Deconstructing the electron clock

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The Dirac equation is reinterpreted as a constitutive equation for the vacuum, with the electron modeled as a point singularity in a lightlike toroidal vortex. The singularity generates electric and magnetic potentials with circular zitterbewegung around the spin axis. These fields are then propagated by Maxwell's equation. The result is an integrated *Maxwell-Dirac* field theory proposed as a non-perturbative approach to quantum electrodynamics with implications for the electron's anomalous magnetic moment and the structure of the photon. Extension to a "Standard Model" of elementary particles as vacuum singularities is discussed.

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I. INTRODUCTION

The spectacular success of quantum electrodynamics (QED) gives physicists great confidence in Maxwell's equation on the one hand and Dirac's equation on the other, yet something is missing in relations between them. With his usual penetrating insight, Einstein focused on the crux of the problem [1, 2]: "It is a delusion to think of electrons and the fields as two physically different, independent entities. Since neither can exist without the other, there is only *one* reality to be described, which happens to have two different aspects; and the theory ought to recognize this from the start instead of doing things twice."

This paper proposes a synthesis of Maxwell and Dirac theories based on a new model for singularities in the electromagnetic vacuum. The model is suggested by a remarkable relation between electron mass and vacuum polarization proposed by Seymour Blinder. The only requirement is consistency with Maxwell's equation. No changes in the form of Dirac or Maxwell equations are necessary, but the two are fused at the source. The solutions seamlessly integrate electron field and particle properties along lines proposed by de Broglie. They answer Einstein's call for a unified electron theory with a unified *Maxwell-Dirac* theory.

Singular toroidal solutions of the Dirac equation constitute a new class of wave functions, fairly called *ontic states* (or "states of reality" as Einstein might have put it), because they have a definite physical interpretation in terms of local observables of the electron and associated deformation of the vacuum. No probabilities are involved. Electron states are thus characterized by a literal *field-particle duality*: field and particle coexist as a real physical entity. This appears to finesse the notorious self-energy problem. It implies there is no such thing as the electron's own field acting on itself, because particle and field are two different aspects of one and the same thing.

Section II reviews the formulation of classical electromagnetic theory in terms of Spacetime Algebra (STA) to provide a context for two important new developments. The first is Blinder's concept of a classical vacuum singularity. The second is Antonio Rañada's discovery of toroidal solutions to Maxwell's equation. The two provide complementary inputs to a new theory of the electron and the electromagnetic vacuum.

Section III begins with a review and extension of the Zitter particle model for the electron clock in [3] and updated in the preceding paper [4]. That sets the stage in Section IV for the main subject of this paper, namely, a reconstruction of the Dirac equation as a constitutive equation for the vacuum with the electron as a point singularity. The singularity generates the electron's Coulomb field with circular zitterbewegung and a toroidal magnetic field with the spin vector as its axis. I call this approach *Maxwell-Dirac* theory. It is complementary to the conventional *Born-Dirac* theory discussed in [4] in the sense that motion is described by the same Dirac equation in both. In Born-Dirac the electron charge is inert and responds only to action of external fields. In Maxwell-Dirac the charge is active and generates an electromagnetic field. In this sense, Maxwell-Dirac may be regarded as an alternative to second quantization, though we do not directly consider its relation to standard QED. We do show, however, that Maxwell-Dirac incorporates Oliver Consa's physical explanation for the electron's anomalous magnetic moment with impressive quantitative accuracy [5]. That itself should be sufficient to justify further study of QED implications. Moreover, it leads directly to a proposed new theory of the photon as an electron-positron singular state.

Section V discusses possibilities for modeling all elementary particles as vacuum singularities, and implications for gravitation theory are pointed out.

Considering the enormous scope of Maxwell-Dirac theory, our treatment has many loose ends and is best regarded as a guide for further research.

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II. CLASSICAL ELECTROMAGNETICS

To place our analysis in the most general context, we begin with an STA formulation of Maxwell's electrodymamics in accord with the authoritative presentation by Sommerfeld [6].

In STA an *electromagnetic field* is represented by a bivector-valued function $F = F(x)$ on spacetime, appropriately called the *Faraday*. Its split into electric and magnetic fields is given by

$$F = \mathbf{E} + i\mathbf{B}. \tag{1}$$

In a polarizable medium, the *electromagnetic field density* is a bivector field $G = G(x)$ with the split into electric and magnetic densities given by

$$G = \mathbf{D} + i\mathbf{H}. \tag{2}$$

The important distinction between "field" and "field density" or "excitation" is emphasized by Sommerfeld. We are interested in G for describing properties of the vacuum.

The most general possible version of *Maxwell's equation* for the electromagnetic field is

$$\Box F = J_e + iJ_m\,, \tag{3}$$

where $J_e = J_e(x)$ is the *electric charge current* and $J_m = J_m(x)$ is a *magnetic charge current*. Separating vector and pseudovector parts, we get

$$\Box \cdot F = J_e \tag{4}$$

and

$$\Box \wedge F = iJ_m. \tag{5}$$

Using the *duality* between divergence and curl

$$(\Box \wedge F)i = \Box \cdot (iF) \tag{6}$$

and the anticommutivity of the pseudoscalar with vectors, the latter equation can be written

$$\Box \cdot (iF) = J_m. \tag{7}$$

In the most general case, the Faraday F can be derived from a "complex" pair of vector potentials $A = A(x)$ and $C = C(x)$, so we have

$$F = \Box(A + Ci). \tag{8}$$

The scalar and pseudoscalar parts of this equation give us

$$\Box \cdot A = 0 = \Box \cdot C, \tag{9}$$

while equations (4) and (7) give us separate equations for fields produced by electric and magnetic charges:

$$\Box^2 A = J_e, \tag{10}$$

$$\Box^2 C = J_m. \tag{11}$$

Though we shall dismiss the magnetic monopole current J_m as unphysical, we shall see good reason to keep the complex vector potential for radiation fields.

Squaring the Faraday gives us scalar and pseudoscalar invariants which can be expressed in terms of electric and magnetic fields:

$$F^2 = (\mathbf{E} + i\mathbf{B})^2 = \mathbf{E}^2 - \mathbf{B}^2 + 2i\mathbf{E} \cdot \mathbf{B}. \tag{12}$$

As first shown in [7], if either of these invariants is nonzero, they can be used to put the Faraday in the unique invariant form:

$$F = \mathbf{f}e^{i\varphi} = \mathbf{f}\cos\varphi + i\mathbf{f}\sin\varphi, \tag{13}$$

where $\mathbf{f}$ is a simple timelike bivector, and the exponential specifies a duality transformation through an angle given by

$$\tan 2\varphi = \frac{2\mathbf{E} \cdot \mathbf{B}}{\mathbf{E}^2 - \mathbf{B}^2} = \frac{F \cdot F}{iF \wedge F}, \tag{14}$$

Note that (13) determines a rest frame in which the electric and magnetic fields are parallel without using a Lorentz transformation. In addition, the squared magnitude of $\mathbf{f}$ is

$$\mathbf{f}^2 = [(\mathbf{E}^2 + \mathbf{B}^2) - 4(\mathbf{E} \times \mathbf{B})^2]^{\frac{1}{2}}, \tag{15}$$

which is an invariant of the Poynting vector for F.

A null field can also be put in the form

$$F = \mathbf{f}e^{i\varphi} \quad \text{with} \quad \mathbf{f}^2 = 0, \tag{16}$$

so we can write

$$\mathbf{f} = \mathbf{e} + i\mathbf{b} = \mathbf{e}(1 + \hat{\mathbf{k}}) = \mathbf{f}\hat{\mathbf{k}}. \tag{17}$$

Hence we can put the null field into the form

$$F = \mathbf{f}e^{i\varphi} = \mathbf{f}e^{i\hat{\mathbf{k}}\varphi}, \tag{18}$$

showing that the duality rotation is equivalent to a rotation of vectors in the null plane. Of course, this is significant for description of radiation fields.

A. Conservation Laws

Using the identity

$$\Box \cdot (\Box \cdot F) = (\Box \wedge \Box) \cdot F = 0, \tag{19}$$

from (4) and (7), we get the current conservation laws

$$\Box \cdot J_e = 0 \quad \text{and} \quad \Box \cdot J_m = 0. \tag{20}$$

An energy-momentum tensor $T(n) = T(n(x), x)$ describes the energy-momentum flux in direction of a unit normal n at spacetime point x. As discussed elsewhere

[8, 9], the electromagnetic energy-momentum tensor is given by

$$T(n) = \frac{1}{2}\langle Fn\widetilde{G}\rangle_1 = \frac{1}{4}[Fn\widetilde{G} + Gn\widetilde{F}]. \qquad (21)$$

where n is a unit normal specifying the direction of flux. With some algebra, the tensor can be expressed in the alternative form, which separates normal and tangential fluxes:

$$T(n) = \frac{1}{2}[(G \cdot \widetilde{F})n + G \cdot (n \cdot \widetilde{F}) + (G \cdot n) \cdot \widetilde{F}]. \qquad (22)$$

Whence,

$$\partial_n T(n) = \partial_n \cdot T(n) + \partial_n \wedge T(n) = 0. \qquad (23)$$

The vanishing of both scalar and bivector parts in this expression tells us the linear function $T(n)$ is traceless and symmetric.

The divergence of the energymomentum tensor is given by

$$\dot{T}(\dot{\Box}) = \tfrac{1}{2}\langle \dot{F}\dot{\Box}\widetilde{G} + F\Box\widetilde{G}\rangle_1 = \tfrac{1}{2}[G \cdot J_e + F \cdot J_f], \qquad (24)$$

where current J_f includes any polarization or magnetization currents. Note use of the overdot to indicate differentiation to the left.

Physical interpretation of the energymomentum tensor is perhaps facilitated by using a *v-split* to put it in the form

$$T(v) = \frac{1}{4}[FG^\dagger + GF^\dagger]v. \qquad (25)$$

From this we find the energy density

$$T(v) \cdot v = \tfrac{1}{2}F \cdot \widetilde{G} = \tfrac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}), \qquad (26)$$

and the momentum density

$$T(v) \wedge v = \tfrac{1}{2}(\mathbf{D} \times \mathbf{B} + \mathbf{E} \times \mathbf{H}). \qquad (27)$$

For a unit normal $\mathbf{n} = nv$ orthogonal to v, we have

$$T(n)v = \frac{1}{4}[F\mathbf{n}G^\dagger + G\mathbf{n}F^\dagger], \qquad (28)$$

so

$$T(n) \cdot v = \tfrac{1}{2}(\mathbf{D} \times \mathbf{B} + \mathbf{E} \times \mathbf{H}) \cdot \mathbf{n}, \qquad (29)$$

and the spatial flux in direction $\mathbf{n}$ is

$$\boldsymbol{T}(\boldsymbol{n}) = T(n) \wedge v = \frac{1}{4}\langle \mathbf{EnD} + \mathbf{BnH}\rangle_{\mathbf{1}}. \qquad (30)$$

The treatment of energymomentum conservation in this subsection is completely general, applying to models of the electromagnetic vacuum considered next as well as material media. It sets the stage for specific applications considered next as well as extensions to be considered in the future.

B. The Classical Vacuum

In a thorough analysis of constitutive equations in Maxwell's electrodynamics, E. J. Post [10] identified a hitherto unrecognized degree of freedom in Maxwell's equation for the vacuum. Regarding the vacuum as a dielectric medium with variable permittivity $\varepsilon = \varepsilon(x)$ and permeability $\mu = \mu(x)$ at each spacetime point x, Maxwell's condition for the propagation of light in a vacuum is given by

$$\varepsilon\mu = 1/c^2 = \varepsilon_0\mu_0. \qquad (31)$$

Obviously, this leaves the *impedance*

$$Z = \sqrt{\frac{\mu}{\varepsilon}} = Z(x) \qquad (32)$$

as an undetermined function. To ascertain its value, Post further argues that charge should be regarded as an independent unit q_e rather than the derived unit e. The standard rule for changing units is

$$e^2 = \frac{q_e^2}{4\pi\varepsilon_0}. \qquad (33)$$

Accordingly, the *fine structure constant* is given by

$$\alpha_e = \frac{e^2}{\hbar c} = \frac{q_e^2}{4\pi\varepsilon_0\hbar c} = \frac{q_e^2}{4\pi\hbar}\sqrt{\frac{\mu_0}{\varepsilon_0}}. \qquad (34)$$

Hence, as Post observes, the fine structure constant can be expressed as a ratio of two generally invariant impedances:

$$\alpha_e = \frac{Z_0}{Z_H}, \quad \text{where} \quad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}, \quad Z_H = \frac{2h}{q_e^2}. \qquad (35)$$

This suggested to Post that the *Hall impedance* Z_H is an intrinsic property of the electromagnetic vacuum.

Blinder [11, 12] has shown that polarization of the vacuum in the neighborhood of a classical electron is uniquely determined by the very simple assumptions that (1) the energy density of the electron field is proportional to the charge density, and (2) the total energy in the field determines the electron mass. We review Blinder's argument to serve as a guide for generalizing it to the Dirac electron.

For a point charge in its rest frame the electric field $\mathbf{E}$ and electric displacement $\mathbf{D}$ are given by

$$\mathbf{E} = \frac{q_e}{4\pi\varepsilon(r)}\frac{\mathbf{r}}{r^3}, \qquad \mathbf{D} = \varepsilon(r)\mathbf{E} = \frac{q_e}{4\pi}\frac{\mathbf{r}}{r^3}, \qquad (36)$$

where $r = |\mathbf{r}|$. The total energy in the field is

$$W = \frac{1}{2}\int \mathbf{E} \cdot \mathbf{D}\, d^3r = \frac{1}{32\pi^2}\int_0^\infty \frac{1}{\varepsilon(r)}\frac{q_e^2}{r^4}\, 4\pi r^2 dr. \qquad (37)$$

The charge density ϱ is determined by

$$\frac{\varrho}{\varepsilon_0} = \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{-q_e}{4\pi}\left[\frac{\varepsilon'(r)}{r^2[\varepsilon(r)]^2} + \frac{\delta^3(\mathbf{r})}{\varepsilon(0)}\right] \qquad (38)$$

with $\varepsilon' = \partial_r \varepsilon$ and the normalization

$$\int_0^\infty \varrho(r)\, 4\pi r^2 dr = -q_e \int_0^\infty \frac{\varepsilon_0 \varepsilon'(r) dr}{[\varepsilon(r)]^2} = q_e, \qquad (39)$$

which require that $\varepsilon(\infty) = \varepsilon_0$ and $\varepsilon(0) = \infty$.

Finally, assuming that $W = m_e c^2$ and the charge density in (38) is proportional to the energy density in (37), we get

$$\frac{-\varepsilon_0 \varepsilon'(r)}{r^2 [\varepsilon(r)]^2} = \frac{q_e^2}{32\pi^2 m_e c^2 \varepsilon(r) r^4}. \qquad (40)$$

Whence,

$$\varepsilon(r) = \varepsilon_0 \exp\left(\frac{\lambda_0}{r}\right), \qquad (41)$$

where

$$\lambda_0 = \frac{q_e^2}{8\pi\varepsilon_0} \frac{1}{m_e c^2} = \frac{1}{2} \frac{e^2}{m_e c^2} \qquad (42)$$

is recognized as half the *classical electron radius* and puts that quantity into new perspective as a radius of vacuum polarization.

Now we have an explicit expression for the vacuum charge density:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{q_e}{4\pi\varepsilon_0} \frac{\lambda_0}{r^4} e^{-\lambda_0/r} = \frac{\varrho(r)}{\varepsilon_0}. \qquad (43)$$

Sommerfeld [6] emphasizes that this does not have the dimensions of charge, and he interprets it simply as "divergence of the electric field." Of course, he was not privy to our notion of vacuum polarization or its expression as a manifestly nonsingular quantity. However, the dimensions of charge and its singularity are explicit in

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \frac{-q_e}{4\pi}\, \boldsymbol{\nabla}^2 \left(\frac{1}{r}\right) = q_e \delta^3(\mathbf{r}). \qquad (44)$$

Also note that $\mathbf{E} = -\boldsymbol{\nabla}\varphi_e$, where

$$\varphi_e(r) = \frac{-q_e}{4\pi\varepsilon(r)\lambda_0}. \qquad (45)$$

This suggests a straightforward generalization of Blinder's argument.

By interpreting the variable r as the retarded distance between each spacetime point x and the path $z = z(\tau)$ of a point charge with velocity $v = \dot{z} = c^{-1} dz/d\tau$, the simple form for the scalar potential in (45) leads immediately to the following generalization of the Lìenard-Wiechert potential:

$$A(x) = \frac{q_e}{4\pi\varepsilon(r)} \frac{v}{\lambda_0}. \qquad (46)$$

Let's call this the *"Coulomb vector potential"* to emphasize its relation to the Coulomb scalar potential. The retarded distance is defined explicitly by

$$r = (x - z(\tau)) \cdot v = |\mathbf{r}| \quad \text{with} \quad \mathbf{r} = (x - z(\tau)) \wedge v \qquad (47)$$

subject to the constraint

$$(x - z(\tau))^2 = 0. \qquad (48)$$

Hence, it generates the electromagnetic field

$$F = \Box \wedge A = \frac{q_e}{4\pi\varepsilon(r)} \left\{ v \wedge \Box\frac{1}{r} + \frac{\Box \wedge v}{\lambda_0} \right\}. \qquad (49)$$

To evaluate the derivatives, we use the constraint (48), which implies proper time $\tau = \tau(x)$ as a function of position with gradient

$$\Box\tau = \frac{x - z}{c\, r} \equiv k, \qquad (50)$$

where null vector k is independent of distance r. Consequently, the curl of $v = v(\tau(x))$ has the simple form

$$\Box \wedge v = \Box\tau \wedge \partial_\tau v = c\, k \wedge \dot{v}. \qquad (51)$$

Similiarly, the gradient of (47) gives us

$$\begin{aligned} \Box r &= v(v \wedge \Box r) + \Box\tau\, (x - z) \cdot \dot{v} c \\ &= v\hat{\mathbf{r}} + crkk \cdot \dot{v}. \end{aligned} \qquad (52)$$

Inserting all this into (49), we get

$$\begin{aligned} F &= \Box \wedge A \\ &= \frac{q_e}{4\pi\varepsilon(r)} \left\{ \frac{\mathbf{r}}{r^3} + \frac{ck \cdot \dot{v}}{r} v \wedge k + \frac{ck \wedge \dot{v}}{\lambda_0} \right\}. \end{aligned} \qquad (53)$$

This differs from the classical retarded field [13] only in the last term, wherein the distance r in the denominator is replaced by λ_0. At first sight this seems wrong, because a spherical electromagnetic wave surely attenuates with distance. That may be why Blinder did not associate his vacuum impedance with the electron's vector potential. On the other hand, if Blinder's idea has any relevance to quantum mechanics it must be expressed through the vector potential, because that is the only mechanism for electromagnetic interaction. Indeed, if the last term contributes to photon emission, it should not depend on distance. Let us therefore withhold judgment until we examine how the vacuum impedance might fit into quantum mechanics.

To wind up our discussion of classical theory, we define a generalized displacement field by $G = \varepsilon F$. Whence its divergence for a charge at rest is

$$\Box \cdot G = q_e \delta^3(\mathbf{r}) v = q_e \int_{-\infty}^{\infty} dz \delta^4(x - z(\tau)) = J_e, \qquad (54)$$

in agreement with (44) and (4). These expressions for F and $G = \varepsilon F$ suffice to fit Blinder's model for a point charge in the vacuum into the general formulation of classical electrodynamics in the preceding section.

For a more realistic model of the electron, we need to incorporate electron spin and magnetic moment as well. Blinder [12] tried that with a dipole model of the magnetic moment, and he deduced an exponential form for the magnetic permeability μ analogous to that for ε, but with a different functional dependence. However, that approach is inconsistent with Maxwell's condition (31) relating μ and ε to the speed of light. Instead, from now on, we impose Maxwell's condition by introducing the dimensionless vacuum impedance

$$\epsilon(x) = \sqrt{\frac{\varepsilon}{\mu}}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{\varepsilon}{\varepsilon_0} = \frac{\mu_0}{\mu} = e^{\lambda_0/r} \tag{55}$$

I call the inverse of this function with some given value for λ_0 a *Blinder function* in recognition of Blinder's seminal contribution.

C. Toroidal Radiation Fields

A remarkable new family of null solutions to Maxwell's equation was discovered by Antonio Rañada in 1989 [14, 15] and subsequently reviewed and extended in [16, 17]. He called them "knotted radiation fields" because electric and magnetic field lines are interlocked in toroidal knots that persist as fields propagate. As beautifully described in [18, 19], this has opened up an exciting new thread of research on electromagnetic radiation.

Many researchers have discovered advantages in formulating the radiation field as a "complex vector" $F = \mathbf{E} + i\mathbf{B}$ with the null condition $F^2 = F \circ F = 0$, but only Enk [20] has formulated it with geometric algebra to show that the imaginary unit must be interpreted as the unit pseudoscalar of spacetime. We shall demonstrate that STA has further advantages preparing the way for extending the theory to include toroidal fields described by the Dirac equation in a later Section.

As emphasized by Kholodenko [21, 22], a crucial element in an electromagnetic knot is a "self-generating" vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$ described by the "eigenvector equation"

$$\boldsymbol{\nabla} \times \mathbf{v} = \kappa\mathbf{v}. \tag{56}$$

This equation has been known since the nineteenth century as the *Beltrami equation* and employed to model vorticity in fluids. It appears again in magnetohydrodynamics, where it is called the *force-free equation.* And in superconductivity it is known as the *London equation.* Kholodenko [23] has reviewed the vast literature on the subject across mathematics as well as physics from the unifying perspective of *contact geometry.*

The Beltrami equation is even inherent in the free field Maxwell equation [22]. Indeed, using the spacetime split, Maxwell's equation can be written

$$\partial_0 F = -\boldsymbol{\nabla} F. \tag{57}$$

For the unique reference frame and form specified by (13), we can write

$$F(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t)e^{i\varphi(\mathbf{x},t)}. \tag{58}$$

Inserting this into Maxwell's equation and using an overdot for the time derivative, we have

$$(\dot{\mathbf{v}} + i\dot{\varphi}\mathbf{v})e^{i\varphi} = -(\boldsymbol{\nabla}\mathbf{v} + i(\boldsymbol{\nabla}\varphi)\mathbf{v})e^{i\varphi}. \tag{59}$$

Cancelling exponentials and separately equating even and odd parts, we get

$$+i\dot{\varphi}\mathbf{v} = -\boldsymbol{\nabla}\mathbf{v}, \tag{60}$$

and

$$\dot{\mathbf{v}} = -i(\boldsymbol{\nabla}\varphi)\mathbf{v}. \tag{61}$$

Writing $\kappa = -\dot{\varphi}$, we put (60) in the form

$$+\kappa\mathbf{v} = -i\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{\nabla} \times \mathbf{v}. \tag{62}$$

As advertised, this is precisely the Beltrami equation, with a side condition $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$. If κ is constant, we can differentiate again to get an equation of familiar "Helmholtz type:"

$$(\boldsymbol{\nabla}^2 + \kappa^2)\mathbf{v} = 0. \tag{63}$$

Time dependence of the vector field is governed by eq. (61), which can be written

$$\dot{\mathbf{v}} = -\mathbf{v} \times \boldsymbol{\nabla}\varphi, \tag{64}$$

with the side condition $\mathbf{v} \cdot \boldsymbol{\nabla}\varphi = 0$. Besides identifying a role for Beltrami's equation, this analysis serves to demonstrate some advantages of geometric calculus and the way it incorporates standard vector calculus.

As emphasized by Enk [20], it follows immediately from Maxwell's equation (57), that there is a whole hierarchy of "parabivector" fields $F_n \equiv \boldsymbol{\nabla}^n F$ that satisfy

$$\partial_0 F_n = -\boldsymbol{\nabla} F_n. \tag{65}$$

In particular, with $F = \boldsymbol{\nabla} F_0$ we can define a vector potential

$$F_0 = \mathbf{A} + i\mathbf{C} \tag{66}$$

so that $\mathbf{A}$ is the usual magnetic vector potential given by

$$i\mathbf{B} = \boldsymbol{\nabla}\mathbf{A} = i(\boldsymbol{\nabla} \times \mathbf{A}) \tag{67}$$

with $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, and $\mathbf{C}$ is an analogous vector potential for the electric field $\mathbf{E}$.

For radiation fields the relation to Beltrami's equation is especially simple. When $F^2 = 0$, we can write

$$F = \mathbf{E} + i\mathbf{B} = \rho(\mathbf{e} + i\mathbf{b})e^{i\varphi}, \tag{68}$$

where ρ is a single scale factor for both electric and magnetic fields so we can set $\mathbf{e}^2 = \mathbf{b}^2 = 1$. Then, from (25), the *Poynting paravector* $\mathcal{P}$ is given by

$$\mathcal{P} = \frac{1}{2}FF^\dagger = \rho^2[\frac{1}{2}(\mathbf{e}^2 + \mathbf{b}^2) - i\mathbf{e}\mathbf{b}] = \rho^2(1 + \mathbf{e} \times \mathbf{b}). \tag{69}$$

For monochromatic radiation, all the time dependence is in the phase, so we can write $\kappa = -\dot{\varphi}$ as before, and Maxwell's equation (57) becomes

$$\boldsymbol{\nabla} F = i\kappa F. \tag{70}$$

Defining "complex" inner and outer products in terms of commutator and anticommutator parts, we can split this into two equations

$$\boldsymbol{\nabla} \times F = \kappa F, \tag{71}$$

$$\boldsymbol{\nabla} \circ F = 0. \tag{72}$$

Thus we see Beltrami's equation in a more fundamental role. Further, defining $F_n = \boldsymbol{\nabla}^n F$, we get

$$\boldsymbol{\nabla} F_n = (i\kappa)^n F_n \tag{73}$$

from (70) with constant κ. Thus we have a whole nest of Beltrami fields.

With this prelude on the structure of radiation fields, we turn to Rañada's seminal insight into the toroidal structure of magnetic fields provided by the celebrated *Hopf fibration*.

Hopf studied smooth maps from the 3-sphere $\mathcal{S}^3$ onto the 2-sphere $\mathcal{S}^2$ using classical techniques of complex variable theory. However, it is simpler and more informative to exploit the fact that $\mathcal{S}^3$ is a 3-dimensional manifold isomorphic to the group $SU(2) = \{U\}$ of unitary quaternions or *rotors*, to use a more descriptive term.

Accordingly, we define a *Hopf map* as a rotor-valued function $U = U(\mathbf{r})$ defined on a dimensionless representation of physical space $\mathcal{R}^3$ with a fixed origin $\mathbf{r} = 0$. Expressed in terms of *Euler variables* $(u_0, \mathbf{u})$ defined and discussed in [24], the rotor is normalized by

$$U = u_0 + i\mathbf{u} \quad \text{with} \quad UU^\dagger = u_0^2 + \mathbf{u}^2 = 1. \tag{74}$$

This determines the orientation of an orthonormal frame of vectors

$$\mathbf{e}_k = U\boldsymbol{\sigma}_k U^\dagger, \tag{75}$$

though the Hopf map requires only that one of them, say $\boldsymbol{\sigma}_3$ (designated by dropping the subscript), serves as a *pole* $\boldsymbol{\sigma}$ and unit normal $\mathbf{n} = U\boldsymbol{\sigma}U^\dagger$ for the sphere $\mathcal{S}^2$.

Accordingly, fixing the pole $\boldsymbol{\sigma}$ on $\mathcal{S}^2$ determines a smooth mapping $\mathbf{n}(\mathbf{r})$ of a unit normal on the surface specified by the rotor function $U(\mathbf{r})$:

$$\begin{aligned}\mathbf{n} = U\boldsymbol{\sigma}U^\dagger &= u_0^2\boldsymbol{\sigma} + 2u_0\boldsymbol{\sigma} \times \mathbf{u} + \mathbf{u}\boldsymbol{\sigma}\mathbf{u} \\ &= (u_0^2 - \mathbf{u}^2)\boldsymbol{\sigma} + 2[u_0\boldsymbol{\sigma} \times \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\sigma})\mathbf{u}],\end{aligned} \tag{76}$$

where the identity $\mathbf{u}\boldsymbol{\sigma} = -\boldsymbol{\sigma}\mathbf{u} + 2\mathbf{u}\cdot\boldsymbol{\sigma}$ has been used to reorder noncommuting vectors. It is crucial to note, however, that the function $\mathbf{n}(\mathbf{r})$ in (76) is uniquely specified by the rotor function $U(\mathbf{r})$ only up to a rotation about the pole, as specified by

$$U_\varphi \boldsymbol{\sigma} U_\varphi^\dagger = \boldsymbol{\sigma} \quad \text{where} \quad U_\varphi = e^{i\boldsymbol{\sigma}\varphi/2}. \tag{77}$$

Let's call this *toroidal gauge invariance*. This implies that $\mathbf{n} = UU_\varphi\boldsymbol{\sigma}U_\varphi^\dagger U^\dagger = U\boldsymbol{\sigma}U^\dagger$, and thus reduces the degrees the freedom for $\mathbf{n}(\mathbf{r})$ from three to two, while maintaining its smooth covering of the entire sphere. This completes our formulation of the Hopf map using rotors in geometric algebra.

Rañada had the great insight to use the Hopf map to model magnetic field lines. The essential idea is already contained in Hopf's original example for a map. Hopf recognized that $\mathcal{S}^3$ is isomorphic to $\mathcal{R}^3$ by stereographic projection, as expressed by

$$2\mathbf{x}/\lambda_0 = \frac{\mathbf{u}}{1 - u_0}, \tag{78}$$

where λ_0 is a length scale factor. Using (74) and writing $2\mathbf{r} = \mathbf{x}/\lambda_0$, this can be inverted to give

$$\mathbf{u} = \frac{2\mathbf{r}}{r^2 + 1}, \quad u_0 = \frac{r^2 - 1}{r^2 + 1} \quad \text{with} \quad r^2 = \mathbf{r}^2. \tag{79}$$

Thus, we have the explicit function

$$U(\mathbf{r}) = (r^2 + 1)^{-1}[(r^2 - 1) + 2i\mathbf{r}]. \tag{80}$$

This can be inserted into (76) to give us an explicit example of a Hopf map, which has been thoroughly studied in [25].

Actually, we can eliminate the stereographic projection to produce an even simpler version of the Hopf map where the rotor is normalized by

$$U(\mathbf{r}) = \sqrt{\lambda}(1 + i\mathbf{r}) \quad \text{with} \quad UU^\dagger = \lambda(1 + \mathbf{r}^2) = 1. \tag{81}$$

Then, the normal map $\mathbf{n}(\mathbf{r})$ on $\mathcal{S}^2$ is given by

$$\begin{aligned}\mathbf{n} = U\boldsymbol{\sigma}U^\dagger &= \lambda\{\boldsymbol{\sigma} + 2\boldsymbol{\sigma} \times \mathbf{r} + \mathbf{r}\boldsymbol{\sigma}\mathbf{r}\} \\ &= \lambda\{(1 - \mathbf{r}^2)\boldsymbol{\sigma} + 2[\boldsymbol{\sigma} \times \mathbf{r} + (\mathbf{r}\cdot\boldsymbol{\sigma})\mathbf{r}]\}.\end{aligned} \tag{82}$$

Thus, the normalization for the Hopf map is completely determined by the simple scaling factor $\lambda(\mathbf{r}) = 1/(1 + r^2) = 1/(1+\mathbf{r}^2)$. The significance of this remarkable scaling factor was implicit in Rañada's treatment of knotted radiation fields from the beginning, as is evident in his treatment of a static magnetic field in [16, 25]. In fact, Hopf's original example already suffices to model a magnetic field with a suitable power of the scale factor, as we now demonstrate for purposes of comparison.

Consider the following candidate vector potential:

$$\mathbf{A}(\mathbf{r}) = \tfrac{1}{2}\lambda^2(\boldsymbol{\sigma} \times \mathbf{r} + \boldsymbol{\sigma}). \tag{83}$$

Differentiating

$$\boldsymbol{\nabla}\lambda^2 = -2\lambda^3\boldsymbol{\nabla}r^2 = -4\lambda^3\mathbf{r} \tag{84}$$

and

$$\boldsymbol{\nabla}(\boldsymbol{\sigma} \times \mathbf{r}) = i\boldsymbol{\nabla}(\boldsymbol{\sigma} \wedge \mathbf{r}) = 2i\boldsymbol{\sigma}, \tag{85}$$

we obtain

$$\boldsymbol{\nabla}\mathbf{A} = -2\lambda^3\mathbf{r}(\boldsymbol{\sigma} \times \mathbf{r} + \boldsymbol{\sigma}) + \lambda^2 i\boldsymbol{\sigma}. \tag{86}$$

Whence

$$\boldsymbol{\nabla}\mathbf{A} = -2\lambda^3 i[\mathbf{r}^2\boldsymbol{\sigma} - \mathbf{r}(\mathbf{r}\cdot\boldsymbol{\sigma}) + i\mathbf{r}\boldsymbol{\sigma}] + \lambda^2 i\boldsymbol{\sigma}. \tag{87}$$

The scalar part of this equation gives us a single term

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 2\lambda^3\mathbf{r}\cdot\boldsymbol{\sigma}, \tag{88}$$

which we will need to eliminate to achieve toroidal gauge invariance. For the moment, though, we are only interested in the curl of the vector potential $\boldsymbol{\nabla}\times A = i\boldsymbol{\nabla}\wedge A$. Accordingly, from (87) we get

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{A} &= -2\lambda^3[\mathbf{r}^2\boldsymbol{\sigma} - \mathbf{r}(\mathbf{r}\cdot\boldsymbol{\sigma}) + \boldsymbol{\sigma}\times\mathbf{r}] + \lambda^2\boldsymbol{\sigma}\\ &= \lambda^3[2\boldsymbol{\sigma}\times\mathbf{r} + 2\mathbf{r}(\mathbf{r}\cdot\boldsymbol{\sigma}) - 2r^2\boldsymbol{\sigma} + (1+r^2)\boldsymbol{\sigma}]\\ &= \lambda^3[2\boldsymbol{\sigma}\times\mathbf{r} + 2\mathbf{r}(\mathbf{r}\cdot\boldsymbol{\sigma}) + (1-r^2)\boldsymbol{\sigma}].\end{aligned} \tag{89}$$

This is proportional to the normal field $\mathbf{n}(\mathbf{r})$ defined in (82). Hence, with a suitable choice of units, we can identify it with a magnetic field

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \lambda^2\mathbf{n}(\mathbf{r}) = \lambda^2 U\boldsymbol{\sigma}U^\dagger. \tag{90}$$

To make the vector potential (83) gauge invariant, we simply need to eliminate terms orthogonal to the pole $\boldsymbol{\sigma}$. This is achieved by

$$\mathbf{A}(\mathbf{r}) = \tfrac{1}{2}\lambda^2(\boldsymbol{\sigma} + \boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\mathbf{r})), \tag{91}$$

where the last term is just a fancy way of writing

$$\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\mathbf{r}) = \boldsymbol{\sigma}\cdot(\boldsymbol{\sigma}\times\boldsymbol{\sigma}) = \mathbf{r} - (\boldsymbol{\sigma}\cdot\mathbf{r})\boldsymbol{\sigma} \equiv \mathbf{r}_\perp. \tag{92}$$

This now satisfies the condition (77) for toroidal gauge invariance:

$$U_\varphi\mathbf{A}(\mathbf{r})U_\varphi^\dagger = \mathbf{A}(\mathbf{r}). \tag{93}$$

And the toroidal magnetic field is given still by

$$i\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{A} = \lambda^2 Ui\boldsymbol{\sigma}U^\dagger. \tag{94}$$

This prepares us for a straightforward generalization to toroidal radiation fields.

As Rañada has demonstrated [15], the structure in a monochromatic radiation field can be generated by an orthogonal pair of static vector potentials, as specified by (66), where

$$F_0^2 = (\mathbf{A} + i\mathbf{C})^2 = 2i\mathbf{A}\mathbf{C} = 2\mathbf{C}\times\mathbf{A}. \tag{95}$$

Whence the radiation field has the form

$$F = (\boldsymbol{\nabla}F_0)e^{i\varphi}. \tag{96}$$

Despite its appearance, this quantity is a relativistic invariant. Our main interest in this result is its relevance to modeling the photon, which is considered in a subsequent Section.

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D. Geometric Calculus and differential forms

Cartan's calculus of differential forms is used widely in mathematics and increasingly in physics, despite some significant drawbacks. Consequently, it is worth pointing out here that there is a more general *Geometric Calculus* (GC) that articulates smoothly with standard vector calculus and applies equally well to spinor-valued functions. As a detailed exposition of GC is given in [13, 26], it suffices here to illustrate how it relates to differential forms in the simplest case of applications to electrodynamics.

In GC, the concept of *directed integral* is fundamental, and the volume element for a k-dimensional integral is a (simple) k-vector $d^k\mathbf{r} = I_k d^k r$ with magnitude $|d^k\mathbf{r}| = d^k r$ and direction at $\mathbf{r}$ given by unit k-vector $I_k = I_k(\mathbf{r})$. For a closed k-dimensional surface $\mathcal{S}^k$ with boundary $\mathcal{B}^k$, The *Fundamental Theorem of Integral Calculus* has the general form

$$\int_{\mathcal{S}^k} d^k\mathbf{r}'\cdot\boldsymbol{\nabla}' f(\mathbf{r}'-\mathbf{r}) = \oint_{\mathcal{B}^k} d^{k-1}\mathbf{r}' f(\mathbf{r}'-\mathbf{r}), \tag{97}$$

where $f(\mathbf{r}'-\mathbf{r})$ is an arbitrary (differentiable) function, not necessarily scalar-valued. This reduces to the fundamental theorem for differential forms when the integrands are scalar-valued. Note that the inner product with the volume element projects away any component of the vector derivative normal to the surface.

For a multivector-valued function $A = A(\mathbf{r})$ with k-vector parts $A_k =< A(\mathbf{r}) >_k$, a *differential k-form*, or just a "*k-form*," can be defined by

$$\alpha_k =< d^k\mathbf{r}A(\mathbf{r}) >= d^k\mathbf{r}\cdot A_k. \tag{98}$$

The *exterior derivative* of a k-form is a $(k-1)$-form defined by

$$d\alpha_k =< d^k\mathbf{r}\cdot\boldsymbol{\nabla}A >= (d^k\mathbf{r}\cdot\boldsymbol{\nabla})\cdot A_{k-1}. \tag{99}$$

The *Hodge dual* can be defined (up to a choice of sign for the pseudoscalar i) by

$$*\alpha_k =< d^k\mathbf{r}A(\mathbf{r})i >= d^k\mathbf{r}\cdot A_{n-k}. \tag{100}$$

Here of a couple of important examples of differential forms.

The unit "outward" normal $\mathbf{n}$ of the directed area element $d^2\mathbf{r}$ is defined by $d^2\mathbf{r} = -i\mathbf{n}\,d^2r$. Accordingly, the element of flux for a magnetic field $\mathbf{B}(\mathbf{r})$ is given by

$$d^2\mathbf{r}\cdot(i\mathbf{B}) = \mathbf{B}\cdot\mathbf{n}\,d^2r = d^2\mathbf{r}\cdot(\nabla\wedge\mathbf{A}), \tag{101}$$

which integrates to a familiar form of Stokes' theorem.

An important example of a 3-form is the *magnetic helicity* h_m [20], defined as an integral over all space:

$$h_m = \int d^3r\,\mathbf{A}\cdot\mathbf{B}. \tag{102}$$

To make the directed volume element and the vector potential explicit, we write

$$\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = \mathbf{A}\cdot(-i\boldsymbol{\nabla}\wedge\mathbf{A}) = -i(\mathbf{A}\wedge\boldsymbol{\nabla}\wedge\mathbf{A}). \tag{103}$$

Whence,

$$h_m = \int d^3\mathbf{r} \cdot (\mathbf{A} \wedge \boldsymbol{\nabla} \wedge \mathbf{A})^\dagger. \tag{104}$$

Rañada [15, 16] recognized that the h_m can be identified with the *Hopf index* for a multi-valued vector potential with integer values.

III. KINEMATICS OF THE ELECTRON CLOCK

As proposed in the *Zitter Particle Model* for the electron in [4], we regard the worldline of the electron as a lightlike circular helix $z_e = z_e(\tau)$ with velocity $u = \dot{z}_e$. The helix is centered as on a timelike path $z = z(\tau)$ with velocity $v = \dot{z}$ normalized to $v^2 = 1$. This sets a time scale for the proper time parameter τ. A length scale is set by identifying the radius of the helical path with

$$|z_e(\tau) - z(\tau)| = \lambda_e = c/\omega_e, \tag{105}$$

where $2\lambda_e = \hbar/m_e$ is the reduced Compton wavelength and ω_e is the frequency of the circular motion called *zitter*. We refer to the point $z_e(\tau)$ as the *center of charge* (CC) and to $z(\tau)$ as the *center of mass* (CM).

We define a comoving frame of *local observables* attached to the CM by

$$e_\mu(\tau) = R\gamma_\mu\widetilde{R}, \tag{106}$$

where rotor $R = R(\tau)$ is normalized to $R\widetilde{R} = 1$. Then we identify $e_0 = v$ and define $r_e = z_e - z = \lambda_e e_1$ as the radius vector for the zitter. Thus we can regard e_1 as the "hand of the electron clock."

We identify the unit vector e_2 as the tangential velocity of the zitter. There are two distinct senses for the circulation which we naturally identify with the *electron/positron* distinction called "*chirality*." Accordingly, we have two null vector tangential velocities:

$$e_\pm = v \pm e_2 = R\gamma_\pm\widetilde{R}, \quad \text{with} \quad \gamma_\pm = \gamma_0 \pm \gamma_2. \tag{107}$$

Unless otherwise noted, we restrict our attention to the electron case here, and our choice of sign for the chirality is in agreement with [27]. Accordingly, we define the electron "*chiral velocity*" u by

$$u = R\gamma_+\widetilde{R} = v + e_2. \tag{108}$$

The *rotational velocity* of the zitter is a spacelike bivector defined by

$$\Omega = 2\dot{R}\widetilde{R},. \tag{109}$$

so that

$$\dot{e}_\mu = \Omega \cdot e_\mu. \tag{110}$$

In particular,

$$\dot{r}_e = \Omega \cdot r_e = \lambda_e\Omega \cdot e_1 = (e_1e_2) \cdot e_1 = e_2, \tag{111}$$

in agreement with (108).

As shown in [4], zitter mechanics can be described in terms of mass current $m_e v$, momentum p and spin angular momentum S. The *chiral* spin bivector S can be expressed in several equivalent forms:

$$S = ud = v(d + is) = ius, \tag{112}$$

Note that the null velocity $u^2 = 0$ implies null spin bivector $S^2 = 0$. It follows that the free particle zitter motion can be reduced to a single equation:

$$\Omega S = pv. \tag{113}$$

The bivector part of this expression gives us the spin equation of motion:

$$\dot{S} = \Omega \times S = p \wedge v. \tag{114}$$

And the scalar part gives us

$$m_e c = p \cdot v = \pm\Omega \cdot S, \tag{115}$$

where the sign distinguishes electron/positron chirality.

Note that these equations are consistent with identifying momentum p with either the timelike vector $m_e cv$ or the null vector $m_e cu$. This ambiguity is resolved in the next Section, where eu is identified as a charge current. On the other hand, the opposite choice was tacitly introduced in [4] by dividing (114) v to get

$$p = m_e cv + \dot{S} \cdot v. \tag{116}$$

Of course, we could not instead divide by the null vector u. The fact that introduction of electron charge seems to be needed to resolve this ambiguity about mass may be an important clue about the relation to charge and mass in the theory.

The basic solution of the above zitter equations of motion is a constant spacelike bivector given by:

$$\Omega = \omega_e e_2 e_1, \quad \text{where} \quad \omega_e = 2m_e c^2/\hbar \tag{117}$$

is the zitter frequency. There is, however, a more subtle solution that was recognized only recently by Oliver Consa [5], who also fully explained its significance. Before discussing that solution, some preparatory analysis should be helpful.

Since we are concerned with free particle solutions only, it is convenient to use the spacetime split defined by $v = e_0$ to introduce a comoving frame of relative vectors

$$\mathbf{e}_k = \mathbf{e}_k(\tau) = U\boldsymbol{\sigma}_k U^\dagger, \tag{118}$$

so that

$$\dot{\mathbf{e}}_k = \boldsymbol{\omega} \times \mathbf{e}_k, \quad \text{where} \quad -i\boldsymbol{\omega} = 2c\dot{U}U^\dagger \tag{119}$$

is the angular velocity. In the simplest solution, called *circular zitter*, the particle orbit is generated by rotating vector $\mathbf{r}_e = \lambda_e \mathbf{e}_1$ with period $\tau_e = 2\pi/\omega_e$.

In the new solution, called *toroidal zitter*, the particle orbit $\mathbf{r} = \mathbf{r}(\tau)$ is composed of a pair of orthogonal rotating vectors

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2 = r_1\mathbf{e}_1 + r_2\mathbf{e}_2, \tag{120}$$

and the path circulates on a torus with radius r_2. The motion is governed by a single rotor $U = U(\tau)$ in

$$\mathbf{r} = U(r_1\boldsymbol{\sigma}_1 + r_2\boldsymbol{\sigma}_2)U^\dagger, \tag{121}$$

with the specific form

$$U = U_1U_2 = e^{-i\boldsymbol{\omega}_1\tau/2}e^{-i\boldsymbol{\omega}_2\tau/2}, \tag{122}$$

where $\boldsymbol{\omega}_1 = \omega_e\boldsymbol{\sigma}_3$ and $\boldsymbol{\omega}_2 = \omega_e\boldsymbol{\sigma}_1$. This differentiates to

$$\dot{U} = \dot{U}_1U_2 + U_1\dot{U}_2 = -\tfrac{1}{2}[i\boldsymbol{\omega}_1 U + Ui\boldsymbol{\omega}_2]. \tag{123}$$

Since the electron circulates at the speed of light, we have

$$\dot{\mathbf{r}}^2 = \left(\frac{1}{c}\frac{d\mathbf{r}}{d\tau}\right)^2 = 1. \tag{124}$$

Since the circular and toroidal velocities are orthogonal but have the same frequency, we have

$$\dot{\mathbf{r}}^2 = \dot{\mathbf{r}}_1^2 + \dot{\mathbf{r}}_2^2 = v_1^2 + v_2^2 = (r_1\omega_e)^2 + (r_2\omega_e)^2. \tag{125}$$

Consequently, the velocity of the circular motion is reduced to less than the speed of light by a factor

$$v_1 = c/g, \tag{126}$$

Consta calls the quantity

$$g = \sqrt{1 + (r_1/r_2)^2} = \sqrt{1 + (v_1/v_2)^2} \tag{127}$$

the *helical g-factor.*

This result enabled Consta to calculate the electron's anomalous magnetic moment precisely and unambiguously [5]. Indeed, it appears to improve on Schwinger's famous QED calculation. More important, it provides a clear physical explanation for its origin that is likely to have further implications for elementary particle theory.

Rotor methods for magnetic resonance measurement are given in [24], along with many other coordinate-free applications to rotational dynamics. Generalization to rotor Frenet equations for lightlike particles is given in [3].

Though a properly tuned magnetic resonance measurement may activate toroidal zitter, external fields are not necessary to maintain it. Instead, the electron motion is analogous to that of a freely precessing top with toroidal zitter described as nutation. If the toroidal zitter can be quenched and activated at will, then the electron has at least two distinguishable internal states and might thereby serve as the ultimate magnetic storage device.

The kinematic model of the electron formulated in this Section has been memorably described by Consa [5] as a superconducting LC circuit composed of two indivisible elements: "*a quantum of electric charge and a quantum of magnetic flux, the product of which is equal to Plank's constant. The electron's magnetic flux is simultaneously the cause and the consequence of the circular motion of the electric charge:*"

$$e\phi = h. \tag{128}$$

Note, however, that this model of the electron is completely self-contained, ignoring any interaction with external electromagnetic fields. Addressing that limitation is our main task in the next Section.

IV. MAXWELL–DIRAC THEORY

Born-Dirac theory supports the "Pilot–Wave" interpretation of quantum mechanics originally proposed by de Broglie [4]. But that is only half of de Broglie's proposal, which he called *double solution* theory [28]. In consonance with his realist perspective on quantum mechanics, he proposed that there must be two distinct kinds of solution to the wave equation. Besides the pilot wave, there must be some kind of *singular solution* describing a real particle without involving probability. In other words, he proposed a unique kind of *wave–particle duality*, or *complementarity* if you will.

De Broglie insisted that relativity is an essential ingredient of fundamental quantum mechanics, and he noted that monochromatic plane wave solutions of the relativistic wave equation determine well–defined particle paths, just as we have seen for the Dirac equation in Born-Dirac theory. However, he never found a convincing way to define a singularity that represents a physical particle. In this section we introduce a new physical interpretation of the Dirac wave function Ψ that seems to do the trick. *The essential idea is to give back to the electron its charge and electromagnetic field*, which were ignored in the original neutered version of Dirac theory. This necessarily localizes the electron to the point source of its Coulomb field. The trick is to do it in a way that is consistent with well established features of Dirac theory.

According to the Born rule, the Dirac current $\Psi\gamma_0\widetilde{\Psi} = \psi\gamma_0\widetilde{\psi} = \rho v$ is to be interpreted as a probability current so the dimensionless function $\rho = \rho(x) = \psi\widetilde{\psi}$ must be a probability density; therefore $e\rho(x)$ should be interpreted as a "probable" charge density. In the early days of QED, Furry and Oppenheimer [29] called this proportionality into question by asserting that $e\rho(x)$ must be interpreted as a physically real charge density to enable comparison with "real charges" in classical electrodynamics that produce real electromagnetic fields. Indeed, second quantization was soon invented to do that, but it involves some dislocation from the original Dirac theory. However, there is a subtle alternative to this approach that has not been heretofore considered.

Our formulation of Born-Dirac as a classical field theory in [4] facilitates comparison with Maxwell's electromagnetics. To coordinate the two to produce a fully integrated Maxwell-Dirac theory, we need some educated guesses to guide us. To that end, I propose three fundamental anzatz's,

First, we invoke *"de Broglie's clock ansatz"* already articulated as a kinematic model of the electron clock in the preceding Section.

Second, we note from Section II that Blinder's assumption that *charge density is proportional to mass density* in Maxwell theory can be carried over to Dirac theory where both quantities are associated with the Dirac current. Hence Blinder's argument relating electron mass to impedance of the vacuum should apply to the interpretation of the Dirac wave function in some way. Let me call this the "*Blinder ansatz.*"

Third, we recall London's assumption that for electrons in a superconductor the magnetic vector potential is proportional to the charge current [30]. As all interactions in the Dirac equation are mediated by vector potentials, we look for a comparable relation to the electron's magnetic vector potential. Let's call that the "*London ansatz.*"

We begin by reviewing a general form the *real Dirac equation* for the electron that has been thoroughly discussed in the preceding paper [4]:

$$\Box\Psi\mathbf{i}\hbar - \frac{e}{c}A\Psi = m_e c\Psi\gamma_0\,. \tag{129}$$

Here $A = A(x) = A_\mu\gamma^\mu$ is the electromagnetic vector potential for external sources and the unit bivector $\mathbf{i} = i\boldsymbol{\sigma}_3$ encodes the crucial property of spin. The spinor "*wave function*" $\Psi = \Psi(x)$ has the *Lorentz invariant* decomposition

$$\Psi = \rho^{\frac{1}{2}}Re^{i\beta}, \tag{130}$$

where $\rho = \rho(x)$ is scalar-valued density and "*rotor*" $R = R(x)$ is normalized to $R\widetilde{R} = \widetilde{R}R = 1$. The rotor R has a unique decomposition into the product

$$R = VUe^{-i\boldsymbol{\sigma}_3\varphi}, \tag{131}$$

where rotor $V = (v\gamma_0)^{\frac{1}{2}}$ defines a boost to the electron's center of mass, the spatial rotor

$$U = U_1U_2 = e^{-i\boldsymbol{\sigma}_3\varphi_1}e^{-i\boldsymbol{\sigma}_1\varphi_2}, \tag{132}$$

describes kinematics of spin, and $\varphi = \varphi(x)$ is identified as the quantum mechanical phase of the wave function.

Now, for reasons explained in [4], we eliminate the Lorentz invariant "duality factor" $e^{i\beta}$ from (130). Then, to reformulate the Dirac equation (129) in terms of local observables, we multiply it on the right by $\widetilde{\Psi}$ to get

$$\Box\Psi i\boldsymbol{\sigma}_3\hbar\widetilde{\Psi} - \frac{e}{c}A\rho = m_e c\Psi\gamma_0\widetilde{\Psi}, \tag{133}$$

where $\rho = \Psi\widetilde{\Psi}$ and we recognize

$$\Psi\gamma_0\widetilde{\Psi} = \rho R\gamma_0\widetilde{R} = \rho V\gamma_0\widetilde{V} = \rho v \tag{134}$$

as the conserved Dirac current. Finally, by symmetrizing the first term in (133) and making the phase $\varphi = \varphi(x)$ therein explicit, we reduce the Dirac equation to the equivalent form

$$\rho(P - (e/c)A) = m_e c\rho v - \Box\cdot(\rho S), \tag{135}$$

where the phase gradient $P = \hbar\Box\varphi$ is called the *canonical momentum*, and the quantity on the right is a conserved vector field known as the *Gordon current*:

$$\rho S = \frac{\hbar}{2}\Psi i\boldsymbol{\sigma}_3\widetilde{\Psi} \tag{136}$$

is the bivector spin density.

This completes our reformulation of the Dirac equation in terms of local observables. Now, to incorporate zitter into the Dirac equation in accord with *"de Broglie's clock ansatz,"* we simply replace the timelike velocity $v = R\gamma_0\widetilde{R}$ in the Dirac current (134) with the lightlike velocity of the electron's helical path:

$$u = R(\gamma_0 + \gamma_2)\widetilde{R} = e_0 + e_2 = \dot{z}_e(\tau), \tag{137}$$

This decomposes the zitter into the timelike CM velocity

$$v = R\gamma_0\widetilde{R} = \dot{z}(\tau) = \bar{u}(\tau) \tag{138}$$

and the fluctuating spacelike velocity $e_2 = R\gamma_2\widetilde{R}$ of the circular zitter with $\bar{e}_2 = 0$. In a similar way we identify the spacelike spin bivector in (136) as the zitter average $\bar{S}$ of the lightlike spin bivector (112).

With the electron's helical path embedded in the wave function $\Psi(x)$, its functional form reduces to $\Psi(x - z_e(\tau))$ and factors into the product:

$$\Psi = VU\psi, \tag{139}$$

where zitter kinematics is incorporated in the rotor $U = U(\tau)$ given by (122), while

$$\psi = \rho^{1/2}e^{-\mathbf{i}\varphi} = e^{-(\alpha + i\boldsymbol{\sigma}_3\varphi)}. \tag{140}$$

has the familiar form for a Schrödinger wave function, but, of course, with an imaginary unit specified by the bivector $\mathbf{i} = i\boldsymbol{\sigma}_3$.

A truly remarkable implication of the wave function $\psi = \psi(x - \dot{z}(\tau))$ given by (140) is its reduction of coupling to the electromagnetic vacuum to a pair of retarded electric and magnetic scalar functions $\alpha(x - \dot{z}(\tau))$ and $\varphi(x - \dot{z}(\tau))$. Details are provided by the complementary ansatzes of Blinder and London.

The *Blinder ansatz* identifies the Dirac current $m_e c\rho v$ with the Blinder form for the retarded potential of a point charge (46) by assuming that the Dirac *density* $\rho = \rho(x)$ is reciprocal impedance ϵ of the vacuum. Thus, it holds that

$$\rho = \epsilon^{-1} = e^{-\alpha} \quad \text{where} \quad \alpha = \lambda_c/r, \tag{141}$$

while the *classical electron radius* $\lambda_c = e^2/m_e c^2$ serves as a charge/mass scaling length and $r = (x - z_e(\tau)) \cdot v$ is the *classical retarded distance* from a point singularity at the position $z_e(\tau)$ of the electron. In other words, we identify

$$\frac{e}{c}A_c \equiv \frac{e^2}{c\lambda_c \epsilon}v = m_e c\rho v, \tag{142}$$

or, more simply,

$$\lambda_c A_c = e\rho v, \tag{143}$$

with the classical "*Coulomb vector potential*" A_c for a point charge. Acceptance of this "Blinder ansatz" commits us to solutions of the Dirac equation as a function of retarded position. Note that the source CC of the electric field $z_e(\tau)$ is displaced from the CM $z(\tau)$ by the radius vector

$$r_e = z_e(\tau) - z(\tau) = \lambda_e e_1, \tag{144}$$

where e_1 is the electron "clock vector" and $\lambda_e = \hbar/2m_e c$ is the zitter radius.

A crucial feature of the Blinder function is that $\rho(z_e(\tau)) = 0$ everywhere along the electron path $z_e(\tau)$, and thus at a single point on any 3-D spacelike hypersurface. At that point the phase in the wave function is undetermined, so it can be multivalued. This mechanism can also serve to pick out a particle path in Pilot Wave theory.

If indeed the electron's Coulomb potential is already inherent in the Dirac equation as the Blinder ansatz requires, then we should expect to find the electron's magnetic potential there as well. This leads us to examine the Gordon current, where we note that it includes a "magnetization current." Proposing this as a specific realization of the "*London ansatz*," we reinterpret that term as a "magnetic potential." Thus, in analogy with the Blinder potential (143) we write

$$\frac{e}{c}A_m \cong -\Box \cdot (\rho S). \tag{145}$$

The congruence sign $\cong$ serves to indicate that the two quantities are mathematically equivalent but have different physical interpretations. In Born-Dirac theory presented in [4] the spin density ρS represents a distribution of spins associated with distinct particle paths. As will become evident, in the present Maxwell-Dirac theory the analogous quantity represents a density of magnetic field lines. Combining the electron's electric and magnetic vector potentials, we have a complete analogy with the entire Gordon current:

$$\frac{e}{c}A_e = \frac{e}{c}(A_c + A_m) \cong m_e c\rho v - \Box \cdot (\rho S). \tag{146}$$

According to (143), therefore, this reduces the Dirac equation to:

$$\rho p = \rho(P - \frac{e}{c}A) = \frac{e}{c}A_e. \tag{147}$$

Thus, we have boiled down the Dirac equation to a relation between the electron's vector potential A_e and what we can now identify as an energymomentum density of the vacuum ρP. We note that, as an equation for momentum balance, it comes close to the ideal of putting the electron vector potential A_e on equal footing with the vector potential A for external interactions. The only difference is that the density ρ is an essential part of A_e but not of A. Later on, we take this as a clue to a many particle generalization. For present purposes, we can ignore external interactions, though we maintain the electron's interaction with the vacuum.

Electric and magnetic components of the electron potential A_e are separated by a spacetime split with respect to the CM velocity v; thus

$$A_e v = A_c \cdot v + A_m \wedge v = e\rho/\lambda_c + \mathbf{A}_m. \tag{148}$$

Then a v-split of the canonical momentum P gives us

$$Pv = P \cdot v + P \wedge v = P_0 + \mathbf{P} = m_e c + \hbar\boldsymbol{\nabla}\varphi. \tag{149}$$

Hence, electron energy is identified with the electric potential

$$P_0 = m_e c = \frac{e}{c}A_c \cdot v\rho^{-1}, \tag{150}$$

while momentum is identified with the magnetic potential:

$$\mathbf{P} = \hbar\boldsymbol{\nabla}\varphi = \frac{e}{c}\mathbf{A}_m\rho^{-1}. \tag{151}$$

Note that, though electric and magnetic fields are separable in the Gordon current (146), they are not independent. They are intimately coupled by the vacuum density $\rho = \rho(x)$, as we now show.

Regarding the zitter average of spin $\mathbf{s}$ as constant, we calculate the magnetic vector potential from:

$$\begin{aligned}\mathbf{A}_m\rho^{-1} &= \frac{-c}{e\rho}\boldsymbol{\nabla} \cdot (\rho i\mathbf{s}) = \frac{c}{e\rho}\boldsymbol{\nabla} \times (\rho\mathbf{s}) \\ &= \frac{c}{e}\boldsymbol{\nabla} \times \left(\frac{-\lambda_c\mathbf{s}}{r}\right) = g_e\frac{\mathbf{r} \times \mathbf{s}}{r^3},\end{aligned} \tag{152}$$

which is the familiar potential for a magnetic dipole, where $\lambda_c = e^2/m_e c^2$ is the "classical electron radius" and

$$g_e = c\lambda_c/e = e/m_e c \tag{153}$$

is the correct g-factor for a Dirac electron. What a surprise! For g_e has been derived here from a singular wave function unknown in conventional Dirac theory. Indeed, conventional free particle solutions do not even have a magnetic moment! Moreover, the derivation connects the magnetic vector potential to the Coulomb potential, as expected in an integrated model of electric and magnetic forces in an electron.

Comparison of equation (151) with (152) shows that ρ serves as a Lagrange multiplier relating gradient to curl in the form

$$\hbar\boldsymbol{\nabla}\varphi = \boldsymbol{\nabla}\times\boldsymbol{\pi}. \tag{154}$$

The general question of when the curl of a vector field $\boldsymbol{\pi}$ is equivalent to the gradient of a scalar has been studied by Kleinert [31]. He shows that is possible when the scalar function is the projection of an area integral – in physical terms, a flux generated by a current loop. Moreover, in general, the flux is multivalued, so we can conclude that

$$\frac{e}{c}\oint d\mathbf{x}\cdot\mathbf{A}_m\rho^{-1} = \oint d\mathbf{x}\cdot\mathbf{P} = \hbar\oint d\varphi = nh, \tag{155}$$

This assigns a definite quantum of flux to the electron's magnetic field, as anticipated by London and evidently measured in the quantum Hall effect [32]. As presented here, it can be regarded as one more prediction of Dirac theory, provided the above identification of the electron vector potential with the Gorden current is accepted. In that case, *the flux quantum ranks with electric charge as a fundamental property of the electron*, as, indeed, many since London have suggested. More precisely, it supports interpreting Dirac theory as modeling the electron as a fundamental singularity of the vacuum.

Since we have concluded that internal quantized states are inherent in the structure of the electron, some further remarks about quantized particle states may be helpful. Two necessary conditions for a stationary state appear to be: (1) A constant energy $P_0 = P\cdot\gamma_0$ in an inertial reference frame specified by a constant timelike vector γ_0, and (2) the integrability condition $\Box\wedge P = 0$, which implies

$$P = p + \frac{e}{c}A = \hbar\Box\varphi \tag{156}$$

and, by Stokes Theorem,

$$\oint P\cdot dx = 0. \tag{157}$$

for *any closed spacetime curve*. The spacetime split $P\cdot dx = P_0 dt - \mathbf{P}\cdot d\mathbf{x}$ of this equation then suggests *a sufficient condition for quantized states*:

$$\int_0^{T_n} P_0\, dt = \oint \mathbf{P}\cdot d\mathbf{x} = nh/2, \tag{158}$$

which incorporates the spatial quantization. Then we have

$$\int_0^{T_n} P_0\, dt = P_0 T_n = nh/2, \tag{159}$$

which relates the energy to period counts T_n of the electron clock. This connection between spatial quantization and temporal period was first proposed and proved in a special case by Post [33]. It plays a role in the quantum conditions for hydrogen discussed in [4].

Having established a connection of zitter to quantum conditions, we can postpone analysis of its implications to later. In most of quantum mechanics the high frequency zitter fluctuations in electron phase are negligible, in which case we can adopt the dipole approximation for the magnetic vector potential.

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A. Photon topology

Hard on the spectacular successes of Dirac's electron theory, de Broglie applied it to model the photon. De Broglie must have had high hopes for his photon theory, because he took the unusual step of announcing it with fanfare and immediately translating it into English [34]. Nevertheless, despite his deep insight and clever analysis, he was never able to bring it to a successful conclusion. Now, however, Maxwell-Dirac theory offers a new approach to reach the goal.

We have seen how a singularity of the vacuum Dirac equation can serve as the seat of the electron's electromagnetic potential. For the sake of possible modification or generalization, let us frame our assumptions in the most general terms. To that end, we exploit the multiplicative structure of the Dirac wave function to cast it in the general form

$$\Psi = R\psi = VUZ^{-1}, \tag{160}$$

where

$$\psi = \rho^{1/2}e^{-\mathbf{i}\varphi/2} = Z^{-1}, \tag{161}$$

and rotor R describes the spacetime kinematics of the electron singularity. Here we propose to interpret Z as a *complex impedance* of the vacuum, so ψ is the *vacuum admittance.* Let us refer to the multiplicative form (160) for a singular solution of the Dirac equation as *vacuum separability.*

It is noteworthy that the complex admittance ψ has the form of the Schrödinger wave function, which is indeed an approximation to the Dirac wave function [35, 36]. That suggests that Schrödinger theory is fundamentally about vacuum singularities.

Later on we shall consider possibilities for generalizing the "vacuum impedance." To retain essential physical features that we have already identified, we place the following two restrictions on the functional form of ψ. First, *vacuum positivity*:

$$\rho(r) = \psi\widetilde{\psi} \geq 0, \tag{162}$$

with $\rho(r) = 0$ only at $r = 0$. Second, for agreement with electrodynamics, we require that ρ reduce to the Blinder function in the asymptotic region, that is:

$$\rho(r) = e^{-\lambda_c/r} \quad \text{for} \quad r \gg \lambda_c. \tag{163}$$

These restrictions leave open the possibility of a more complex functional form for $\psi(r)$ due to short range interactions in the neighborhood of the singularity to be discussed later.

We note that the Blinder function has an alternative formulation that strongly suggests it describes a general property of vacuum singularities, not limited to the electron or even to charged particles. We write the electron's Blinder exponent in the form

$$\lambda_c/r = \frac{e^2/\hbar c}{m_e c/\hbar v \cdot (x - z_e)} = \frac{\alpha_e}{k_e \cdot (x - z_e(\tau))}. \quad (164)$$

This suggests that any particle with kinetic momentum $k = p/\hbar$ and position $z(\tau)$ will have a Blinder function of the form

$$\rho = e^{-\alpha_e/k\cdot(x-z(\tau))}, \quad (165)$$

so the particle is located at $\rho = 0$, so we drop the subscript on k_e and allow k to be a null as well as timelike. There is no longer a suggestion here that the exponent is the Coulomb potential of a charged particle. Here the fine structure constant α_e acts as a kind of general scaling constant for vacuum singularities, so it may play that role even in strong interactions, as argued by MacGregor [37]. Throughout this article we will identify the presence of an electron with a zero of its Blinder function, and we propose that the same for other elementary particles. In particular, we propose that there must be a bifurcation in zeros of the Blinder function for electron and photon in the photon emission process.

Accepting the formulation of the vacuum impedance Z^{-1} in (161) just described for the photon reduces the photon wave function to the kinematic factor $R = VU$ in (160). This factor couples electron to positron with a phase difference to compose the electrically neutral photon, as we now explain.

As specified in (120), the electron circulates on a periodic toroidal path

$$\mathbf{r}(\tau) = \mathbf{r}_1(\tau) + \mathbf{r}_2(\tau) = \mathbf{r}(\tau + \tau_e). \quad (166)$$

We assume the positron circulates on the complimentary path

$$\mathbf{r}_+(\tau) = \mathbf{r}_1(\tau) - \mathbf{r}_2(\tau) = \mathbf{r}_+(\tau + \tau_e). \quad (167)$$

Taken together, the electron-positron pair composes an electric dipole with a fixed charge separation $2r_2$. As the dipole circulates it rotates with a frequency commensurate with its orbital frequency. The circulating dipole sweeps out a 2-dimensional strip of width $2r_2$ and its edge is a closed curve with period $2\tau_e$, as described in [38].

This completes the description of our photon model. Of course, many details remain to be worked out, such as analysis of photon emission and absorption. That will be left for another occasion.

B. Field-particle duality

Maxwell-Dirac theory exhibits an ontic version of wave-particle duality that might be better described as *field-particle duality*. We can describe this duality with the fundamental field equation (147) for balance of momentum densities:

$$\rho p = \rho(P - \frac{e}{c}A), \quad (168)$$

where canonical momentum $P = P(x)$ and external vector potential $A = A(x)$ are defined as before. With the electron momentum density ρp identified with its electromagnetic field :

$$\rho p = \frac{e}{c}A_e \cong m_e c\rho u - \Box \cdot (\rho S), \quad (169)$$

this equation describes a remarkable duality between the charge current along the electron path and the electromagnetic field it generates and propagates by Maxwell's equation:

$$\Box(\rho^{-1}\Box A_e) = 0. \quad (170)$$

Recall that the Blinder function for ρ, given by (141), vanishes on the electron path, so it must be factored out to get an equation for the path from the momentum balance equation (168).

Because zitter fluctuations are so localized in space and time, it is often convenient to suppress them. That can be done systematically by taking a zitter average of Eq. (169):

$$\rho\bar{p} = \frac{e}{c}\bar{A}_e \cong m_e c\rho v - \Box \cdot (\rho\bar{S}). \quad (171)$$

This equation corresponds to the original Gordon current in Dirac theory. We have identified $\bar{u} = v = \dot{z}$ as a *center of mass* (CM) or (*center of momentum*) velocity for electron motion and its image propagated elsewhere by the wave equation.

According to (184), at the center of mass $z(\tau)$, the Blinder function has the constant value

$$\rho(z(\tau)) = \exp[-\lambda_c/\lambda_e] = e^{-2\alpha_e}. \quad (172)$$

Hence, it can be factored out of (171) and (168) when evaluated on the CM path to give us an equation of motion

$$p = m_e cv + \dot{S} \cdot v = P - \frac{e}{c}A. \quad (173)$$

This may be recognized as a generalized Hamilton-Jacobi (H-J) equation. Indeed, if spin is neglected and $P = \hbar\Box\varphi$ is given by the gradient of electron phase φ, then its square gives us the well-known classical relativistic H-J equation:

$$(\hbar\Box\varphi - \frac{e}{c}A)^2 = m_e^2 c^2. \quad (174)$$

We conclude that the H-J equation (173) is a suitable equation for the electron path embedded in its own electromagnetic field.

Note that zero values of the Blinder function determine a timelike tube with radius λ_e around the CM path, which might be related to similar tubes considered in string theory.

The particle equation (173) should be compared to the the guidance equation in Pilot wave theory discussed in [4] It differs in the absence of the “quantum potential” $S\cdot\Box\ln\rho$, because the density ρ used here is defined to be a specific function for a single electron by (184), whereas density ρ in Born-Dirac theory has a probability interpretation. In the Section on many particle theory, the two viewpoints on ρ will be merged and applied to analysis of electron diffraction.

For the moment, it is worth noting that we now have two complementary kinds of wave-particle duality to interpret the Dirac equation: the ontic Maxwell-Dirac electromagnetic wave, and the epistemic Born-Dirac probability wave. They are united by a common equation describing particle paths. Born-Dirac treats that equation as a mechanism for guiding motion of a passive particle by a *“Pilot Wave”*. Whereas Maxwell-Dirac regards it as a particle current actively generating electric and magnetic fields – a kind of “*Pilot Particle,*” if you will, that generates a “real physical wave.”

To summarize Maxwell-Dirac theory to the present point: We have proposed a singular solution of the Dirac equation that models the electron as the seat of an electromagnetic field:

$$\frac{e}{c}A_e = \frac{e}{c}(A_c + A_m) \cong m_e c\rho u - \Box\cdot(\rho S). \tag{175}$$

that fluctuates with zitter frequency. The field is propagated from the source by the wave equation, in essential agreement with de Broglie’s original proposal. Thus, *the electron’s de Broglie wave is electromagnetic!* Since zitter fluctuations are tied to the source and the zitter velocity is orthogonal to its radius, they do not radiate energy. The wave equation propagates these fluctuations without a net transfer of energy. Energy transfer requires emission and absorption of photons, which must be a separate process involving acceleration of the electron’s zitter center.

The “Maxwell-Dirac” theory seems to answer Einstein’s call (quoted at the beginning of this paper) to unify the electron with its electromagnetic field. It is noteworthy that Dirac [39] made a similar proposal to modify classical electrodynamics by assuming that the charge current is proportional to the vector potential (like the London ansatz).

C. Electron Self-Energy and zitter

Zitterbewegung is said to contribute to electron self-energy in QED, though that can be questioned because the integrals are divergent and must be removed by renormalization. Weisskopf [40] was the first to discuss the role of zitterbewegung in QED explicitly. Expressed in our lingo, he argues that zitter generates a fluctuating electric field. But when he calculates the zitter contribution to the energy in the field he gets a divergent result. In contrast, we show here that calculation with the zitter model is not only simpler, but the result is finite and equal to the expected result m_ec^2. This is one reason to suspect that the zitter model may generate finite results in QED.

According to the *Blinder ansatz* proposed in Section IV the vacuum is characterized by the Blinder form (141) for the Dirac density $\rho=\rho(x)$ given by

$$\rho = \epsilon^{-1} = e^{-\lambda_c/r}, \tag{176}$$

where the *polarization radius* $\lambda_c = e^2/m_ec^2$ is a charge/mass scaling length, and

$$r = (x - z_e(\tau))\cdot v \tag{177}$$

is the classical retarded distance from a point singularity at the position of the electron.

To study zitter fluctuations, we shift electron position to the zitter CM. That shift must be done in a way that preserves the ”retarded distance” property of r.

With the backward Taylor expansion $z(\tau-\tau_c) \approx z(\tau) + v(\tau)\tau_c$, we get

$$z_e(\tau) = z(\tau - z_e) + r_e(\tau), \tag{178}$$

hence

$$z_e = z(\tau) + v\tau_c + r_e, \tag{179}$$

where all functions are evaluated at time τ. Requiring $[v\tau_c + r_e]^2 = 0$, we find

$$\tau_c = \lambda_e/c \tag{180}$$

as the time for a light signal to propagate from the zitter center to the circulating particle. Accordingly, we have

$$z_e = z + v\tau_c + r_e = z + \lambda_e(v + e_1). \tag{181}$$

Then requiring $(x-z_e)^2 = r^2 - \mathbf{r}^2 = 0$, for the the zitter *retarded position* we get

$$\mathbf{r} = (x - z_e)\wedge v = \mathbf{x} - (\mathbf{z} + \mathbf{r}_e) \tag{182}$$

with *retarded distance*

$$r = |\mathbf{r}| = (x - z_e)\cdot v = (x - z)\cdot v + \lambda_c. \tag{183}$$

This completes our characterization of the zitter vacuum.

Now, to incorporate the effect of zitter into the electron’s electromagnetic field, we simply replace velocity v with $u = v + e_2$ in (137) to get a *Coulomb vector potential* of the form

$$\frac{e}{c}A_c \equiv \frac{e^2}{c\lambda_c}\frac{u}{\epsilon} = m_e c\rho u \tag{184}$$

To ascertain the implications of this change on the electron self energy, we need only consider how it modifies the Coulomb field of a free electron:

$$F = \frac{m_e c^2}{e}(\Box\rho)\wedge u = e\rho\,\frac{\hat{r}}{r^2}\wedge u = \mathbf{E} + i\mathbf{B}. \tag{185}$$

Note that $\hat{r}\cdot u = \hat{r}\cdot(v+e_2) = 0$, so we have $\hat{r}\wedge u = \hat{r}u$. Hence,

$$F^2 = \mathbf{E}^2 - \mathbf{B}^2 + i\mathbf{E}\cdot\mathbf{B} = 0. \tag{186}$$

Since $G = \rho^{-1}F = \mathbf{D} + i\mathbf{H}$, this implies that

$$\tfrac{1}{2}\mathbf{E}\cdot\mathbf{D} = \tfrac{1}{2}\mathbf{B}\cdot\mathbf{H}. \tag{187}$$

Hence, the total energy in the field is

$$\begin{aligned} W &= \tfrac{1}{2}\int(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})\,d^3r \\ &= \int \mathbf{E}\cdot\mathbf{D}\,d^3r = m_e c^2, \end{aligned} \tag{188}$$

exactly twice the result obtained if the electron velocity were v instead of u. As first suggested by Slater [41], the reason for the difference is expressed by (187), which tells us the potential energy density of the circulating charge is equal to its kinetic energy density. However, this cannot be regarded as a fully satisfactory resolution of the notorious electron self-energy problem until the relation of gravitational to inertial mass is understood.

More generally, we note that $\hat{r}uvu\hat{r} = uvu = 2u$, hence the field F has an energymomentum density

$$T(v) = \tfrac{1}{2}Fv\,\widetilde{G} = \frac{e^2\rho}{r^4}\,u, \tag{189}$$

where $u = v + \dot{r}_e$ exhibits momentum fluctuations due to the rapidly rotating vector $\dot{r}_e$. Note that these zitter fluctuations are not radiating, because the zitter velocity is orthogonal to the zitter radius vector. It remains to be seen if they can be identified with vacuum fluctuations of QED or the ubiquitous ground state oscillators proposed by Planck. Toward that end, let us consider how zitter fluctuations might account for important QED results.

It appears that zitter fluctuations will not alter the quantum conditions proposed for stationary states of a Pilot particle as long as they are resonant with the quantum periods. Indeed, resonance of zitter fluctuations with orbital motion may be the most fundamental criterion for stationary states.

Though zitter will not alter quantum conditions, it should alter energy levels by smearing out the Coulomb potential over the zitter radius λ_e. This kind of explanation for the *Lamb shift* was first proposed by Welton [42]. Moreover, in s-states the Coulomb oscillator solutions of the zitter will carry the electron around the nucleus at the distance λ_e instead of right through it. In the ground state of hydrogen the nucleus just sits inside the zitter circle. This is an analog of the Darwin term in wave mechanics. Calculations of the ground state energy are therefore especially sensitive to the model for the nucleus.

At a deeper level, if zitter resonances are characteristic of quantized states, they may play a role in electron-electron interactions. Indeed, it has been suggested [43] that the *Pauli principle* and *Exchange forces* may be explained by zitter resonances. Possibilities for experimental test of those ideas are noted in the many electron theory proposed below.

V. VACUUM UNIVERSALITY

The simplicity and power of modeling the electron as a vacuum singularity strongly suggests that the spacetime vacuum can be regarded as a universal medium for the physical world, so all elementary particles can be regarded as vacuum singularities of various types. Success in explaining quantum mechanics and QED for the electron promises strong support for the general thesis that the Dirac equation describes spacetime dynamics of vacuum singularities. Thus we have new prospects for a *unified vacuum field theory* of elementary particles. Here are two avenues for research in this direction.

The first is a generalization to many electron theory with a proposal to explain the Pauli principle as a consequence of zitter resonance. A crucial test will be accounting for the "exchange force" in Helium. Passing that test would provide strong grounds for studying effects of zitter resonance in superconductivity and the rest of many particle physics.

The second avenue is generalizing the "complex impedance" of the electron to include electroweak interactions and relate it to gravitational field equations. The general idea is that gravity is about deformation of the vacuum due to presence and propagation of singularities described by the Dirac equation. The implication that all elementary particles and their interactions can be described by variations and excitations of the vacuum impedance promises closure to the search for a Unified Field Theory. This prospect appears be consistent with the Standard Model but does not leave much room for extension beyond that.

A. Many Particle Theory

We start with a minimal generalization of *Vacuum Dirac Theory* to a many particle theory. We consider a system of N charged particles regarded as particle singularities in the vacuum with zitter velocities u_k and CM spacetime paths $z_k = z_k(\tau_k)$ with proper velocities $v_k = \dot{z}_k$. We suppose their motions are determined by a spinor wave function for the vacuum $\Psi = \Psi(x, z_1, z_2, ..., z_N)$.

The vacuum density is then given by

$$\Psi\widetilde{\Psi} = \rho = \prod_{k=1}^{N} \rho_k = \check{\rho}_k \rho_k, \tag{190}$$

where $\check{\rho}_k$ designate the product with the kth factor omitted, and, as before,

$$\rho_k = e^{-\alpha_k}, \tag{191}$$

where $\alpha_k = \alpha(r_k)$ is a Blinder potential or its generalization with retarded position $r_k = (x - z_k) \cdot v_k$.

As the basic equation for energymomentum density in the vacuum, we consider a straightforward generalization of the Gordon current in Born-Dirac theory, namely

$$\rho P = \frac{e}{c} \sum_{k=1}^{N} A_k \check{\rho} = \rho \sum_{k=1}^{N} p_k, \tag{192}$$

where, as before,

$$\frac{e}{c} A_k = \rho_k p_k = m_e c \rho_k u_k - \Box \cdot (\rho_k S_k) \tag{193}$$

is the electromagnetic *vector potential* (Gordon current) of the kth particle modeled with or without zitter, and

$$\rho P_\mu = \hbar \langle (\partial_\mu \Psi) \mathbf{i} \widetilde{\Psi} \rangle \tag{194}$$

defines components $P_\mu = \gamma_\mu \cdot P$ of the *canonical momentum.*

Since equation (192) is the core synthesis of Maxwell's electrodynamics with Dirac's electron theory, consolidating what we have discussed already and providing a platform for extensions to follow, let me christen it with the name *Maxwell-Dirac Equation.*

Generalization to include other fermions is discussed later. Restricting our attention to electrons for the moment, we note that equation (192) has obvious implications for the Helium atom, where it will treat both electrons on equal footing and imply correlations similar to the "exchange interaction." Carrying out the calculations would provide a stringent test of (192) with implications for the Pauli principle.

Equation (192) also meets Carver Mead's objective for a many electron quantum state determined entirely by vector potentials of all particles in the system [2, 44]. It goes beyond Mead in anchoring the electron state in a Dirac wave function, in principle including the contribution of positive charges in the lattice of a superconductor. Note that P in (192) can be regarded as kind of "*superpotential*" for the entire system. It follows, then, that

$$F \equiv \Box \wedge P = \sum_{k=1}^{N} \Box \wedge p_k \tag{195}$$

can be regarded as the total electromagnetic field for the entire system.

Inside a superconductor we have $F = \mathbf{E} + i\mathbf{B} = 0$ (Meissner effect). Hence, as we have seen before, Stokes Theorem implies that for *any closed curve* in the region

$$\oint P \cdot dx = 0. \tag{196}$$

And, as in the single particle case, we get a many particle quantization condition

$$\int_0^{T_n} P_0\, dt = \oint \mathbf{P} \cdot d\mathbf{x} = (n + \tfrac{1}{2})h, \tag{197}$$

with integer n. This agrees with Mead's formulation of phase and flux quantization in a superconductor [2, 44]. The relevance of this argument to the *Aharonhov–Bohm effect* is also worth noting [45].

Specific application to superconductors is beyond the purview of this exploratory discussion. However, before dropping the subject, it is worth noting that the present model satisfies the additivity of electron phases $\varphi_k = \varphi_k(x - z_k)$ that is essential for superconductivity. That can be made manifest by writing the wave function in the form

$$\Psi = R \prod_{k=1}^{N} \Psi_k \Lambda, \tag{198}$$

where $R\widetilde{R} = 1,\ \Lambda^2 = 0$ and

$$\Psi_k = e^{-\alpha_k - \mathbf{i}\varphi_k}. \tag{199}$$

Evidently, the *Pauli principle* can be incorporated in symmetries of the wave function in the usual way, and that would identify it as a property of the vacuum! The symmetries need not apply to all particle variables, but only to particles whose motions are resonant in some sense, as in the quantized atomic states discussed earlier.

B. Electron Diffraction

Maxwell-Dirac theory has unique implications for the problem of electron diffraction. We point them out here without delving into detailed calculations or experimental tests.

The first and most important point is that, according to (190), the density $\rho = \rho(x)$ of a single electron factors into a product

$$\rho = \prod_{k=1}^{N} \rho_k = \check{\rho}_e \rho_e, \tag{200}$$

where $\rho_e(x)$ is the Blinder function of Maxwell-Dirac theory, and $\check{\rho}_e(x) = \check{\rho}_e(x, x_1, \ldots, x_{N-1})$ is the density expressed with Blinder functions of all other particles with influence. Since the Blinder function satisfies $0 \le |\rho_k| \le 1$, we also have $0 \le |\rho(x)| \le 1$. So there are no normalization issues, and sufficiently distant particles automatically have insignificant influence.

One consequence is that the "Quantum potential" in the Pilot Wave guidance law must have a causal source in matter composing the diffraction slits. To get the "acausal density" of Pilot Wave theory, the matter coordinates must be integrated out with some sort of average $\langle\rho\rangle$. That leaves the possibility open for fluctuations in path density, for example, from heating material in the slits.

We still have the problem of identifying a plausible mechanism for momentum exchange between each diffracted particle and the slits, a causal link which is missing from all accounts of diffraction by standard wave mechanics or by Pilot Wave theory. Note that momentum transfer is observable for each scattered particle, whereas the diffraction pattern conserves momentum only as a statistical average. Evidently the only way to account for this fact is by reducing diffraction to quantized momentum exchange between each particle and slit.

Duane was the first to offer a quantitative explanation for electron diffraction as quantized momentum exchange [46, 47]. A more general argument using standard quantum mechanics has been worked out by Van Vliet [48, 49]. These explanations suffer from the same disease as Old Quantum Mechanics in failing to account for the density distribution in the diffraction pattern. However, we now have the possibility of curing that disease with relativistic Pilot Wave theory. We only need to explain how the momentum exchange is incorporated into the Pilot Wave guidance law.

Now, presuming vanishing electric and magnetic fields outside the diffraction slits as before, we have $\Box\wedge A = 0$, so locally, at least, A is a gradient. Assuming the same for P, we have a gauge invariant *phase gradient*

$$\Box\Phi = P - \frac{e}{c}A. \qquad (201)$$

This provides a promising mechanism for quantized momentum transfer in diffraction. For we know that quantized states in QM are determined by boundary conditions on the phase. Successful calculation of diffraction patterns along these lines would provide strong evidence for the following claim: the vacuum surrounding electromagnetically inert matter is permeated by a vector potential with vanishing curl. Remarkably, the same mechanism would explain the extended Aharonov-Bohm (AB) effect [50]. Evidently, then, the causal agents for diffraction and the AB effect are one and the same: a universal vector potential permeating the vacuum (or, *Aether*, if you will) of all spacetime, much as proposed by Dirac [51].

Considering the similarity of electron and photon diffraction patterns, we should expect the same mechanism to explain photon diffraction. Indeed, the evolution of path density for the electron is determined by the Dirac equation, which gives

$$\Box^2\Phi = -m_e c\dot{z}\cdot\Box\ln\rho. \qquad (202)$$

For a photon with propagation vector k, the analog is

$$k\cdot\Box\ln\rho = \Box^2\Phi/\hbar, \qquad (203)$$

where, of course, ρ is the path density for photons, just as it is for electrons. Accordingly, we conclude that diffraction is "caused" by the vacuum surrounding material objects. In other words, *diffraction is refraction by the vacuum*!

We have seen that the Blinder form for the vacuum density $\rho = \rho(x)$, which was originally introduced to generalize the Coulomb potential, is actually determined by the momentum at each vacuum singularity independent of any charge. Evidently it applies to photons as well as electrons and protons, so it should be regarded as a universal property of the vacuum. This suggests association with a gravitational field. That possibility is best approached by a gauge theory as proposed below.

Strictly speaking, the density (impedance) of the vacuum should be incorporated into any vector potential by writing $\mathcal{A} = \rho A$, with a new notation to distinguish it from the usual vector potential, whether or not it is the gradient of a scalar field. The Aether can then be regarded as a conserved fluid (with $\Box\cdot\mathcal{A} = 0$) flowing through spacetime with particle singularities (electron, photon or whatever) in the density swept along. This picture has a beautiful macroscopic analog describing diffraction of a macro particle in a classical fluid [52].

C. Vacuum Topology

With electrons modeled as vacuum singularities, it is natural to consider the topology of more complex vacuum singularities to model the whole zoo of elementary particles, including photons. A promising possibility is based on the fact that, in a certain sense, the electroweak gauge group is already inherent in the Dirac equation, and a gauge theory version of gravitational interactions is readily included as well. A unified "gravelectroweak theory" of that kind has already been described in [53]. It suffices to summarize its main features here and discuss what Vacuum Dirac theory has to add.

A natural extension of the Dirac equation to include weak interactions rests on the unique fact that the electroweak gauge group SU(2)⊗U(1) is the maximal group of gauge transformations $\Psi\to\Psi' = \Psi U$ that leave the velocity observable invariant:

$$\rho u = \Psi\gamma_0\widetilde{\Psi} = \Psi'\gamma_0\widetilde{\Psi'}, \qquad (204)$$

This gives the gauge group geometric significance as the invariance group of the Dirac current, thereby insuring a spacetime path for the zitter center. To incorporate both gravitational and electroweak interactions in vacuum Dirac theory, we require invariance under the group

$$\Psi \quad\to\quad \Psi' = L\Psi U, \qquad (205)$$

where $L\widetilde{L} = 1$. The corresponding gauge invariant derivative is

$$D_\mu \Psi = (\partial_\mu + \tfrac{1}{2}\omega_\mu)\Psi - \Psi i W_\mu, \tag{206}$$

where the geometric "connexion" ω_μ expresses gravitational interaction and the W_μ express electroweak interactions. See [53] for details.

Now let us speculate on possibilities for refining electroweak theory by incorporating geometry enabled by the present formulation. A simple generalization of the vacuum impedance has the form

$$\psi = \rho^{1/2} e^{-i\boldsymbol{\sigma}_3\varphi}\, e^{-i\boldsymbol{\sigma}_1\chi}. \tag{207}$$

Now suppose that both angles are harmonically related functions of proper time $\varphi(m\tau)$ and $\chi(n\tau)$, where (n, m) is a pair of coprime integers known in knot theory as *writhe* and *rotation* numbers respectively. The angle φ generates circular zitter in the spin plane, while the angle χ tilts the plane, so together the two angles can be adjusted to generate a family of closed toroidal curves (or helices) with periods in ratio n/m.

Regarding each helix as the path of a point singularity like the electron, we have here a family of singularity types distinguished by quantum numbers (m, n). Since the interactions are electroweak, we adopt it as a candidate for the family of Leptons. We know that the simplest member is the electron, to which the quantum numbers $(1, 1)$ might be assigned to express double-valuedness of electron phase. The curve $(2, 3)$ is corresponds to a trefoil knot. But it would be premature to try classifying elementary particles here, as our objective is only to open up possibilities.

Others have proposed knot theory for classifying elementary particles, though not with such a direct tie to the Dirac equation and geometry of the vacuum. Jehle [54, 55], for one, proposed a classification based on quantized flux that has much in common with the present approach, but it is weak on connection with field equations. Finkelstein [56, 57] has developed a detailed match of knot topology with structure of the Standard Model. Knot topology can be taken as supplementing the present approach, which is based on differential geometry.

To the extent that our speculations on geometric grounding of leptonic states and interactions are credible, we should surely expect similar grounding for baryons and strong interactions. So let us indulge in one more round of speculation to survey possibilities for hadronic structure.

Since quarks have the same electroweak interactions as leptons, they might be regarded as leptons that are permanently bound into knots such as nucleons. Strong interactions would then be about tying and untying knots.

Since nucleons have the same spin as leptons we guess that they are also helical current loops, but with three singularities (quarks) on the loop instead of one. Quarks are distinguished by different helical structures (gluons) that connect them. Since the proton's charge is presumably divided up among the quarks, perhaps charge should be regarded as a property of the loop rather than singularities on the loop. Perhaps, then, the loop should be regarded as a stringlike vacuum defect in which the particle is embedded, rather than a path that the particle traverses. Shades of string theory!

As to the structure of gauge bosons, we would expect weak bosons to be open strings, like the photon, that peel off helical structure from a lepton on emission. Perhaps gluons are similar strings (or cuts in the fabric of spacetime, if you will), but with different types of attachment at quark endpoints.

Concerning the density of the vacuum in general, it should be derivable from (or, at least, consistent with) the Einstein-Maxwell gravitational field equations. Thus, the Blinder potential $\rho = exp(\lambda_c/|x - z_e|)$ must be modified to include gravity, though it will differ from the Kerr-Neuman solution in the way it treats the electron singularity. Presumably, the ideal solution will involve something like the Higgs mechanism to explain mass, so it will involve complex structure of the impedance.

Derivation of equations of motion for singularities from gravitational field equations has been an important theoretical objective since Einstein, Infeld and Hoffman attacked it [58]. We have seen something like that for Dirac theory. Conversely, if we develop a rich theory of singularities along the lines suggested here, that might require modification of the field equations. Gauge theory may then be regarded as a means for coordinating singularities with equations of motion. "Always keeping one principle object in view, to preserve their symmetrical shape!"

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