1, y_2 \in \mathbb{F}$
- $r(x \otimes y) = rx \otimes y = x \otimes ry \quad \forall x \in \mathbb{E}, y \in \mathbb{F}, r \in \mathbb{R}.$

The following facts are easily verified

4.2. Proposition.

1. $\mathbb{E} \otimes \mathbb{F} \cong \mathbb{F} \otimes \mathbb{E}, \quad \mathbb{E} \otimes \mathbb{R} \cong \mathbb{E}.$
2. $(\mathbb{E} \otimes \mathbb{F}) \otimes \mathbb{P} \cong \mathbb{E} \otimes (\mathbb{F} \otimes \mathbb{P}).$
3. $\mathbb{E} \otimes (\mathbb{F} \oplus \mathbb{P}) \cong \mathbb{E} \otimes \mathbb{F} \oplus \mathbb{E} \otimes \mathbb{P}.$
4. *If* $\{e_i\}, \{f_j\}$ *are bases for* $\mathbb{E}, \mathbb{F}$ *respectively, then* $\{e_i \otimes f_j\}$ *is a basis for* $\mathbb{E} \otimes \mathbb{F}$. *In particular, if* $\mathbb{E}$, $\mathbb{F}$ *are finite dimensional,* $\dim(\mathbb{E} \otimes \mathbb{F}) = \dim(\mathbb{E}) \dim(\mathbb{F})$.
5. *If* $\mathbb{E}$ *is finite dimensional,* $\mathbb{E}^* \otimes \mathbb{E}^* \cong \mathbb{E}_2$.

Let $\pi : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{E} \otimes \mathbb{F}$ be the bi-linear extension of $\pi(x, y) = x \otimes y$.

4.3. Proposition. *The following* universal property of the tensor product *holds*

• (UP$\otimes$) *If* $\mathbb{K}$ *is a vector space and* $b : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{K}$, *is a bilinear map, then there exists a unique linear map* $l : \mathbb{E} \otimes \mathbb{F} \longrightarrow \mathbb{K}$ *such that* $l \circ \pi = b$.

Proof. Set $l(x \otimes y) = b(x, y)$. By the "calculus rules", l extend to a linear map of $\mathbb{E} \otimes \mathbb{F}$ into $\mathbb{K}$ such that $l \circ \pi = b$. If $l' : \mathbb{E} \otimes \mathbb{F} \longrightarrow \mathbb{K}$ is a linear map with $l' \circ \pi = b$, then $l'(x \otimes y) = b(x, y) = l(x \otimes y)$. Since the elements of the type $x \otimes y$ span $\mathbb{E} \otimes \mathbb{F}$, we have $l = l'$. $\square$

The general philosophy is that objects defined by *universal properties* are *unique.*

4.4. Proposition. *Let $\mathbb{H}$ be a vector space and let $\tilde{\pi} : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{H}$ be a bi-linear map such that* UP$\otimes$ *holds for $(\tilde{\pi}, \mathbb{H})$. Then $\mathbb{H} \cong \mathbb{E} \otimes \mathbb{F}$.*

Proof. From the universal property for $\pi : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{E} \otimes \mathbb{F}$ follows that there is a unique linear map $l : \mathbb{E} \otimes \mathbb{F} \longrightarrow \mathbb{H}$ such that $l \circ \pi = \tilde{\pi}$. From the universal property of $\tilde{\pi} : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{H}$ follows that there is a unique map $l' : \mathbb{H} \longrightarrow \mathbb{E} \otimes \mathbb{F}$ such that $l' \circ \tilde{\pi} = \pi$. Now, $l \circ l' : \mathbb{H} \longrightarrow \mathbb{H}$ is such that $\tilde{\pi} \circ (l \circ l') = \tilde{\pi}$. But also $\tilde{\pi} \circ \mathbb{1} = \tilde{\pi}$. Hence, by uniqueness, $(l \circ l') = \mathbb{1}$. Analogously $l' \circ l = \mathbb{1}$, hence l and l' are inverse isomorphisms. $\square$

The important feature of the tensor product is that it allows us *to transform a bi-linear problem into a linear problem*, which is, in general, easier to solve.

Let $\mathbb{E}_i,\ \mathbb{F}_i,\ \ i = 1, 2$ be vector spaces and let $L_i : \mathbb{E}_i \longrightarrow \mathbb{F}_i$ be linear map. We define

$$L_1 \otimes L_2 : \mathbb{E}_1 \otimes \mathbb{E}_2 \longrightarrow \mathbb{F}_1 \otimes \mathbb{F}_2, \qquad L_1 \otimes L_2(v \otimes w) := L_1(v) \otimes L_2(w).$$

We will need the following result, whose proof we will leave to the reader (Exercise 7.32)

4.5. Proposition. *Let*

$$\cdots \longrightarrow \mathbb{E}_1 \xrightarrow{\phi} \mathbb{E}_2 \xrightarrow{\psi} \mathbb{E}_3 \longrightarrow \cdots$$

be an exact sequence and let $\mathbb{F}$ be a vector space. Then the sequence

$$\cdots \longrightarrow \mathbb{E}_1 \otimes \mathbb{F} \xrightarrow{\phi \otimes \mathbb{1}} \mathbb{E}_2 \otimes \mathbb{F} \xrightarrow{\psi \otimes \mathbb{1}} \mathbb{E}_3 \otimes \mathbb{F} \longrightarrow \cdots$$

is exact.

We go back now to our question. Let $U_1 \subseteq \mathbb{R}^n,\ U_2 \subseteq \mathbb{R}^m$ be open sets. Set $U = U_1 \times U_2 \subseteq \mathbb{R}^{n+m}$. Let $\pi_i : U \longrightarrow U_i$ be the projection maps. Define a map

$$\kappa_U : \Omega^p(U_1) \otimes \Omega^q(U_2) \longrightarrow \Omega^k(U),\ k = p + q, \qquad \kappa_U(\omega \otimes \tau) = \pi_1^* \omega \wedge \pi_2^* \tau.$$

Since the π_i^*'s commutes with d, κ_U induces a morphism in cohomology. Summing up these morphism for $p + q = k$, we get a map, still denoted by κ_U,

$$\kappa_U : \oplus_{p+q=k} H^p(U_1) \otimes H^q(U_2) \longrightarrow H^k(U).$$

4.6. Definition. The map κ_U is called the *Künneth map*

An important feature of the Künneth's map is that it is *natural* in the following sense (which is an immediate consequence of the definitions)

4.7. Lemma. *Let $U_i \subseteq \mathbb{R}^{n_i},\ \ V_i \subseteq \mathbb{R}^{m_i},\ i = 1, 2,$ be open sets in the corresponding spaces, $U = U_1 \times U_2,\ \ V = V_1 \times V_2$. Let $F_i : U_i \longrightarrow V_i$ be smooth maps. Then*

$$\kappa_U(F_1^* \omega \otimes F_2^* \tau) = (F_1 \times F_2)^* \kappa_V(\omega \otimes \tau)$$

The result we have promised, called the *Künneth Theorem*, or also the *Künneth formula*, is the following

4.8. Theorem. [Künneth's Theorem] *κ_U is an isomorphism*

PROOF. We will use the Mayer-Vietoris argument very much as the the proof of the Theorem of de Rham. Let $W \subseteq U_1$ be an open set and consider the statement

$$\mathcal{P}(W) = \kappa_{W\times U_2} \text{ is an isomorphism.}$$

We want to prove that $\mathcal{P}(U_1)$ is true. For this we will show that the conditions of the Lemma 3.1 are verified. Clearly $\mathcal{P}(W)$ is true if W is convex. Also, if W_α are disjoint open sets for which $\mathcal{P}(W_\alpha)$ is true, the same holds for $W = \cup_\alpha W_\alpha$. It remains to show that if $V, W \subseteq U_1$ are open sets such that $\mathcal{P}(V)$, $\mathcal{P}(W)$ and $\mathcal{P}(V\cap W)$ are true, them $\mathcal{P}(V\cup W)$ is true. Consider the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^p(V\cup W) \longrightarrow H^p(V)\oplus H^p(W) \longrightarrow H^p(V\cap W) \xrightarrow{\Delta^*} H^{p+1}(V\cup W) \longrightarrow \cdots$$

Tensoring with $H^q(U_2)$ and summing for $p+q=k$ we obtain the diagram

$$\begin{array}{ccccc}
\cdots\longrightarrow & \oplus_{p+q=k}H^p(V\cap W)\otimes H^q(U_2) & \xrightarrow{\oplus\Delta^*\otimes 1\!\!1} & \oplus_{p+q=k}H^{p+1}(V\cup W)\otimes H^q(U_2) & \longrightarrow\cdots \\
 & \downarrow \kappa_{(V\cap W)\times U_2} & & \downarrow \kappa_{(V\cup W)\times U_2} & \\
\cdots\longrightarrow & H^k((V\cap W)\times U_2) & \xrightarrow{\Delta^*} & H^{k+1}((V\cup W)\times U_2) & \longrightarrow\cdots
\end{array}$$

The upper line is exact by Proposition 4.5 and the lower one is exact being the Mayer-Vietoris sequence of $V\times U_2, W\times U_2 \subseteq (V\cup W)\times U_2$. Moreover $\kappa_{(V\cap W)\times U_2}$ and $\kappa_{V\times U_2}\oplus\kappa_{W\times U_2}$ are isomorphisms, by hypothesis. So we can use the five Lemma to conclude that $\kappa_{(V\cup W)\times U_2}$ is an isomorphism, once we show that the squares commute. This is simple to show. For example, for the square above, we have

- $\kappa_{(V\cup W)\times U_2}\circ(\Delta^*\otimes 1\!\!1)(\omega\otimes\phi) = \pi_1^*\Delta^*\omega\wedge\pi_2^*\phi$,
- $\Delta^*\circ\kappa_{(V\cap W)\times U_2}(\omega\otimes\phi) = \Delta^*\pi_1^*\omega\wedge\pi_2^*\phi$.

and conclusion follows from the fact that Δ^* commutes with induced maps (Remark 4.14 in Chapter 1). □

4.9. REMARK. An important feature of the Künneth's map is that it is multiplicative. We explain what this means. Consider $H^*(U_i) = \oplus H^p(U_i)$. Then $H^*(U_i)$ is a vector space and we can consider the tensor product $H^*(U_1)\otimes H^*(U_2) = \oplus_k[\oplus_{p+q=k}H^p(U_1)\otimes H^q(U_2)]$. We define a product in $H^*(U_1)\otimes H^*(U_2)$, denoted by $\cdot$, suitably extending $([\alpha]\otimes[\beta])\cdot([\gamma]\otimes[\delta]) := (-1)^{|\beta||\gamma|}[\alpha\wedge\gamma]\otimes[\beta\wedge\delta]$ where $|\tau| = p$ if $\tau\in\Omega^p$. This operation induces an algebra structure in $H^*(U_1)\otimes H^*(U_2)$ and, as it is easily seen, the Künneth's map $\kappa_U : H^*(U_1)\otimes H^*(U_2)\longrightarrow H^*(U)$ is an algebra isomorphism.

4.10. EXAMPLE. Consider $U_1 = \mathbb{R}^n\backslash\{0\}$, $U_2 = \mathbb{R}^m\backslash\{0\}$. Let α, β be generators of $H^{n-1}(U_1)$, $H^{m-1}(U_2)$. Then $H^*(U_1)\otimes H^*(U_2)$ is generated by $1, \alpha\otimes 1, 1\otimes\beta$ and $\alpha\otimes\beta$, with the relations $(\alpha\otimes 1)^2 = 0 = (1\otimes\beta)^2$.

4.11. REMARK. Two open sets may have cohomologies that are isomorphic, as vector spaces, but *not* as algebras. So the algebra structure helps differentiating *not* homotopically equivalent open sets. We will propose, in Exercise 7.33, an example of an open set with the same cohomology, *as vector space*, of the one of $(\mathbb{R}^2\setminus\{0\})\times(\mathbb{R}^2\setminus\{0\})$, but *not as algebras*. In particular these two spaces are *not* homotopy equivalent.

5. Integration of 1-forms and some applications

Let $U \subseteq \mathbb{R}^n$ be an open set. A smooth curve $\gamma : [a,b] \longrightarrow U$ can be seen as the smooth 1-simplex $\tilde{\gamma} = \gamma \circ L(a,b)$ where $L(a,b) = (1-t)a + tb$. If $\omega \in \Omega^1(U)$ is a 1-form, we define

$$\int_\gamma \omega := \int_{\tilde{\gamma}} \omega = \int_0^1 \left[\sum \omega_i(\tilde{\gamma}(t))\dot{\tilde{\gamma}}_i(t)\right] \mathrm{d}t = \int_a^b \left[\sum \omega_i(\gamma(t))\dot{\gamma}_i(t)\right] \mathrm{d}t,$$

where the second integral is the integral of ω on the 1-simplex $\tilde{\gamma}$ and the last equality comes from the formula of change of variable in 1-dimensional integrals (see also Example 1.8). For the rest of this section, when clear from the context, *we will make no difference between the curve γ and the 1-simplex $\tilde{\gamma}$.*

Let $\gamma : [a,b] \subseteq \mathbb{R} \longrightarrow U$ be a piecewise smooth curve, i.e. a continuous curve such that there exists a partition $t_0 = a < t_1 < \cdots < t_k = b$ of $[a,b]$ such that $\gamma_i := \gamma|[t_i, t_{i+1}]$ is smooth. Then γ can be viewed either as the (smooth) 1-chain $\gamma = \sum \gamma_i$ or as a continuous 1-simplex. Clearly, in both cases, $\partial\gamma = \gamma(b) - \gamma(a)$.

Let $\gamma : [a,b] \subseteq \mathbb{R} \longrightarrow U$ be a continuous *closed* curve, i.e. $\gamma(a) = \gamma(b)$. Consider the map $\pi : [a,b] \longrightarrow S^1 := \{x \in \mathbb{R}^2 : \|x\| = 1\}$, $\pi((1-t)a + tb)) = (\cos 2\pi t, \sin 2\pi t)$. Since γ is closed, $\overline{\gamma} = \gamma \circ \pi^{-1}$ is a well defined continuous map of S^1 into U. Conversely, any such map defines a continuous closed curve. From this point of view, continuous closed curves and continuous maps of the circle into U look like the same thing. However, there are some differences:

• If γ is a smooth curve $\overline{\gamma}$ will be just piecewise smooth. It will be smooth if and only if the derivatives of all orders of γ at a, coincide with the derivatives of the corresponding order of γ at b.

• Any curve $\gamma : [a,b] \longrightarrow U$ is homotopic to a constant (see Exercise 7.4). This is not the case for maps of S^1 into U. The following result,whose proof is quite obvious, relates the two situations:

5.1. Lemma. *Let $\overline{\gamma}_i : S^1 \longrightarrow U,\ i = 0,1$ be continuous maps and γ_i be the corresponding closed curves. Then $\overline{\gamma}_0 \sim \overline{\gamma}_1$ if and only if there is a homotopy $H : [a,b] \times [0,1] \longrightarrow U$ between γ_0 and γ_1 such that $H(a,s) = H(b,s)\ \ \forall s \in [0,1]$.*

5.2. Remark. A homotopy like the one in Lemma 5.1 is called a *free homotopy* and the maps $\overline{\gamma}_i$ are said to be *freely homotopic*. The word "free" is to distinguish this concept from the one of *based homotopy*, frequently used in homotopy theory, for example in the definition of the fundamental group.

When clear from the context we will make no distinction between γ and $\overline{\gamma}$.

Let $\gamma : [a,b] \longrightarrow U$ be a closed piecewise smooth curve. If we think of γ as a smooth 1-chain, $\partial\gamma = 0$ and therefore γ determines an element $[\gamma] \in H_1(U)$.

5.3. Lemma. *If γ_0 and γ_1 are freely homotopic piecewise smooth closed curves, then $[\gamma_0] = [\gamma_1]$ in $H_1(U)$.*

Proof. Let $H : [a,b] \times [0,1] \longrightarrow U$ be a free homotopy between the two curves. Subdividing $[a,b] \times [0,1]$ into triangles and using linear simplices as in the proof of homotopy invariance for singular homology (see Theorem 2.10), we get a chain $\tilde{H}$ with $\partial\tilde{H} = \gamma_1 - \gamma_0$. □

The following (well known) facts follow easily from the Theorem of de Rham (we invite the reader to give a more elementary proof, see Exercise 7.14).

5.4. Proposition. *Let $\omega \in \Omega^1(U)$ be a closed 1-form.*

(1) *If* $\gamma_i,\ i = 0, 1$ *are freely homotopic piecewise smooth closed curves then:*

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

(2) ω *is exact if and only if for all closed curves* γ

$$\int_{\gamma} \omega = 0.$$

5.5. Definition. A connected open set $U \subseteq \mathbb{R}^n$ is *simply connected* if every closed curve is freely homotopic to a constant curve [8].

From Proposition 5.4 we have:

5.6. Corollary. *If* U *is simply connected, then* $H^1(U) = \{0\}$.

5.7. Remark. A natural question is whether $H^1(U) = \{0\}$ implies that U is simply connected. The answer to this question is affirmative for $n = 2$ (see Exercise 7.18) and negative if $n \geq 3$. For example there are, in $\mathbb{R}^3$, (complicated) closed sets, homeomorphic to the 3-dimensional closed disk, whose complements are not simply connected (for example the so called "horned disks"). The complement of such a disk has, by the Jordan-Alexander duality (see Theorem 5.4 of Chapter 1), the same cohomology as the complement of the standard 3-dimensional disk, hence vanishing first cohomology group (see Example 4.15 of Chapter 1). We do not know of any simpler example in dimension 3. For $n \geq 4$ there are simpler examples that we will discuss in Chapter 4.

We will focus now on closed curves in $U = \mathbb{R}^2 \setminus \{0\}$. In U there is an important 1-form, the *angle form*

$$\omega = \frac{-y}{x^2 + y^2}\mathrm{d}x + \frac{x}{x^2 + y^2}\mathrm{d}y.$$

It is easily seen that $\mathrm{d}\omega = 0$, in fact, locally, $\omega = \mathrm{d}\arctan(y/x)$ (see Exercise 7.30 of Chapter 1). But ω is not exact since, if $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$,

$$\int_{\gamma} \omega = \int_0^1 2\pi[\sin^2(2\pi t) + \cos^2(2\pi t)]\mathrm{d}t = 2\pi \neq 0.$$

In particular, $dR_U([\omega])([\gamma]) = 2\pi$. Since $H^1(U) \cong \mathbb{R}$, by Examples 4.15 of Chapter 1, $[\omega]$ spans $H^1(U)$ and $[\gamma]$ spans $H_1(U) \cong \mathbb{R}$.

5.8. Definition. Let $\gamma : [0, 1] \longrightarrow U$ be a piecewise smooth curve. An *angular function* for γ is a piecewise smooth function $\theta : [0, 1] \longrightarrow \mathbb{R}$ such that $\theta(t)$ is one of the determinations, in radians, of the (oriented) angle between e_1 and $\gamma(t)$.

5.9. Lemma. *Any piecewise smooth curve* $\gamma : [0, 1] \longrightarrow U$ *admits angular functions and two angular functions for* γ *differ by an entire multiple of* 2π.

[8] The concept of simply connectedness is usually defined in terms of the vanishing of the fundamental group. In this group, two freely homotopic closed curves are in the same conjugacy class (and conversely), but they may not be the same element of the group. However, the *vanishing* of the fundamental group is equivalent to the fact that every two closed curves are freely homotopic (see Remark in Exercise 7.28).

PROOF. Let $\theta_0 \in [0, 2\pi)$ be the angle between e_1 and $\gamma(0)$, and ω the angle form. Consider

$$\theta(t) = \int_{\gamma|[0,t]} \omega \; + \; \theta_0.$$

Since, locally, $\omega = \mathrm{d} \arctan(y/x)$ (see Exercise 7.30 of Chapter 1), θ is an angular function for γ. Finally we observe that two angular functions are, at a given time, determinations of the same angle, so they differ, at that time, by an entire multiple of 2π. This multiple does not depend on the time since the difference of the two angular functions, divided by 2π, is an integers valued continuous function defined on a connected set, hence constant. $\square$

5.10. REMARK. The advantage of having angular functions is that we can write γ in *polar coordinates*

$$\gamma(t) = \|\gamma(t)\| \; e^{i\theta(t)} = \|\gamma(t)\| \; (\cos\theta(t), \sin\theta(t)).$$

Let $\gamma : [0, 1] \longrightarrow U$ be a closed curve and θ an angular function. Since $\gamma(0) = \gamma(1)$, $\theta(1) - \theta(0)$ is an entire multiple of 2π.

5.11. DEFINITION. The *winding number* of γ is the integer

$$w(\gamma) = \frac{\theta(1) - \theta(0)}{2\pi}.$$

5.12. REMARK. Since two angular functions differ by a multiple of 2π, the winding number does not depend on the particular angular function. Moreover

$$w(\gamma) = \frac{1}{2\pi} \int_\gamma \omega.$$

5.13. EXAMPLE. Consider the curve $\xi_n(t) = (\cos 2\pi nt, \sin 2\pi nt)$, $t \in [0, 1]$, n a given integer. Then $\theta(t) = 2\pi nt$ is an angular function and $w(\xi_n) = n$.

The main fact about winding numbers is the following

5.14. THEOREM. [Homotopy classification] *Two piecewise smooth closed curves $\gamma_i : [0, 1] \longrightarrow U$, $i = 0, 1$, are freely homotopic if and only if they have the same winding number.*

PROOF. If the two curves are freely homotopic, by Proposition 5.4 and Remark 5.12, they have the same winding number. Let γ be a piecewise smooth closed curve with angular function θ and winding number $w(\gamma) = n \in \mathbb{Z}$. Let ξ_n be as in Example 5.13. Define

$$H : [0, 1] \times [0, 1] \longrightarrow U, \quad H(t, s) = [s\|\gamma(t)\| + (1 - s)](\cos(s\theta(t) + (1 - s)2\pi nt), \sin(s\theta(t) + (1 - s)2\pi nt)).$$

Then $H(t, 0) = \xi_n(t)$, $H(t, 1) = \gamma(t)$ and the condition $w(\gamma) = n$ implies $H(0, s) = H(1, s)$, $\forall\, s \in [0, 1]$. Hence H is a free homotopy between ξ_n and γ. This concludes the proof since the relation of being freely homotopic is an equivalence relation. $\square$

5.15. REMARK. Any *continuous* curve in U admits *continuous* angular functions. Once we have angular functions, we can define the winding number for a continuous closed curve. Theorem 5.14 holds true in this more general situation (see Exercise 7.25).

We will see now some applications of the homotopy invariance of the winding number.

Let $D^2(r) := \{x \in \mathbb{R}^2 : \|x\| \le r\}$ be the disk of radius r and let $S^1(r) := \{x \in \mathbb{R}^2 : \|x\| = r\}$ be its boundary. Consider a smooth function[9] $f : D^2(r) \longrightarrow \mathbb{R}^2$. A basic question is to find solutions of the equation $f(x) = 0$. In the case of a function $f : [-r, r] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, the celebrated Theorem of Bolzano states that if $f(r)f(-r) < 0$ then the equation $f(t) = 0$ has a solution in $(-r, r)$. We will prove a similar result for our case, similar in the sense that we shall give a condition on f, at the boundary of the disk, that is sufficient (but not necessary, in general) for the existence of solutions of our equation.

5.16. Definition. Let $f : D^2(r) \longrightarrow \mathbb{R}^2$ be a smooth function. Suppose $f(x) \neq 0$ if $\|x\| = r$. The degree of f, $dg(f)$, is defined as the winding number of the closed curve:

$$\gamma_f : [0,1] \longrightarrow U := \mathbb{R}^2 \setminus \{0\}, \qquad \gamma_f(t) = f(r(\cos 2\pi t, \sin 2\pi t))$$

.

5.17. Example. Consider the complex plane $\mathbb{C} \cong \mathbb{R}^2$ with complex variable $z = x + iy$, and the map $g(z) = z^n$. Then $\gamma_g(t) = r(\cos 2\pi nt, \sin 2\pi nt)$. Hence $dg(g) = n$.

The announced result is the following

5.18. Theorem. *If $dg(f) \neq 0$ then the equation $f(x) = 0$ has a solution.*

Proof. Suppose $dg(f) \neq 0$, $f(x) \neq 0 \ \forall\, x \in D^2(r)$. Consider the map

$$H : [0,1] \times [0,1] \longrightarrow \mathbb{R}^2 \setminus \{0\}, \qquad H(t,s) = f(sr(\cos 2\pi t, \sin 2\pi t)).$$

Since $f(x) \neq 0$, for $\|x\| \le r$, H is a free homotopy, in $\mathbb{R}^2 \setminus \{0\}$, between γ_f and the constant curve $\alpha(t) = f(0)$. Therefore, by Theorem 5.14, $dg(f) := w(\gamma_f) = w(\alpha) = 0$, a contradiction. $\square$

In order to compute degrees, the following fact is often useful

5.19. Lemma. [Poincaré-Bohl] *Let $\gamma_i : [0,1] \longrightarrow \mathbb{R}^2 \setminus \{0\}$, $i = 0, 1$ be two closed curves. If $\|\gamma_0(t) - \gamma_1(t)\| < \|\gamma_0(t)\| \ \ \forall\, t \in [0,1]$, the two curves are freely homotopic.*

Proof. Consider the map:

$$H : [0,1] \times [0,1] \longrightarrow \mathbb{R}^2, \qquad H(t,s) = s\gamma_1(t) + (1-s)\gamma_0(t).$$

The condition $\|\gamma_0(t) - \gamma_1(t)\| < \|\gamma_0(t)\|$ implies that the segment joining $\gamma_0(t)$ and $\gamma_1(t)$ does not contain the origin. Hence $H([0,1] \times [0,1]) \subseteq \mathbb{R}^2 \setminus \{0\}$ and H is a free homotopy (in U) between the two curves. $\square$

As an application of Theorem 5.18, we prove now the *Fundamental Theorem of Algebra*:

5.20. Theorem. *Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ be a polynomial in the complex variable z. If $n \ge 1$, f has a complex root.*

[9] By Remark 5.15 we only need continuity of the function.

PROOF. Let $r > 1 + \sum_1^n |a_i|$. If $f(z) = 0$, for some $z \in S^1(r)$, there is nothing to prove. Suppose $f(z) \neq 0$ for $\|z\| = r$ and consider the function $g(z) = z^n$. For $\|z\| = r$ we have:

$$\|f(z) - g(z)\| \leq \sum_1^n |a_i| \|z\|^{n-i} < r^n = \|g(z)\|.$$

Hence, by Lemma 5.19, f and g have the same degree and $dg(g) = n \neq 0$, by Example 5.17. Hence, by Theorem 5.18, the polynomial f has a root in $D^2(r)$. □

5.21. REMARK. We can take a slightly different approach to the winding number. Let $\gamma : S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}$ be a closed smooth curve. Then we can extend γ to a map

$$\Gamma : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \{0\}, \quad \Gamma(tx) = t\gamma(x), \;\; x \in S^1.$$

Hence we have an induced map

$$\Gamma_* : H_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R} \longrightarrow H_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R}$$

which is multiplication by a real number, which is, as it is easily seen, the winding number of γ. In this context the winding number is also called the *degree* of γ and is denoted by $dg(\gamma)$. This point of view is useful in extending the concept to higher dimensions (see Exercise 7.29).

6. Appendix: baricentric subdivision and the proof of Theorem 2.13

Let U_1, U_2 be open sets in $\mathbb{R}^n$ and $U = U_1 \cup U_2$. We want to show that the inclusion $i : C_p(U_1 + U_2) \longrightarrow C_p(U)$ induces an isomorphism in homology. The idea of the proof goes as follow: consider a singular simplex $\sigma : \Delta^p \longrightarrow U$ and the covering of Δ^p given by $\sigma^{-1}(U_i)$, $i = 1, 2$. We "subdivide" Δ^p into subsimplices of very small diameter so that σ sends each one of the small simplices into U_1 or U_2. With this operation we pass from a chain $c \in C_p(U)$ to a chain $\tilde{c} \in C_P(U_1 + U_2)$. Finally we must show that if c is a cycle, $\tilde{c}$ is also a cycle and represents, in $H_p(U)$, the same class of c and that, if $\tilde{c}$ is the boundary of a chain in $C_{p+1}(U)$, it is also the boundary of a chain in $C_{p+1}(U_1 + U_2)$.

We will describe the subdivision process in three steps.

STEP 1. Barycentric subdivision of a simplex.

Let $\Gamma = [v_0, \dots, v_p]$ be a p-simplex in $\mathbb{R}^n$ and let $b := b_\Gamma := (p+1)^{-1} \sum_0^p v_i$ be its barycenter.

6.1. DEFINITION. A *barycentric subsimplex* of Γ is a simplex of the form $[b_0, \dots b_k]$ where b_i is the barycenter of a face F_i of Γ with $F_k \supsetneq F_{k-1} \supsetneq \cdots \supsetneq F_0$.

The set of barycentric subsimplices of Γ, denoted by $\Gamma^{(1)}$, is called the *(first) barycentric subdivision* of Γ. Inductively we define the r^{th} *barycentric subdivision* of Γ, $\Gamma^{(r)}$, as the collection of barycentric subsimplices of simplices in $\Gamma^{(r-1)}$.

The effect of the barycentric subdivisions of Γ, that we are interested in, is that the diameters of the simplices of $\Gamma^{(r)}$ go to zero when $r \to \infty$ (Corollary 6.3).

6.2. PROPOSITION. *Let* $\Gamma = [v_0, \dots, v_p]$. *If* $\Delta \in \Gamma^{(1)}$, $\operatorname{diam}(\Delta) \leq p(p+1)^{-1} \operatorname{diam}(\Gamma)$.

PROOF. First we observe that, since the diameter of a subset of $\mathbb{R}^n$ coincides with the one of its convex hull, the diameter of a simplex is the maximum of the distance between its vertices.

We will proceed by induction. Clearly the claim is true for $p = 0$. Suppose the claim true for $k-$simplices, $k < p$ and set $\Delta = [b_0, \dots, b_h]$, where b_i is the barycenter of a face F_i of Γ, $F_0 \supsetneqq \cdots \supsetneqq F_h$. If $\dim(F_0) = k < p$, we have, by induction,

$$\|b_i - b_j\| \leq k(k+1)^{-1} \operatorname{diam}(F_0) \leq p(p+1)^{-1} \operatorname{diam}(F_0) \leq p(p+1)^{-1} \operatorname{diam}(\Gamma).$$

Let us suppose now that $b = b_0$ is the barycenter of Γ.

<u>CLAIM.</u> $\|v_i - b\| \leq p(p+1)^{-1} \operatorname{diam}(\Gamma)$.

PROOF. We can assume $i = 0$. Then

$$\|v_0 - b\| = \|v_0 - (p+1)^{-1} \sum_0^p v_i\| = \|v_0 - (p+1)^{-1} v_0 - (p+1)^{-1} \sum_1^p v_i\| =$$

$$= \|p(p+1)^{-1} v_0 - (p+1)^{-1} \sum_1^p v_i\| \leq (p+1)^{-1} \| \sum_1^p (v_0 - v_i)\| \leq p(p+1)^{-1} \sup\{\|v_0 - v_j\|\} \leq p(p+1)^{-1} \operatorname{diam}(\Gamma).$$

□

In particular Γ, hence Δ, is contained in the ball with center b and radius $p(p+1)^{-1} \operatorname{diam}(\Gamma)$ and $\|b_0 - b_i\| \leq p(p+1)^{-1} \operatorname{diam}(\Gamma)$. Since, by the inductive hypothesis, the distance between any other two vertices of Δ is also, at most, $p(p+1)^{-1} \operatorname{diam}(\Gamma)$, the conclusion follows.

□

6.3. COROLLARY. *Given* $\epsilon > 0$, *there exists* r_0 *such that* $\operatorname{diam}(\Delta) < \epsilon, \quad \forall \quad \Delta \in \Gamma^{(r)}, \quad r \geq r_0.$

PROOF. The claim follows from the fact that, if $\Delta \in \Gamma^{(s)}$, $\operatorname{diam}(\Delta) \leq p^s(p+1)^{-s} \operatorname{diam}(\Gamma)$ (by induction using Proposition 6.2) and the latter quantity is monotone decreasing with limit zero as s goes to ∞. □

<u>STEP 2.</u> Barycentric subdivision of linear chains.

Let $U \subseteq \mathbb{R}^n$ be a *convex* open set. We will denote by $LC_p(U)$ the linear subspace of $C_p(U)$ spanned by the linear simplices. Clearly $\partial(LC_p(U)) \subseteq LC_{p-1}(U)$, (see Example 1.12) and therefore $\{LC_p(U), \partial|_{LC_p(U)}\}$ is a subcomplex of $\{C_p(U), \partial\}$.

Let $v \in U$. We define the *cone operator* with vertex v extending, by linearity, the map

$$L(v_0, \dots, v_p) \rightsquigarrow C_v L(v_0, \dots, v_p) := L(v, v_0, \dots, v_p).$$

Since U is convex, C_v is well defined.

6.4. LEMMA. *If* $c \in LC_p(U)$, $\partial C_v(c)) = \sigma - C_v(\partial c)$.

PROOF. It is sufficient, by linearity, to prove the Lemma in the case that $c = L(v_0, \dots, v_p)$ is a linear simplex. Since the "boundary of the cone" is the base, essentially c, union the cone on the boundary of c, the formula is "morally true". Formally,

$$\partial C_v(c) = \partial L(v, v_0, \dots, v_p) = L(v_0, \dots, v_p) + \sum_{i=0}^p (-1)^{i+1} L(v, v_0, \dots, \hat{v}_i, \dots, v_p) =$$

$$= c - C_v(\sum_{i=0}^{p}(-1)^i L(v_0, \dots, \hat{v}_i, \dots, v_p)) = c - C_v(\partial c).$$

□

6.5. Definition. The *barycentric subdivision* $S_p : LC_p(U) \longrightarrow LC_p(U)$ is defined, inductively, extending, by linearity,

$$S_0(\sigma) = \sigma, \quad S_p(\sigma) = C_{b_\sigma}(S_{(p-1)}(\partial\sigma)), \text{ if } p > 0,$$

where σ is a linear p-simplex and b_σ is its barycenter, i.e. if $\sigma = L(v_0, \dots, v_p)$, $b_\sigma = (p+1)^{-1}\sum v_i$.

Our next task is to show that S is a chain morphism that induces an isomorphism in homology. This will follow from the next two Lemmas.

6.6. Lemma. *If $c \in LC_p(U)$, $\partial S(c) = S(\partial c)$.*

Proof. If $p = 1$ and $\sigma = L(v_0, v_1)$, both sides are equal to $\partial\sigma$ and the formula is true. Suppose, by induction, that the formula is true for linear chains of dimension $\leq p - 1$ and let σ be a linear p-simplex.

$$\partial S(\sigma) = \partial(C_{b_\sigma} S(\partial\sigma)) = S\partial\sigma - C_{b_\sigma}\partial S(\partial\sigma) = S\partial\sigma - C_{b_\sigma} S(\partial\partial\sigma) = S\partial\sigma,$$

where the second equality follows from Lemma 6.4 and the third one from the inductive hypothesis. □

We define, by induction, a linear map $T : LC_p(U) \longrightarrow LC_{p+1}(U)$,

$$T(L(x)) = L(x, x), \text{ if } p = 0, \quad T(\sigma) = C_{b_\sigma}(\sigma - T(\partial\sigma)), \text{ if } p \geq 1.$$

6.7. Lemma. $\partial T(c) + T(\partial c) = c - S(c)$. *In particular T is an algebraic homotopy between S and $\mathbb{1}$.*

Proof. Again by induction. The formula holds for $p = 0$. Suppose that it holds for linear chains of dimensions at most $p - 1$. Then, if σ is a linear p-simplex,

$$\partial T(\sigma) = \partial(C_{b_\sigma}(\sigma - T(\partial\sigma)) = \sigma - T(\partial\sigma) - C_{b_\sigma}\partial(\sigma - T(\partial\sigma)) = \sigma - T(\partial\sigma) - C_{b_\sigma}(\partial\sigma) + C_{b_\sigma}(\partial T(\partial\sigma)) =$$

$$= \sigma - T(\partial\sigma) - C_{b_\sigma}(\partial\sigma) + C_{b_\sigma}(\partial\sigma - S(\partial\sigma) - T\partial(\partial\sigma)) = \sigma - T(\partial\sigma) - C_{b_\sigma} S(\partial\sigma) = \sigma - T(\partial\sigma) - S(\sigma),$$

where the second equality follows from Lemma 6.4, the fifth by induction and the last by definition of S.

Again, by linearity, the Lemma holds for linear chains. □

Step 3. Subdivision of singular chains.

We will extend the operators S, T to act on singular chains in an arbitrary open set $U \subseteq \mathbb{R}^n$.

Let $\mathbb{1} : \Delta^p \longrightarrow \Delta^p$ be the identity singular simplex. Since Δ^p is convex, $S(\mathbb{1}) \in LC_p(\Delta^p)$ and $T(\mathbb{1}) \in LC_{p+1}(\Delta^p)$ are well defined[10].

If $\sigma : \Delta^p \longrightarrow U$ is a singular simplex, we define $S(\sigma) := \sigma_*(S(\mathbb{1}))$, $T(\sigma) := \sigma_*(T(\mathbb{1}))$. Then, by linear extensions, we have maps

$$S : C_p(U) \longrightarrow C_p(U), \quad T : C_p(U) \longrightarrow C_{p+1}(U).$$

As for the case of linear chains we have

[10] To be more precise we should consider $\mathbb{1}$ as a singular simplex on a convex *open* neighborhood of $\Delta^p \subseteq \mathbb{R}^p$.

6.8. LEMMA. *S is a chain map and T an algebraic homotopy between S and $\mathbb{1}$.*

PROOF. Let $\sigma : \Delta^p \longrightarrow U$ be a singular simplex. Then

$$\partial S(\sigma) = \partial\sigma_* S(\mathbb{1}) = \sigma_* \partial S(\mathbb{1}) = \sigma_* S(\partial \mathbb{1}) = S(\partial\sigma),$$

where the tird equality follows from Lemma 6.6. This prove the first claim. For the second one we have

$$\partial T(\sigma) + T(\partial\sigma) = \partial\sigma_*(T(\mathbb{1})) + T\partial\sigma_*(\mathbb{1}) = \sigma_*(\partial T(\mathbb{1}) + T(\partial\mathbb{1})) = \sigma_*(\mathbb{1} - S(\mathbb{1})) = \sigma - S(\sigma),$$

where the third equality follows from Lemma 6.7. □

.

We define the *support of a singular simplex* $\sigma : \Delta^p \longrightarrow U$ as $\text{support}(\sigma) = \sigma(\Delta^p)$. The *support of a singular chain* $c = \sum r_i\sigma_i$ will be the union of the supports of the simplices σ_i. Then we have

6.9. LEMMA. *If $c \in C_p(U)$, $\text{support}(S(c)) = \text{support}(c)$, $\text{support}(T(c)) = \text{support}(c)$.*

We go back now to our problem. Let $U_1,\ U_2$ be open sets in $\mathbb{R}^n$ and $U = U_1 \cup U_2$.

6.10. LEMMA. *If $z \in C_p(U)$ is a cycle, then $S(z)$ is a cycle and $z - S(z) = \partial w,\ w \in C_{p+1}(U)$. In particular $[z] = [S(z)$. Moreover, if $z \in C_p(U_1 + U_2),\ w$ can be chosen in $C_{p+1}(U_1 + U_2)$.*

PROOF. Since $\partial T(z) + T(\partial z) = z - S(z)$, it follows that $z - S(z) = \partial T(z)$. Moreover, if $z \in C_p(U_1 + U_2),\ T(z) \in C_{p+1}(U_1 + U_2)$, by Lemma 6.9. □

We define, by induction, the r^{th} subdivision operator $S^{(r)} : C_p(U) \longrightarrow C_p(U),\quad S^{(0)} = S,\quad S^{(r)} = S \circ S^{(r-1)}$. Clearly, also $S^{(r)}$ is a chain map and, in particular, if z is a cycle, $S^{(r)}(z)$ is also a cycle.

6.11. LEMMA. *If $z \in C_p(U)$, there exists $r = r(z)$ such that $S^{(r)}(z) \in C_p(U_1 + U_2)$.*

PROOF. Suppose first that $z = \sigma : \Delta^p \longrightarrow U$ is a singular simplex. If $m \geq 1,\ S^{(m)}(\sigma) = \sigma_*(S^{(m)}(\mathbb{1}))$. The simplices that appears in $S^{(m)}(\mathbb{1})$ are linear homeomorphisms between Δ^p and barycentric subsimplices of Δ^p. By the Lebesgue Lemma[11] and Corollary 6.3 there exist an $r = r(\sigma)$ such that the simplices of $[\Delta^p]^{(r)}$ are contained in one of the open sets $\sigma^{-1}(U_i)$. Hence $S^{(r)}(\sigma) \in C_p(U_1 + U_2)$, by Lemma 6.9. This shows that the lemma holds for simplices. If $z = \sum_{i=1}^k t_i\sigma_i$, where $\sigma_i : \Delta^p \longrightarrow U$ are singular simplices, we can take $r(z) = \max\{r(\sigma_i)\}$ and the result follows, again, from Lemma 6.9. □

At this point we can prove our result

2.13. THEOREM. [Small simplices Theorem] *The inclusion $C_p(U_1 + U_2) \longrightarrow C_p(U)$ induces an isomorphism in homology.*

PROOF. We have to show two things

(1) If z is a cycle in $C_p(U)$ then z is equivalent in $H_p(U)$ to a cycle in $C_p(U_1 + U_2)$.
(2) If z is a cycle in $C_p(U_1 + U_2)$ which is the boundary of a chain in $C_{p+1}(U)$, then it is also the boundary of a chain in $C_p(U_1 + U_2)$.

[11] The Lemma states that if $K \subseteq \mathbb{R}^n$ is a compact set and $\mathcal{U}$ is an open covering of K, then there exists $\delta > 0$ such that any subset of K of diameter less than δ is contained in one of the sets of the covering.

For (1) we consider $z' = S^{(r)}(z)$, with r as in Lemma 6.11. Then z' is a cycle in $C_p(U_1 + U_2)$ equivalent, in $H_p(U)$, to z (Lemma 6.10 and induction).

For (2), we suppose $z = \partial c,\ c \in C_{p+1}(U)$. Let r be such that $S^{(r)}(c) \in C_{p+1}(U_1 + U_2)$. Then $\partial S^{(r)}(c) = S^{(r)}(z)$. Also, by Lemma 6.10 (and induction), $z - S^{(r)}(z)$ is the boundary of a chain $d \in C_{p+1}(U_1 + U_2)$ and therefore $S^{(r)}(c) + d \in C_{p+1}(U_1 + U_2)$. Finally

$$\partial(S^{(r)}(c) + d) = S^{(r)}(z) + \partial d = S^{(r)}(z) + z - S^{(r)}(z) = z.$$

□

7. Exercises

7.1. Consider the p-simplex $\tilde{\Delta}^p = [e_1, \dots, e_{p+1}] \subset \mathbb{R}^{p+1}$ where $\{e_i\}$, is the canonical basis of $\mathbb{R}^{p+1}$.

(1) Prove that $\tilde{\Delta}^p = \{(t_1, \dots, t_{p+1}) \in \mathbb{R}^{p+1} : t_i \ \in [0,1],\ \sum_{i=1}^{p+1} t_i = 1\}$.

(2) Let $\{v_1, \dots v_{p+1}\} \subseteq \mathbb{R}^n$ be points in general position. Prove that the map $\tilde{L}(t_1, \dots, t_{p+1}) = \sum_1^{p+1} t_i v_i$ is a homeomorphism of $\tilde{\Delta}^p$ onto $[v_1, \dots, v_{p+1}]$.

(3) Deduce the existence of barycentric coordinates for the points of a p-simplex (see Remark 1.4).

Remark. In many texts it is $\tilde{\Delta}^p$ that is called the *standard simplex*. This has the advantage that every face of $\tilde{\Delta}^p$ is a "standard $(p-1)$ simplex", which is not true with our definition of standard simplex.

7.2. Let $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p \in \Omega^p(\mathbb{R}^n)$ and let Δ^p be the standard p-simplex. Show that

$$\int_{\Delta^p} \omega = \frac{1}{p!} \qquad (= \text{volume of } \ \Delta^p).$$

7.3. Let $U, V \subseteq \mathbb{R}^n$ be connected open sets and $F : U \longrightarrow V$ be a diffeomorphism. Let $D \subseteq U$ be the closure of a bounded open set and $f : V \longrightarrow \mathbb{R}$ a smooth function. Prove that, if $\omega \in \Omega^n(V)$,

$$\int_{F(D)} \omega = \pm \int_D F^*\omega,$$

with the sign $+$ (resp. $-$) if F preserves (resp. reverse) the orientation, i.e. $\det(\mathrm{d}F) > 0$ (resp. $\det(\mathrm{d}F) < 0$).

7.4. Let $U \subseteq \mathbb{R}^m,\ V \subseteq \mathbb{R}^m$ be open sets and $F : U \longrightarrow V$ a continuous map.

(1) Prove that if U (resp. V) is contractible, then F is homotopic to a constant map.

(2) prove that if V is contractible, any two maps $F, G : U \longrightarrow V$ are homotopic.

(3) It is true that, if U is contractible, any two maps $F, G : U \longrightarrow V$ are homotopic?

7.5. Let $D^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}$, $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\} = \partial D^{n+1}$ and $V \subseteq \mathbb{R}^m$. Show that a continuous map $F : S^n \longrightarrow V$ is continuously homotopic to a constant map if and only if it extends to a continuous map $\tilde{F} : D^{n+1} \longrightarrow V$.

7.6. Prove that a map $F : S^n \longrightarrow S^n$ which is not surjective is homotopic to a constant. Give an example of a surjective map $F : S^n \longrightarrow S^n$ which is homotopic to a constant.

7.7. Prove that an open set $U \subseteq \mathbb{R}^n$ is connected if and only if $H_0(U) \cong \mathbb{R}$ (see Example 2.9).

7.8. Let $U \subseteq \mathbb{R}^n,\ V \subseteq \mathbb{R}^m$ be open sets and $F : U \longrightarrow V$ a smooth map. Prove that if U is connected, $F_* : H_0(U) \longrightarrow H_0(V)$ is injective. Study the case in which U is not connected (see Example 2.9).

7.9. For an open set $U \subseteq \mathbb{R}^n$ define the reduced homology, $\tilde{H}_p(U)$, as the homology of the *augmented chain complex*

$$\cdots \longrightarrow C_p(U) \longrightarrow C_{p-1}(U) \longrightarrow \cdots \longrightarrow C_0(U) \longrightarrow \mathbb{R} \longrightarrow \{0\},$$

where the last map sends any singular 0-simplex to $1 \in \mathbb{R}$ and is extended by linearity (the other maps are the usual boundaries). Find the relation between $H_p(U)$ and $\tilde{H}_p(U)$ and prove the homotopy invariance and the exactness of Mayer-Vietoris sequence for reduced homology.

7.10. Compute the homology of $\Sigma_n \subseteq \mathbb{R}^n$ using the Mayer Vietoris sequence for reduced homology (see Example 4.15 of Chapter 1 for the definition of Σ_n).

7.11. Let $U \subseteq \mathbb{R}^n$ be an open set and $p \in U$. Assume known that $H_n(U) = \{0\}$ (see Remark in Exercise 7.32 of Chapter 1). Find the relation between $H_k(U \setminus \{p\})$ and of $H_k(U)$.

7.12. Prove the claim made in Remak 3.4 that the homology of the complex spanned by the *continuous* singular simplices is isomorphic to the homology of the complex spanned by the *smooth* singular simplices.

7.13. Let $U \subseteq \mathbb{R}^n$ be an open set. Two curves $\gamma_i : [0,1] \longrightarrow U$ are *homotopic relative to the end points* if there exists a homotopy $H : [0,1] \times [0,1]$ such that $H(0,s) = \gamma_i(0),\ H(1,s) = \gamma_i(1),\quad \forall\ s \in [0,1]$.

(1) Prove that U is simply connected if and only if any two curves in U, with the same endpoints, are homotopic relative to the endpoints.
(2) Let $\omega \in \Omega^1(U)$ be a *closed* 1-form and $\gamma_i,\ i = 1,2$ be curves homotopic relative to the endpoints. Prove that
$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

7.14. Give a proof of Proposition 5.4 using Stokes Theorem (instead that the Theorem of de Rham).

7.15. Let $\omega = a(x,y)\mathrm{d}x + b(x,y)\mathrm{d}y$ be a smooth closed 1-form in $\mathbb{R}^2 \setminus \{0\}$. Suppose that, for $0 < x^2 + y^2 \leq K$, the functions a, b are bounded. Prove that ω is exact (hint: use homotopy invariance to show that for all closed curves $\gamma : S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}$, $\int_\gamma \omega = 0$).

7.16. Let $\gamma : [0,1] \longrightarrow \mathbb{R}^2 \setminus \{0\}$ be a regular[12] smooth closed curve and ξ a unit vector. Then γ is *transverse* to the ray $r_\xi = \{t\xi : t \geq 0\}$ if, at any intersection point $\gamma(t_0),\quad \dot{\gamma}(t_0)$ and ξ are linearly independent. Moreover, such an intersection point is said to be *positive* if $\{\xi, \dot{\gamma}(t_0)\}$ is a positive basis for $\mathbb{R}^2$, and negative otherwise. Suppose γ intersects r_ξ transversally.

(1) Prove that the intersection points are isolated, in particular the number of such points is finite.
(2) Prove that $w(\gamma) = P - N$ where P (resp. N) is the number of positive (resp. negative) intersection points.

Remark. It is known that "most" rays intersect γ transversally.

7.17. Let $p \in \mathbb{R}^2$. Consider the angle form based at p,

$$\omega_p = \frac{-(y - y(p))}{(x - x(p))^2 + (y - y(p))^2}\mathrm{d}x + \frac{(x - x(p))}{(x - x(p))^2 + (y - y(p))^2}\mathrm{d}y\ \in \Omega^1(\mathbb{R}^2 \setminus \{p\}).$$

[12] i.e. $\dot{\gamma}(t) \neq 0$.

Let $\gamma : [0,1] \longrightarrow \mathbb{R}^2$ be a closed smooth curve and $p \in \mathbb{R}^2 \setminus \gamma([0,1])$. Define the *index of* p *(relative to* γ*)* as

$$i(p) = \int_\gamma \omega_p.$$

(1) Prove that $i(p)$ is a locally constant function, in particular is constant on each connected component of $\mathbb{R}^2 \setminus \gamma([0,1])$.

(2) Prove that, if γ is a smooth Jordan curve, $i(p) = 0$ (resp. $i(p) = \pm 1$), if p is in the unbounded (resp. bounded) component of $\mathbb{R}^2 \setminus \gamma([0,1])$.

7.18. Let $\gamma : [0,1] \longrightarrow \mathbb{R}^2$ be a Jordan curve. Then, by Theorem 5.8 of Chapter 1, $\mathbb{R}^2 \setminus \gamma([0,1])$ has two connected components. It is easily seen that one component is bounded and the other is unbounded (see Exercise 7.16). Assume the following (non trivial) result

THEOREM. [Shoenflies Theorem] *The bounded component of* $\mathbb{R}^2 \setminus \gamma([0,1])$ *is homeomorphic to a disk.*

Let $U \subseteq \mathbb{R}^2$ be an open set such that $H^1(U) = \{0\}$. Prove that any smooth Jordan curve $\gamma : S^1 \longrightarrow U$ is homotopic, in U, to a constant curve (hint: by the Theorem above, $\gamma(S^1)$ is the boundary of a disk $D \subseteq \mathbb{R}^2$. If the disk is contained in U, the curve is contractible in U, by Exercise 7.5. If not, use the angle form based at $p \in D \setminus U$ to get a contradiction).

REMARK: This fact implies that U is simply connected (see Remark 5.7) hence diffeomorphic to $\mathbb{R}^2$, by the Riemann mapping Theorem.

7.19. Let $\gamma : S^1 \longrightarrow R^2 \setminus \{0\}$ be an *odd* closed curve, i.e. $\gamma(-t) = -\gamma(t), \quad \forall\, t \in S^1$. Prove that $w(\gamma)$ is an odd integer.

7.20. Prove the following Theorem of Borsuk: if $f, g : S^2 \longrightarrow \mathbb{R}$ are *odd* continuous functions, there exists $p \in S^2$ such that $f(p) = 0 = g(p)$ (hint: use the projection of the closed upper hemisphere onto the unit disk to define a function of the disk in $\mathbb{R}^2$).

7.21. Let $f, g : S^2 \longrightarrow \mathbb{R}$ be continuous functions. Prove that there exists $p \in S^2$ such that $f(p) = f(-p),\ g(p) = g(-p)$.

7.22. Prove that there are no injective continuous functions $F : S^2 \longrightarrow \mathbb{R}^2$.

7.23. Let $U \subseteq \mathbb{R}^2$ be an open set and $X : U \longrightarrow \mathbb{R}^2$ a smooth vector field. Let $D_\epsilon \subseteq U$ be a disk of radius ϵ, with center $p \in U$, and assume that $X(q) \neq 0,\ \forall q \in D_\epsilon \setminus \{p\}$. The point p is called an (*isolated*) *singularity* of X. The *index* of X at p, $\ i(X,p)$, is defined as the degree of $X|_{D_\epsilon}$, i.e. the winding number of the curve $X(p + \epsilon \cos 2\pi t, p + \epsilon \sin 2\pi t),\ t \in [0,1]$.

(1) Let $\gamma : [0,1] \longrightarrow U$ be a piecewise smooth, positively oriented closed Jordan curve bounding a disk in U and containing p in its interior. Prove that $i(X,p)$ is the winding number of $X \circ \gamma$.

(2) If $X(x,y) = (f(x,y), g(x,y))$, prove that

$$i(x,p) = \frac{1}{2\pi} \int_\gamma \theta,$$

where γ is as in the preceding item and

$$\theta = \frac{-g\mathrm{d}x}{f^2+g^2} + \frac{f\mathrm{d}y}{f^2+g^2} = X^*\omega,$$

where ω is the angle form.

(3) Prove that if $X(p) \neq 0$, then $i(X, p) = 0$.

(4) Let $X : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear isomorphism. Prove that $i(X, 0) = 1$ if $\det X > 0$ and $i(X, 0) = -1$ if $\det X < 0$.

(5) Assume that $X(p) = 0$ and $\mathrm{d}X(p)$ is invertible. In this case we say that p is a *simple* singularity of X, *positive*, if $\det \mathrm{d}X(p) > 0$, *negative* otherwise. Prove that a simple singularity is isolated and $i(X, p) = \pm 1$, depending on whether p is a positive or negative simple singularity (hint: by Taylor's formula $X(q) = \mathrm{d}X(0)(q) + R(q)\|q\|$, with $\lim_{q\to 0} R(q) = 0$. Prove that $H(q, s) = \mathrm{d}X(0)(q) + (1 - t)R(q)\|q\| \neq 0$, if $\|q\|$ is sufficiently small. Hence...).

(6) Prove the following formula, called the *Kronecker formula.*
Let $D \subseteq \mathbb{R}^2$ be a closed disk, with center q and radius r, and $X : D \longrightarrow \mathbb{R}^2$ be a vector field with only simple singularities, none of which is in ∂D. Then
$$\frac{1}{2\pi}\int_\gamma \theta = P - N,$$
where $\gamma(t) = p + r(\cos 2\pi t, \sin 2\pi t)$, P is the number of the positive singularities and N the number of the negative ones.

Remark: The condition $i(X, p) = 0$ does not imply $X(p) \neq 0$ (find an example!). However, if $i(X, p) = 0$, given $\epsilon > 0$, we can find a vector field $\tilde{X}$ which coincides with X outside a disk of radius ϵ and center p and has no zeros in that disk.

7.24. Let $f : U \subseteq \mathbb{C} = \mathbb{R}^2 \longrightarrow \mathbb{C}$ be a holomorphic function (see Exercise 7.33 of Chapter 1), $f = u + iv$.

(1) Prove the following result
Theorem: [Cauchy Theorem] If U is simply connected and $\gamma : S^1 \longrightarrow U$ is a closed piecewise smooth curve then
$$\int_\gamma f(z)\mathrm{d}z := \int_\gamma (u\mathrm{d}x - v\mathrm{d}y) + i\int_\gamma (u\mathrm{d}y + v\mathrm{d}x) = 0.$$

(2) Suppose that $f'(z) \neq 0$ for z in a disk $D \subseteq U$ and $f(z) \neq 0$ for $z \in \partial U$. Prove that the number of zeros in D is given by
$$\frac{1}{2\pi i}\int_{\partial D}\frac{\mathrm{d}f}{f}$$
(hint: prove that the singularities of the vector field $X(x, y) = (u(x, y), (v(x, y))$ are all simple and positive. Then....).

7.25. Prove that any *continuous* curve $\gamma : [a, b] \longrightarrow \mathbb{R}^2 \setminus \{0\}$ admits angular functions (hint: use polar coordinates to prove the claim when the image of γ is contained in a half plane. Then...). Extend Theorem 5.14, Definition 5.16 and Theorem 5.18 to the case of continuous functions.

7.26. (*For this and the next two Exercises, U can be any topological space*) Let $U \subseteq \mathbb{R}^m$ be an open set and let $\alpha, \beta : [0, 1] \longrightarrow V$ be continuous curves with $\alpha(1) = \beta(0)$. Define the *product* $\alpha * \beta$ as
$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & \text{se } 0 \leq t \leq \frac{1}{2} \\ \beta(2t - 1) & \text{se } \frac{1}{2} \leq t \leq 1 \end{cases}$$
and $\alpha^{-1}(t) = \alpha(1 - t)$. Assuming that the products below are well defined, prove that

(1) If $\alpha_1 \sim \beta_1$, $\alpha_2 \sim \beta_2$, then $\alpha_1 * \alpha_2 \sim \beta_1 * \beta_2$.
(2) $(\alpha * \beta) * \gamma \sim \alpha * (\beta * \gamma)$.
(3) $\alpha * \epsilon_p \sim \alpha \sim \epsilon_p * \alpha$.
(4) $\alpha * \alpha^{-1} \sim \epsilon_p \sim \alpha^{-1} * \alpha$.

where the homotopies are relative to the endpoints and ϵ_p is the constant curve $\epsilon_p(t) = p$.
Hint: consider the homotopies

(1)

$$H(t,s) = \begin{cases} H_1(2t,s) & \text{se } 0 \leq t \leq \frac{1}{2} \\ H_2(2t-1,s) & \text{se } \frac{1}{2} \leq t \leq 1 \end{cases}$$

where H_i are homotopies between α_i e β_i.

(2)

$$H(t,s) = \begin{cases} \alpha(\frac{4t}{s+1}) & \text{se } 0 \leq 4t \leq \ s+1 \\ \beta(4t-s-1) & \text{se } s+1 \leq 4t \leq s+2 \\ \gamma(\frac{4t-s-2}{2-s} & \text{se } s+2 \leq 4t \leq 4 \end{cases}$$

(3)

$$H(t,s) = \begin{cases} \epsilon_p(t) & \text{se } 0 \leq 2t \leq 1-s \\ \alpha(\frac{2t-1+s}{1+s}) & \text{se } 1-s \leq 2t \leq 2 \end{cases}$$

(4)

$$H(t,s) = \begin{cases} \alpha(2t) & \text{se } 0 \leq t \leq \frac{1-s}{2} \\ \alpha(1-s) & \text{se } \frac{1-s}{2} \leq t \leq \frac{1+s}{2} \\ \alpha^{-1}(2t-1) & \text{se } \frac{1+s}{2} \leq t \leq 1 \end{cases}$$

7.27. Let $U \subseteq \mathbb{R}^n$ be an open set and $p \in U$. Consider the set $\Omega(U,p) = \{\gamma : [0,1] \longrightarrow U : \gamma \text{ is continuous and } \gamma(0) = \gamma(1) = p\}$. Prove that $*$ induces a group structure on the quotient set $\pi_1(U,p) := \Omega(U,p)$ modulo the equivalence relation $\alpha \sim \beta$ if and only if α, β are homotopic relative to the endpoints (see Exercise 7.13).

Remark. With this structure, $\pi_1(U)$ is called the *fundamental group* of U with respect to p.

7.28. Prove that an open set $U \subseteq \mathbb{R}^n$ is simply connected if and only if $\pi_1(U)$ is trivial.

Remark. The equivalence relation that defines $\pi_1(U,p)$ is the one of *based homotopy*, i.e. two closed curves in $\Omega(U,p)$ are equivalent if there is an homotopy H between them such that $H(0,s) = H(1,s) = p$. This is not the same as free homotopy between closed curves. It is easy to see, and we invite the reader to prove it, that if two curves $\alpha,\ \beta \in \Omega(U,p)$ are freely homotopic, as closed curves, then they define conjugate elements in $\pi_1(U,p)$, i.e. there exist $[\gamma] \in \pi_1(U,p)$ such that $[\alpha] = [\gamma][\beta][\gamma]^{-1}$.

7.29. Let $F : S^n \longrightarrow S^n$ be a smooth function and $\tilde{F} : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}^{n+1} \setminus \{0\}$, $\tilde{F}(tx) = tF(x)$. Then we have an induced linear map $\tilde{F}_* : H_n(\mathbb{R}^{n+1} \setminus \{0\}) \cong \mathbb{R} \longrightarrow H_n(\mathbb{R}^{n+1} \setminus \{0\}) \cong \mathbb{R}$. This map is multiplication by a real number $dg(F)$, called the *degree* of F. It is known that $dg(F) \in \mathbb{Z}$ [13] . Let D^{n+1} be the unit disk and $G : D^{n+1} \longrightarrow \mathbb{R}^{n+1}$ a smooth function not vanishing on the unit sphere $S^n = \partial D^{n+1}$.

[13] It follows, from homotopy invariance, that homotopic maps have the same degree. A basic fact in homotopy theory is the Theorem of Hopf: if two maps from S^n to S^n have the same degree, then they are homotopic.

Then the degree of G, $dg(G)$, is defined as the degree of the map $\tilde{G}(x) = \frac{G(x)}{\|G(x)\|}$. Prove that, if $dg(G) \neq 0$, then the equation $G(x) = 0$ has a solution.

7.30. Prove that there are no smooth maps $F : D^{n+1} \longrightarrow S^n = \partial D^{n+1}$ such that $F(x) = x \ \ \forall x \in S^n$. Use this fact to prove the celebrated result

THEOREM. [Brouwer fixed point Theorem] Any continuous map $G : D^{n+1} \longrightarrow D^{n+1}$ has a fixed point, i.e. a point $x \in D^{n+1}$ such that $G(x) = x$
(hint for the Brouwer fixed point Theorem: suppose $G(x) \neq x \ \ \forall \ x \in D^{n+1}$. For x D^{n+1} consider the ray starting at $G(x)$ containing x and define $F(x)$ to be the intersection of this ray with S^n. Then ...).

7.31. Use Exercise 7.29 to define the index of a vector field $X : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ at a point $p \in U$ and try to extend, as much as you can, the facts claimed in Exercise 7.23 for this situation.

7.32. Let $L : E_1 \longrightarrow \mathbb{E}_2$ be a linear map and let $\mathbb{F}$ be a given vector space. Prove that $\ker(L \otimes \mathbb{1}) = \ker(L) \otimes \mathbb{F}$ and $\mathrm{Im}(L \otimes \mathbb{1}) = \mathrm{Im}(L) \otimes \mathbb{F}$. Prove Proposition 4.5.

7.33. Consider, in $\mathbb{R}^3$, the points $P_0 = (0,0,0),\ P_1 = (-1,0,0),\ P_2 = (1,0,0)$. Let $S_i = P_i + te_3,\ i = 1,2$. Consider the open set $U = \mathbb{R}^3 \setminus \{P_0 \cup S_1 \cup S_2\}$. Let $V_j, \quad j = 0,1,2$ be the open sets

$$V_0 = \{(x,y,z) \in \mathbb{R}^3 : -\frac{2}{3} < x < \frac{2}{3}\} \cap U,\ \ V_1 = \{(x,y,z) \in \mathbb{R}^3 : x < -\frac{1}{3}\} \cap U,\ \ V_2 = \{(x,y,z) \in \mathbb{R}^3 : x > \frac{1}{3}\} \cap U.$$

Clearly $U = V_0 \cup V_1 \cup V_2$.

(1) Prove that $V_0 \sim \mathbb{R}^3 \setminus \{0\},\ V_i \sim \mathbb{R}^3 \setminus S_i,\ i = 1,2,\ V_0 \cap V_i \sim \mathbb{R}^3, i = 1,2.$
(2) Use the Mayer Vietoris sequence (twice) to prove that the restriction homomorphism
$$r_p : \Omega^p(U) \longrightarrow \Omega^p(V_0) \oplus \Omega^p(V_1) \oplus \Omega^p(V_2), \quad r_p(\omega) = (\omega|_{V_0}, \omega|_{V_1}, \omega|_{V_2}),$$
indices an isomorphism in cohomology. Conclude that $H^*(U)$ is spanned, as a vector space, by $1 \in H^0(U), r_1^*([\alpha_1]), r_1^*[\alpha_2] \in H^1(U)$ and $r_2^*([\omega])$ where $[\alpha_i]$ is a generator of $H^1(V_i)$ and $[\omega]$ is a generator of $H^2(V_0)$.
(3) Prove that $[\alpha_1 \wedge \alpha_2] = 0 \in H^2(U)$. Conclude that U and $W = \mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2 \setminus \{0\} \subseteq \mathbb{R}^4$ have cohomology that are isomorphic as vector spaces, but *not* as algebras. In particular the two sets are not homotopy equivalent (see Example 4.10 and Remark 4.11).

REMARK. Naturally U, W do not have the same dimension. To have an example of open sets of the same dimension, with isomorphic cohomology (as vector spaces) but not homotopy equivalent, just take $U \times \mathbb{R}$.

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